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Fundamental properties of generalized intuitionistic fuzzy groups

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Abstract: In this paper, we deal with Molaei's generalized groups. This paper is based on a new algebraic structure called generalized groups and application on intuitionistic fuzzy group. We defined a new structure called generalized intuitionistic fuzzy groups. In this paper, we applied an intuitionistic fuzzy property on a generalized groups.We researched a generalized intuitionistic fuzzy group is a intuitionistic fuzzy group under which conditions.We defined some proposition about relations between identical element and elemnts in G_a means that a set which element have same identical element also it was defined by M.R Molaei in membership function and a-level set.
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1 Introduction

The concept of fuzzy sets was introduced by Zadeh [9] as an extension of crisp sets by expanding the truth value set to the real unit interval $[0, 1]$. Let X be a set. The function $\mu : X \to [0, 1]$ is called a fuzzy set over $X(FS(X))$. For $x \in X$, $\mu(x)$ is the membership degree of x and the non-membership degree is $1 - \mu(x)$. Intuitionistic fuzzy sets have been introduced by Atanassov [3], as an extension of fuzzy sets. If X is a universal then a intuitionistic fuzzy set A, the membership and non-membership degree for each $x \in X$, respectively, $\mu_A(x)$ ($\mu_A : X \to [0, 1]$) and $\nu_A(x)$ (ν_A:$X \to [0, 1]$) such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. The class of intuitionistic fuzzy sets on X is denoted by $IFS(X)$. While the sum of membership degree and non-membership degree is 1 for fuzzy sets, this sum is less than 1 for intuitionistic fuzzy sets.

2 On Molaei's generalized groups

Definition 1 ([6]). *Let G be a non-empty set. For an operation on G, if the following properties are satisfied then G is called generalized group.*

i) $\forall x, y, z \in G, x(yz) = (xy)z$

ii) $\forall x, \exists!\, e(x) \in G, xe(x) = e(x)x = x$

iii) $\forall x, \exists\, I(X) \in G, xI(x) = I(x)x = e(x)$

The elements $e(x)$ and $I(x)$ are called neutral element and inverse of x respectively.

Example 1 ([6]). $\mathbb{R} \times (\mathbb{R} - \{0\})$ *with operation* $(a, b)(c, d) = (bc, bd)$.

In fact $e(a, b) = (\frac{a}{b}, 1)$, $I(a, b) = (\frac{a}{b^2}, \frac{1}{b})$,where $(a, b) \in \mathbb{R} \times (\mathbb{R} - \{0\})$

Solution 1.

i) $\forall (a, b)$, (c, d) and $(e, f) \in \mathbb{R} \times (\mathbb{R} - \{0\})$,

$(a, b)((c, d)(e, f)) = (a, b)(de, df) = (bde, bdf) \quad (*)$
$((a, b)(c, d))(e, f) = (bc, bd)(e, f) = (bde, bdf) \quad (**)$
From $(*)$ and $(**)$ we have, $(a, b)((c, d)(e, f)) = ((a, b)(c, d))(e, f)$

ii) $\forall (a, b) \in \mathbb{R} \times (\mathbb{R} - \{0\}), (a, b)e(a, b) = (a, b)$ and let $e(a, b) = (a^*, b^*)$ then,

$(a, b)(a^*, b^*) = (a, b) \implies (ba^*, bb^*) = (a, b) \implies ba^* = a$ and
$bb^* = b \implies a^* = \frac{a}{b}$ and $b^* = 1$
Now we assume that $e(a, b) = (\frac{a}{b}, 1)$,
$(a, b)e(a, b) = (a, b)(\frac{a}{b}, 1) = (b\frac{a}{b}, b1) = (a, b) \quad (*)$
$e(a, b)(a, b) = (\frac{a}{b}, 1)(a, b) = (a, b) \quad (**)$
From $(*)$ and $(**)$, we get $e(a, b) = (\frac{a}{b}, 1)$

iii) $\forall (a, b) \in \mathbb{R} \times (\mathbb{R} - \{0\})$,

$(a, b)I(a, b) = e(a, b)$, from (ii) we know that $e(a, b) = (\frac{a}{b}, 1)$ and
let $I(a, b) = (a^{'}, b^{'})$ then,
$(a, b)(a^{'}, b^{'}) = (\frac{a}{b}, 1) \implies (ba^{'}, bb^{'}) = (\frac{a}{b}, 1) \implies ba^{'} = \frac{a}{b}$ and
$bb^{'} = 1 \implies a^{'} = \frac{a}{b^2}$ and $b^{'} = \frac{1}{b}$
Now we assume that $I(a, b) = (\frac{a}{b^2}, \frac{1}{b})$,

