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φ–entropy of IF-partitions

Beloslav Riečan[1,2] and Dagmar Markechová[3]

[1] Department of Mathematics, Faculty of Natural Sciences
Matej Bel University
Tajovského 40, SK-974 01 Banská Bystrica, Slovakia
[2] Mathematical Institut, Slovak Academy of Sciences
Štefánikova 49, SK-814 73 Bratislava, Slovakia
e-mail: Beloslav.Riecan@umb.sk

[3] Department of Mathematics, Faculty of Natural Sciences
Constantine the Philosopher University in Nitra
A. Hlinku 1, SK-949 01 Nitra, Slovakia
e-mail: dmarkechova@ukf.sk

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Abstract: In the paper a common formulation is given for two types of entropy of partitions in the intuitionistic fuzzy case: the Shannon-Kolmogorov-Sinai entropy ([6]) and the logical entropy ([4]).
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The paper is dedicated to Professor Janusz Kaczprzyk
on the occasion of his 70th birthday.

1 Introduction

The notion of entropy is a fundamental concept in information theory [5]; it is used as a measure of information which we get from a realization of the considered experiment. As is known, the usual approach in information theory is based on Shannon's entropy [13]. Consider a finite measurable partition $\mathcal{A}$ of a probability space (Ω, S, P) with probabilities $p_1, ..., p_n$ of the corresponding elements of $\mathcal{A}$. We recall that the Shannon entropy of $\mathcal{A}$ is the number

$$H_s(\mathcal{A}) = \sum_{i=1}^{n} s(p_i),$$

where $s:[0,1]\to[0, \infty)$ is the Shannon function defined by $s(x)=-x\log x,$ for every $x\in[0,1]$. Note that we use the convention (based on continuity arguments) that $0\cdot\log 0=0$. The idea of Shannon's entropy was generalized in a natural way to the Kolmogorov–Sinai entropy $h(T)$ of dynamical systems ([6, 14]). Kolmogorov and Sinai applied the entropy $h(T)$ to prove the existence of non-isomorphic Bernoulli shifts. Of course, the theory of Kolmogorov-Sinai entropy has many other applications. That is why various proposals for a generalization of the concept of Kolmogorov–Sinai entropy have been created (see e.g., [7, 8, 11]).

When solving some specific problems, it is more appropriate to use instead of Shannon's entropy an approach based on the concept of logical entropy (see e.g., [4]). In [4], classical logical entropy was discussed by Ellerman as an alternative measure of information. Recall that the logical entropy of a finite measurable partition $\mathcal{A}$ with probabilities $p_1, ..., p_n$ of the corresponding elements is defined as the number

$$H_l(\mathcal{A}) = \sum_{i=1}^{n} l(p_i),$$

where $l:[0,1]\to[0, \infty)$ is the logical entropy function defined by $l(x)=x(1-x),$ for every $x\in[0,1]$.

Both concepts has been developed also in the intuitionistic fuzzy case ([2, 9. 10, 15]). In the paper we introduce an entropy of partitions in the intuitionistic fuzzy case based on a function $\varphi:[0,1]\to[0, \infty)$ containing both mentioned concepts. We shall consider the $\varphi-$ entropy of an IF-partition $\xi=\{A_1,...,A_n\}$ defined by the formula

$$H_\varphi(\xi) = \sum_{i=1}^{n} \varphi(m(A_i)).$$

If we put $\varphi=s,$ then we obtain the Shannon entropy of IF-partition $\xi,$ if we put $\varphi=l,$ we obtain the logical entropy of IF-partition $\xi.$ Our main result is the subadditivity property for IF-partitions ξ,η:

$$H_\varphi(\xi\vee\eta) \le H_\varphi(\xi) + H_\varphi(\eta).$$

2 Basic definitions, notations and facts

An intuitionistic fuzzy set [1] (shortly IF-set) defined on a non-empty set Ω is a mapping $A=(\mu_A,\nu_A):\Omega\to[0, 1]\times[0, 1]$ such that $\mu_A(\omega)+\nu_A(\omega)\le 1,$ for every $\omega\in\Omega.$ The function μ_A is called the membership function of A, the function ν_A is called the non-membership function of A. Denote by $\mathcal{F}$ the family of all IF-sets on Ω. We shall define a partial binary operation $\oplus$ on the family $\mathcal{F}$. If $A=(\mu_A,\nu_A),$ and $B=(\mu_B,\nu_B)$ are two IF-sets from the family $\mathcal{F}$, then $A \oplus B$ is defined in $\mathcal{F}$, if $\mu_A(\omega)+\mu_B(\omega)\le 1,\ \nu_A(\omega)+\nu_B(\omega)\ge 1,$ for every $\omega\in\Omega$, by the formula:

$$A\oplus B=(\mu_A+\mu_B,\nu_A+\nu_B-1_\Omega).$$

Here, 1_Ω denotes the function defined by $1_\Omega(\omega)=1$, for every $\omega\in\Omega$. Similarly, we denote by 0_Ω the function defined by $0_\Omega(\omega)=0$, for every $\omega\in\Omega$. The zero element of operation $\oplus$ is the IF-set $0=(0_\Omega,1_\Omega)$. Indeed, $A\oplus 0 =(\mu_A,\nu_A)\oplus(0_\Omega,1_\Omega) =(\mu_A,\nu_A)=A$, for any $A\in\mathcal{F}$. Moreover, the product $A\cdot B$ is defined for any $A, B\in\mathcal{F}$ by the formula:

$$A\cdot B=(\mu_A\cdot\mu_B,\ 1_\Omega-(1_\Omega-\nu_A)\cdot(1_\Omega-\nu_B))=(\mu_A\cdot\mu_B,\ \nu_A+\nu_B-\nu_A\cdot\nu_B).$$

Put $1=(1_\Omega,\ 0_\Omega)$. Evidently, $A\cdot 1=A$, for any $A\in\mathcal{F}$. It can easily be verified that, for any $A, B, C\in\mathcal{F}$, the following conditions are satisfied:

(F1) $A\oplus B=B\oplus A$ if one side is defined in $\mathcal{F}$ (commutativity);

(F2) $(A\oplus B)\oplus C=A\oplus(B\oplus C)$ if one side is defined in $\mathcal{F}$ (associativity);

(F3) If $A\oplus B$ exists, then $C\cdot A\oplus C\cdot B$ exists, and $C\cdot(A\oplus B)=C\cdot A\oplus C\cdot B$ (the distributive law).

We write $A\le B$ if and only if $\mu_A\le\mu_B$, and $\nu_A\ge\nu_B$. The relation $\le$ is a partial order such that $0\le A\le 1$ for all $A\in\mathcal{F}$.

Definition 1. A map $m:\mathcal{F}\to[0,1]$ is said to be a state if the following conditions are satisfied:
(i) if $A\oplus B$ is defined in $\mathcal{F}$, then $m(A\oplus B)=m(A)+m(B)$;
(ii) $m(1)=1$.

Denote by $\mathcal{M}$ the family of all mappings $A=(\mu_A,\nu_A):\Omega\to[0,1]\times[0,1]$. If $A=(\mu_A,\nu_A)$, and $B=(\mu_B,\nu_B)$ are two elements of $\mathcal{M}$, then we put $A\oplus B=(\mu_A+\mu_B,\nu_A+\nu_B-1_\Omega)$, and $A\cdot B=(\mu_A\cdot\mu_B,\ \nu_A+\nu_B-\nu_A\cdot\nu_B)$.

Theorem 1. Let $m:\mathcal{F}\to[0,1]$ be a state and $\mathcal{M}$ be the family of all mappings $A=(\mu_A,\nu_A):\Omega\to[0,1]\times[0,1]$. Then the mapping $\overline{m}:\mathcal{M}\to[0,1]$ defined, for any element $A=(\mu_A,\nu_A)$ of $\mathcal{M}$, by

$$\overline{m}((\mu_A,\nu_A))=m((\mu_A,0_\Omega))-m((0_\Omega,1-\nu_A))$$

is a state, and $\overline{m}/\mathcal{F}=m$.

Proof :The proof can be found in [12]. □

Proposition 1. Let $A\in\mathcal{F}$ such that $m(A)=1$. Then $m(A\cdot B)=m(B)$, for any $B\in\mathcal{F}$.