$$\begin{aligned}(a,b)I(a,b) &= (a,b)(\frac{a}{b^2},\frac{1}{b}) = (b\frac{a}{b^2}, b\frac{1}{b}) = (\frac{a}{b},1) = e(a,b)\\ &\Longrightarrow (a,b)I(a,b) = e(a,b) \qquad (1)\end{aligned}$$

$$\begin{aligned}I(a,b)(a,b) &= (\frac{a}{b^2},\frac{1}{b})(a,b) = (\frac{1}{b}a, \frac{1}{b}b) = (\frac{a}{b},1) = e(a,b)\\ &\Longrightarrow I(a,b)(a,b) = e(a,b) \qquad (2)\end{aligned}$$

From (1) and (2), we get $I(a,b) = (\frac{a}{b^2},\frac{1}{b})$
$(\mathbb{R}\times(\mathbb{R}-\{0\}),.)$ satisfies (i), (ii) and (iii) so it is a generalized group.

Proposition 1. *[6] If G is a generalized group and $x \in G$ then*

i) $e(e(x)) = e(x)$

ii) $e(I(x)) = e(x) = I(e(x))$

iii) $I(x)$ is unique

iv) $I(I(x)) = x$

Proof. Let $x \in G$ be given. Then

i) $xe(e(x)) = (xe(x))e(e(x)) = x(e(x)e(e(x))) = xe(x) = x \ (*)$

$e(e(x))x = e(e(x))(xe(x)) = e(e(x))(e(x)x) = (e(e(x))e(x))x = e(x)x = x \ (**)$
From $(*)$ and $(**)$ we have, $xe(e(x)) = e(e(x)) = x$.
The uniqueness of $e(x)$ implies that $e(e(x)) = e(x)$.
By use of $e(e(x)) = e(x)$, we have $e(x)I(e(x)) = e(e(x)) = e(x) \implies I(e(x)) = e(x)$

ii) $(e(x)e(I(x)))x = (xI(x)e(I(x)))x = (xI(x))x = e(x)x = x \ (*)$

$x(e(x)e(I(x))) = x(xI(x)e(I(x))) = x(xI(x)) = xe(x) = x \ (**)$
From $(*)$ and $(**)$, we have that $e(x) = e(x)e(I(x))$.
Similarly $e(x) = e(I(x))e(x)$ so $e(x) = e(e(x)) = e(I(x))$

iii) Let $x^*$and x^{**} be inverse elements of x. Then $x^* = e(x^*)x^* = e(x)x^* = (x^{**}x)x^* = x^{**}(xx^*) = x^{**}e(x) = x^{**}e(x^{**}) = x^{**} \implies x^* = x^{**}$

iv) The uniqueness of $I(x)$ implies that $I(I(x)) = x$.

□

Proposition 2 ([8])**.** *Let G be a generalized group and let $xy = yx$ for some $x, y \in G$. Then $e(xy) = e(x)e(y)$.*

Proof. $e(x)(xy) = (e(x)x)y = xy$ and $(xy)e(x) = (yx)e(x) = y(xe(x)) = yx = xy$ so $e(xy) = e(x)$ $(*)$

$e(y)(xy) = e(y)(yx)) = (e(y)y)x = yx = xy$ and $(xy)e(y) = x(y(e(y)) = xy$ so $e(xy) = e(y)$ $(**)$

From $(*)$ and $(**)$, $e(xy) = e(x) = e(y)$ □

Corollary 1 ([8]). *If G is a generalized group and $xy = yx$ for all $x, y \in G$ then G is a group.*

Definition 2 ([8]). *Let G be a generalized group. Then $N \subseteq G$ is a called an e-component of G if $e(x) = e(y)$ for all $x, y \in N$*

N is the maximal subset of G with this property

Lemma 1 ([6]). *Let N be an e-component of generalized group G and $x \in N, e(x) \in N, I(x) \in N$. In particular, N is a group.*

Proof. Let N be e-component of G and $x \in N$. If $e(x) \in G - N$, then $e(e(x)) = e(x)$ makes contradiction with the maximality of N.