Proof: Put $C=(1_\Omega-\mu_A,1_\Omega-\nu_A)$. Then

$$A\oplus C=(\mu_A+1_\Omega-\mu_A,\nu_A+1_\Omega-\nu_A-1_\Omega)=(1_\Omega,0_\Omega)=1,$$

$$A\cdot B\oplus B\cdot C=(\mu_A\cdot\mu_B,\nu_A+\nu_B-\nu_A\cdot\nu_B)\oplus(\mu_B(1_\Omega-\mu_A),\nu_B+1_\Omega-\nu_A-\nu_B(1_\Omega-\nu_A))=B,$$

and

$$1=\overline{m}(A)+\overline{m}(C)=1+\overline{m}(C),$$

hence $\overline{m}(C)=0.$ From the monotonicity of $\overline{m}$ it follows $\overline{m}(B\cdot C)\leq\overline{m}(C)=0.$
Therefore

$$m(B)=\overline{m}(B)=\overline{m}(A\cdot B)+\overline{m}(B\cdot C)=\overline{m}(A\cdot B)=m(A\cdot B). \qquad \square$$

Definition 2. By an IF-partition on $\mathcal{F}$ we mean a finite collection $\xi=\{A_1,\dots,A_n\}$ of elements of $\mathcal{F}$ such that $\oplus_{i=1}^n A_i$ exists, and $m\left(\oplus_{i=1}^n A_i\right)=\sum_{i=1}^n m(A_i)=1.$

Given two IF-partitions $\xi=\{A_1,\dots,A_n\}$, and $\eta=\{B_1,\dots,B_m\}$ their join $\xi\vee\eta$ is defined as the system $\xi\vee\eta=\{A_i\cdot B_j;\ i=1,\dots,n,\ j=1,\dots,m\}$ if $\xi\neq\eta,$ and $\xi\vee\xi=\xi.$

It is easy to see that $\xi\vee\eta$ is an IF-partition on $\mathcal{F}$. Namely, by definition $\oplus_{i=1}^n A_i$, and $\oplus_{j=1}^m B_j$ exist, hence according to (F3) $\oplus_{i=1}^n\oplus_{j=1}^m(A_i\cdot B_j)$ also exists, and $\oplus_{i=1}^n\oplus_{j=1}^m(A_i\cdot B_j)=(\oplus_{i=1}^n A_i)\cdot(\oplus_{j=1}^m B_j).$

By Definition 1 we have

$$m\left(\oplus_{i=1}^n\oplus_{j=1}^m(A_i\cdot B_j)\right)=\sum_{i=1}^n\sum_{j=1}^m m(A_i\cdot B_j).$$

Moreover, using Proposition 1 we get

$$m\left(\oplus_{i=1}^n\oplus_{j=1}^m(A_i\cdot B_j)\right)=m\left((\oplus_{i=1}^n A_i)\cdot(\oplus_{j=1}^m B_j)\right)=m(\oplus_{j=1}^m B_j)=1.$$

3 φ–entropy of IF-partitions

In this section we define the $\varphi-$entropy of IF-partitions. As special cases we obtain the Kolmogorov entropy and the logical entropy of IF-partitions.

Definition 3. If $\varphi:[0,1]\to\Re$ is a mapping, $\xi=\{A_1,\dots,A_n\}$ is an IF-partition, then we define the $\varphi-$entropy of ξ by the formula

$$H_\varphi(\xi)=\sum_{i=1}^n\varphi(m(A_i)).$$

Definition 4. A function $\varphi:[0,1]\to[0,\infty)$ is called a subadditive generator, if the following implication holds: if $c_{ij}\in[0,1]$, $i=1,\dots,n,$ $j=1,\dots,m,$ $\sum_{i=1}^n c_{ij}=b_j,$ $j=1,\dots,m,$ $\sum_{j=1}^m c_{ij}=a_i,$ $i=1,\dots,n,$ and $\sum_{i=1}^n a_i=1,$ $\sum_{j=1}^m b_j=1,$ then

$$\sum_{i=1}^n\sum_{j=1}^m\varphi(c_{ij})\leq\sum_{i=1}^n\varphi(a_i)+\sum_{j=1}^m\varphi(b_j).$$

Example 1. The Shannon function $s:[0,1]\to[0,\infty)$ is a subadditive generator. Namely, if $c_{ij}\in[0,1]$, $i=1,\dots,n, j=1,\dots,m$, $\sum_{i=1}^{n}c_{ij}=b_j$, $\sum_{j=1}^{m}c_{ij}=a_i$, $i=1,\dots,n$, $j=1,\dots,m$, and $\sum_{i=1}^{n}a_i=1$, $\sum_{j=1}^{m}b_j=1$, then

$$\sum_{i=1}^{n}\sum_{j=1}^{m}s(c_{ij})=-\sum_{i=1}^{n}\sum_{j=1}^{m}c_{ij}\log c_{ij}=-\sum_{i=1}^{n}\sum_{j=1}^{m}c_{ij}\log\left(\frac{c_{ij}}{b_j}\cdot b_j\right)$$

$$=-\sum_{i=1}^{n}\sum_{j=1}^{m}c_{ij}\log b_j-\sum_{i=1}^{n}\sum_{j=1}^{m}c_{ij}\log\frac{c_{ij}}{b_j}.$$