Hence we have $e(x) \in N$ similarly $I(x) \in N$.

All elements of N have the same identity element so N is a group. □

Corollary 2 ([6]). *If G is a generalized group then it is a disjoint union of groups. $G = \cup G_a$ where,$G_a = \{x \in G : e(x) = e(a)\}$ are e-components of G.*

Definition 3 ([6]). *(Normal generalized group) A generalized group is called a normal generalized group. if $\forall a, b \in G, e(ab) = e(a)e(b)$*

Theorem 1 ([6]). *Let H be a non-empty subset of G. Then H is a generalized subgroup of G if and only if $\forall a, b \in H, aI(b) \in H$.*

Proof. " $\Longrightarrow$ " Let H be a generalized subgroup of G.

Then $a, b \in H, I(b) \in H \implies aI(b) \in H$.

" $\Longleftarrow$ " Let $H \neq \varnothing$ and, $b \in H$,$aI(b) \in H$.

Then we have $e(b)I(b) = I(b) \in H$ and $ab = aI(I(b)) \in H$. □

3 Generalized intuitionistic fuzzy groups

M. Bakhshi and R. A. Borzoei gave the defininition generalized fuzzy subgroups such that, G denotes a generalized group, $FS(G)$ the set of all fuzzy subsets of G, unless otherwise specified.

Definition 4 ([4]). *Let $\mu \in FS(G)$. μ is called generalized fuzzy subgroup if for all $x, y \in G$,*

i) $\mu(xy) \geq \mu(x) \wedge \mu(y)$

ii) $\mu(I(x)) \geq \mu(x)$

Definition 5. *Let G be a generalized group, $A \in IFS(G)$. Then A is called a generalized intuitionistic fuzzy group if A satisfies propertsies which are given below;*

i) $\forall a, b \in G, \quad A(ab) \geqslant A(a) \wedge A(b)$

ii) $\forall a \in G, \; A(e(a)) \geqslant \sup_{x \in G_a} A(x)$

iii) $\forall a \in G, \quad A(I(a)) \geqslant A(a)$

For the special case of an intuitionistic fuzzy sets seen as a fuzzy set, we get the definition of a fuzzy generalized group as follows,

Definition 6. *Let $\mu \in FS(G)$; then μ is called generalized fuzzy subgroup if for all $x, y \in G$,*

i) $\mu(xy) \geq \mu(x) \wedge \mu(y)$

ii) $\mu(e(x)) = 1$

iii) $\mu(I(x)) \geq \mu(x)$

Example 2. *Let G be a generalized group which is given above. Let $\forall (a, b) \in G$, $\mu : G \longmapsto I$ and $\nu : G \longmapsto I$.*

Let

$$\mu(a, b) = \frac{|a|}{2(|a| + |b|)}$$

and

$$\nu(a, b) = \frac{|a|}{3(|a| + |b|)},$$

then

$$0 \leq \frac{|a|}{2(|a| + |b|)} < \frac{|a|}{|a| + |b|} < \frac{|a|}{|a|} \leq 1 \implies 0 \leq \frac{|a|}{2(|a| + |b|)} < 1 \implies 0 \leq \mu(a, b) < 1$$

and

$$0 \leq \frac{|a|}{3(|a| + |b|)} < \frac{|a|}{|a| + |b|} < \frac{|a|}{|a|} \leq 1 \implies 0 \leq \frac{|a|}{3(|a| + |b|)} < 1 \implies 0 \leq \nu(a, b) < 1.$$

From these inequalities, we get

$$0 \leq \frac{|a|}{2(|a| + |b|)} + \frac{|a|}{3(|a| + |b|)} = \frac{5\,|a|}{6(|a| + |b|)} < \frac{|a|}{|a| + |b|} < 1 \implies 0 \leq \mu(a, b) + \nu(a, b) < 1$$

$$\mu((a, b)(c, d)) = \mu(bc, bd) = \frac{|bc|}{2(|bc| + |bd|)} = \frac{|c|}{2(|c| + |d|)}$$

and

$$\mu(a, b) = \frac{|a|}{2(|a| + |b|)}$$

and

$$\mu(c, d) = \frac{|c|}{2(|c| + |d|)}$$

for $a \geq a \wedge b$.

Thus,

$$\mu((a,b)(c,d)) \geq \mu(a,b) \wedge \mu(c,d).$$

Let $(a,b) \in G$*,*

$$\mu(e(a,b)) = \mu(\frac{a}{b},1) = \frac{\left|\frac{a}{b}\right|}{2(\left|\frac{a}{b}\right|+1)} = \frac{|a|}{(2\,|a|+|b|)} = \mu(a,b)$$

and let $(x,y) \in G_{(a,b)}$*,*

$$\begin{aligned}\mu(x,y) &= \mu((x,y)e(x,y)) = \mu((x,y)(\frac{a}{b},1) = \mu(\frac{ya}{b},y) \\ &= \frac{\left|\frac{ya}{b}\right|}{2(\left|\frac{ya}{b}\right|+|y|)} = \frac{\left|\frac{a}{b}\right|}{2(\left|\frac{a}{b}\right|+1)} = \frac{|a|}{2(|a|+|b|)} = \mu(a,b)\end{aligned}$$

so,

$$\mu(e(a,b)) = \sup_{(x,y)\in G_{(a,b)}} \mu(x,y) = \mu(x,y)$$

$$\implies \mu(e(a,b)) \geq \sup_{(x,y)\in G_{(a,b)}} \mu(x,y).$$

$$\mu(I(a,b)) = \mu(\frac{a}{b^2},\frac{1}{b}) = \frac{\left|\frac{a}{b^2}\right|}{2(\left|\frac{a}{b^2}\right|+\left|\frac{1}{b}\right|)} = \frac{\left|\frac{a}{b}\right|}{2(\left|\frac{a}{b}\right|+1)} = \frac{|a|}{2(|a|+|b|)}$$

$$= \mu(a,b) \implies \mu(I(a,b)) \geq \mu(a,b)$$

$$\nu((a,b)(c,d)) = \nu(bc,bd) = \frac{|bc|}{3(|bc|+|bd|)} = \frac{|c|}{3(|c|+|d|)} = \nu(c,d)$$

and

$$\nu(a,b) = \frac{|a|}{3(|a|+|b|)}.$$

For $a \leq a \vee b, \nu((a,b)(c,d)) \leq \nu(a,b) \vee \nu(c,d)$

$\nu(e(a,b)) = \nu(\frac{a}{b},1) = \frac{\left|\frac{a}{b}\right|}{3(\left|\frac{a}{b}\right|+1)} = \frac{|a|}{3(|a|+|b|)} = \nu(a,b)$ *and arbitrary* $(x,y) \in G_{(a,b)}$*,*

$$\begin{aligned}\nu(x,y) &= \nu((x,y)e(x,y)) = \nu((x,y)e(a,b)) = \nu((x,y)(\frac{a}{b},1) = \nu(\frac{ya}{b},y) \\ &= \frac{\left|\frac{ya}{b}\right|}{3(\left|\frac{ya}{b}\right|+|y|)} = \frac{\left|\frac{a}{b}\right|}{3(\left|\frac{a}{b}\right|+1)} = \frac{|a|}{3(|a|+|b|)} = \nu(a,b)\end{aligned}$$

so,

$$\nu(e(a,b)) = \inf_{(x,y)\in G_{(a,b)}} \nu(a,b) = \nu(a,b) \implies \nu(e(a,b)) \leq \inf_{(x,y)\in G_{(a,b)}} \nu(x,y)$$

$$\nu(I(a,b)) = \nu(\frac{a}{b^2},\frac{1}{b}) = \frac{\left|\frac{a}{b^2}\right|}{3(\left|\frac{a}{b^2}\right|+\left|\frac{1}{b}\right|)} = \frac{\left|\frac{a}{b}\right|}{3(\left|\frac{a}{b}\right|+1)} = \frac{|a|}{3(|a|+|b|)} = \nu(a,b)$$

$$\implies \nu(I(a,b)) \leq \nu(a,b)$$

Therefore, $A \in IFS(G)$ *is a generalized intuitionistic group.*

Proposition 3. *Let G be a generalized group and $A \in IFS(G)$ be a generalized intuitionistic fuzzy group. If $\forall x.y \in G, xy = yx$, then A is a intuitionistic fuzzy group.*

Proof. We know that, if G is commutative then G is a group. Therefore, A is a intuitionistic fuzzy group. □