Of course,

$$-\sum_{j=1}^{m}(\sum_{i=1}^{n}c_{ij})\log b_j=-\sum_{j=1}^{m}b_j\log b_j=\sum_{j=1}^{m}s(b_j).$$

Moreover, since the function $y=s(x)$ is concave, and $\sum_{j=1}^{m}b_j=1$, we have

$$-\sum_{j=1}^{m}c_{ij}\log\frac{c_{ij}}{b_j}=\sum_{j=1}^{m}b_j s\left(\frac{c_{ij}}{b_j}\right)\le s\left(\sum_{j=1}^{m}b_j\frac{c_{ij}}{b_j}\right)=s\left(\sum_{j=1}^{m}c_{ij}\right)=s(a_i).$$

Therefore

$$-\sum_{i=1}^{n}\sum_{j=1}^{m}c_{ij}\log\frac{c_{ij}}{b_j}\le\sum_{i=1}^{n}s(a_i).$$

Hence we get

$$\sum_{i=1}^{n}\sum_{j=1}^{m}s(c_{ij})\le\sum_{j=1}^{m}s(b_j)+\sum_{i=1}^{n}s(a_i).$$

Example 2. Let $l:[0,1]\to[0,\infty)$ be the logical entropy function. If $c_{ij}\in[0,1]$, $i=1,\dots,n, j=1,\dots,m$, $\sum_{i=1}^{n}c_{ij}=b_j$, $\sum_{j=1}^{m}c_{ij}=a_i$, $i=1,\dots,n, j=1,\dots,m$, and $\sum_{i=1}^{n}a_i=1$, $\sum_{j=1}^{m}b_j=1$, then

$$\sum_{i=1}^{n}\sum_{j=1}^{m}l(c_{ij})=1-\sum_{i=1}^{n}\sum_{j=1}^{m}c_{ij}^{2}$$

$$=1-\sum_{j=1}^{m}b_j^{2}+\sum_{j=1}^{m}(\sum_{i=1}^{n}c_{ij})b_j-\sum_{i=1}^{n}\sum_{j=1}^{m}c_{ij}^{2}.$$

But

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{m}c_{ij}(b_j-c_{ij})\right)$$

$$\le\sum_{i=1}^{n}\left(\left(\sum_{j=1}^{m}c_{ij}\right)\left(\sum_{j=1}^{m}(b_j-c_{ij})\right)\right)$$

$$=\sum_{i=1}^{n}a_i\left(1-\sum_{j=1}^{m}c_{ij}\right)=\sum_{i=1}^{n}a_i(1-a_i)=\sum_{i=1}^{n}l(a_i).$$

Hence

$$\sum_{i=1}^{n}\sum_{j=1}^{m}l(c_{ij})\le\sum_{j=1}^{m}l(b_j)+\sum_{i=1}^{n}l(a_i).$$

This means that the logical entropy function is a subadditive generator.

Example 3. Let us consider the function $k:[0,1]\to[0,\infty)$ defined by $k(x)=x(1-x^2)$, for every $x\in[0,1]$. We will show that the function k is a subadditive generator.

Let $c_{ij}\in[0,1]$, $i=1,\dots,s, j=1,\dots,t,$ such that $\sum_{i=1}^{s}\sum_{j=1}^{t}c_{ij}=1$. Put $a_i=\sum_{j=1}^{t}c_{ij}$, $b_j=\sum_{i=1}^{s}c_{ij}$, $i=1,\dots,s, j=1,\dots,t.$ We have to prove the inequality

$$\sum_{i=1}^{s}\sum_{j=1}^{t}k(c_{ij})\le\sum_{i=1}^{s}k(a_i)+\sum_{j=1}^{t}k(b_j),$$

resp. equivalently,

$$1-\sum_{i=1}^{s}\sum_{j=1}^{t}c_{ij}{}^3\le 1-\sum_{i=1}^{s}a_i{}^3+1-\sum_{j=1}^{t}b_j{}^3.$$