Proposition 4. *G is a generalized group and $A \in IFS(G)$ be a generalized intuitionistic fuzzy group, $\forall a \in G, A(I(a)) = A(a)$.*

Proof. In a generalized intuitionistic fuzzy group, We know that $A(I(a)) \geq A(a)$ $(*)$

So, we can see $A(I(I(a)) \geq A(I(a)) \implies A(a) \geq A(I(a))$ $(**)$

From $(*)$ and $(**)$ then we get $A(I(a)) = A(a)$. □

Proposition 5. *All intuitionistic fuzzy groups are generalized intuitionistic fuzzy groups.*

Proof. All groups are generalized groups. Therefore, the proof is clear. □

Definition 7. *$A : G \longrightarrow I$ is a generalized intuitionistic fuzzy group and let A have the property;* $\sup_{a\in E} A(a) = A(c), \exists c \in X$. *Then we define the following sets.*

$$E = \{e(x) : x \in G\}$$
$$\varepsilon = \left\{ e(x) : A(e(x)) = \sup_{a\in E} A(a) \right\}$$

The elements of ε are called universal elements. The set ε is not empty.

Proposition 6. *Let G be a generalized group, e be a universal identity element and $A \in IFS(G)$ be a generalized intuitionistic fuzzy group. Then $\forall x \in G, A(e) \geq A(x)$.*

Proof. From the definition of universal identity elements, $A(e) \geq A(e(x)) \geq \sup_{a\in E} A(a)$ and from the definition E and generalized intuitionistic fuzzy group. Therefore, $\forall a \in E$, $A(a) \geq \sup_{x\in G_a} A(a)$. From this properties, we get

$$A(e) \geq A(e(x)) \geq \sup_{a\in A} \sup_{x\in G_a} A(x) \geq A(x) \implies A(e) \geq A(x).$$

□

Remark 1. *Let G be a group and $A \in IFS(G)$ is a intuitionistic fuzzy group then $\forall x \in G$, $A(x^n) \geq A(x)$, but it may not be like that in the generalized intuitionistic fuzzy group.*

Proposition 7. *Let G be a generalized group, $A \in IFS(G)$ is a generalized intuitionistic fuzzy group and e and I be identity functions. Then $n \in \mathbb{N}, A(e(x^n)) = A(x)$*

Proof. Let us take arbitrary $a, b \in G$. We can easily see that in a generalized group such that e and I are identity functions.

$$e(ab) = ab = e(a)e(b)$$

and

$$a = b \implies e(a^2) = e(aa) = aa = e(a)e(a) = e(a).$$

Let $e(a^{n-1}) = e(a)$, then

$$e(a^n) = e(a^{n-1}a) = e(a^{n-1})e(a) = e(a)e(a) = e(a) \implies e(a^n) = e(a).$$

Therefore, for $a \in G, A(e(a^n)) = A(e(a)) = A(a) \implies A(e(a^n) = A(a)$. □

Proposition 8. *Let G be a generalized group and $A \in IFS(G)$ be a generalized intuitionistic fuzzy group. Then $\forall a \in G, A_{\downarrow G_a}$ is a intuitionistic fuzzy group.*

Proof. If we use the property, G_a is a group. Then the proof is easily seen. □

Theorem 2. *Let G be a generalized group, $A \in IFS(G)$ be a generalized intuitionistic fuzzy group. Then $A_{t,s}$ is a generalized group for every $t, s \in I$, where*

$$A_t = \{x \in G : A(x) \geq (t, s), t, s \in I\}.$$

Proof. For arbitrary $x \in A_{t,s}, \forall a \in G$,

$$A(e(a)) \geq \sup_{x \in G_a} A(x) \geq A(x) \geq (t, s) \implies e(a)) \in A_{t,s} \implies A_{t,s} \neq \varnothing.$$

For $x, y \in A_{t,s}$, $A(x) \geq t$ and $A(y) \geq (t, s) \implies A(x) \wedge A(y) \geq (t, s)$.

$$A(xI(y)) \geq A(x) \wedge A(I(y)) \geq A(x) \wedge A(y) \geq (t, s) \implies xI(y) \in A_{t,s}.$$

Therefore, $A_{t,s}$ is a generalized subgroup of G. Then $A_{t,s}$ is a generalized group. □

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