Let us calculate:

$$\sum_{i=1}^{s}a_i{}^3+\sum_{j=1}^{t}b_j{}^3=\sum_{i=1}^{s}(\sum_{j=1}^{t}c_{ij})^3+\sum_{j=1}^{t}(\sum_{i=1}^{s}c_{ij})^3$$

$$=\sum_{i=1}^{s}\sum_{j=1}^{t}c_{ij}^3+\sum_{i=1}^{s}(3\sum_{k=1}^{t}\sum_{l=1}^{t}c_{ik}^2c_{il})+\sum_{j=1}^{t}\sum_{i=1}^{s}c_{ij}^3+\sum_{j=1}^{t}(3\sum_{k=1}^{s}\sum_{l=1}^{s}c_{kj}^2c_{lj})$$

$$\le\sum_{i=1}^{s}\sum_{j=1}^{t}c_{ij}^3+(\sum_{i=1}^{s}\sum_{j=1}^{t}c_{ij}^3+3\sum_{k=1}^{s}\sum_{l=1}^{t}\sum_{m=1}^{s}\sum_{n=1}^{t}c_{kl}^2c_{mn})$$

$$=\sum_{i=1}^{s}\sum_{j=1}^{t}c_{ij}^3+(\sum_{i=1}^{s}\sum_{j=1}^{t}c_{ij})^3=\sum_{i=1}^{s}\sum_{j=1}^{t}c_{ij}^3+1^3$$

$$=\sum_{i=1}^{s}\sum_{j=1}^{t}c_{ij}^3+1.$$

Thus,

$$\sum_{i=1}^{s}a_i{}^3+\sum_{j=1}^{t}b_j{}^3\le\sum_{i=1}^{s}\sum_{j=1}^{t}c_{ij}^3+1,$$

and therefore

$$1-\sum_{i=1}^{s}\sum_{j=1}^{t}c_{ij}{}^3\le 1-\sum_{i=1}^{s}a_i{}^3+1-\sum_{j=1}^{t}b_j{}^3.$$

Theorem 2. Let φ be a subadditive generator, $m:\mathcal{F}\to[0,1]$ be a state, ξ,η be IF-partitions. Then

$$H_\varphi(\xi\vee\eta)\le H_\varphi(\xi)+H_\varphi(\eta).$$

Proof: Let $\xi=\{A_1,\dots,A_n\}$, $\eta=\{B_1,\dots,B_m\}$. Put $c_{ij}=m(A_i\cdot B_j)$, $a_i=m(A_i)$, $b_j=m(B_j)$. By Proposition 1, (F3), and the additivity of m we have

$$a_i=m(A_i)=m(A_i\cdot(\oplus_{j=1}^{m}B_j))=m(\oplus_{j=1}^{m}(A_i\cdot B_j))=\sum_{j=1}^{m}m(A_i\cdot B_j)=\sum_{j=1}^{m}c_{ij},$$

$$b_j=m(B_j)=m(B_j\cdot(\oplus_{i=1}^{n}A_i))=m(\oplus_{i=1}^{n}(A_i\cdot B_j))=\sum_{i=1}^{n}m(A_i\cdot B_j)=\sum_{i=1}^{n}c_{ij}.$$

In addition, by Definition 2 $\sum_{i=1}^{n}a_i=\sum_{i=1}^{n}m(A_i)=1$, analogously $\sum_{j=1}^{m}b_j=1$.

Therefore

$$H_\varphi(\xi \vee \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} \varphi\big(m(A_i \cdot B_j)\big) = \sum_{i=1}^{n} \sum_{j=1}^{m} \varphi(c_{ij})$$

$$\leq \sum_{i=1}^{n} \varphi(a_i) + \sum_{j=1}^{m} \varphi(b_j)$$

$$= \sum_{i=1}^{n} \varphi(m(A_i)) + \sum_{j=1}^{m} \varphi(m(B_j)) = H_\varphi(\xi) + H_\varphi(\eta). \qquad \square$$

4 Conclusion

In the contribution, we have introduced a general type of entropy of IF-partitions based on a function $\varphi : [0,1] \to [0, \infty)$. As special cases of $\varphi -$entropy we obtain the Shannon entropy as well as the logical entropy of IF-partitions. Further, we have proved that if the function φ is a so-called subadditive generator, then the $\varphi -$entropy of IF-partitions is a subadditive function. It has been shown that the Shannon function and the logical entropy function are subadditive generators. Moreover, we found a subadditive generator different from the Shannon-Kolmogorov-Sinai case and the logical case.

Since we have proved the subadditivity of $\varphi -$entropy of IF-partitions, it is hopeful to construct for the proposed $\varphi -$entropy an isomorphism theory of the Kolmogorov-Sinai type. As a direct consequence of $\varphi -$results we could obtain the Kolmogorov-Sinai entropy theory as well as the logical entropy theory.

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