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Uncertainty inspired by economical models

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Abstract: Some applications of the sets theory in economical problems are presented. Especially the generalized Choquet and Šipoš's integrals are exposed. We present two possibilities how to extend mathematical models of the problem. The first is the Atanassov intuitionistic fuzzy sets theory for the domain, the second one is the Riesz vetor space theory for the range of considered mappings.
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1 Introduction

In economical models similarly as in other areas of theory and practice some uncertainty plays an important role. The classical model was probabilistic, objective, based on measurements (see [5, 8]). 50 years ago another model has been constructed, subjective, based on impressions (see [12]).

In [4] an economical model has been developed with basic notions: prospect, what is a mapping defined on a set S of states of nature and capacity defined on subsets of S. In the present article we extend the mathematical model in two directions. First, the set of states need not be finite. Second, the value of capacity need not be real number but it can be a member of an ordered space. The main instrument in the general situation is integration theory from [2, 3, 11, 12].

2 Prospect theory

In [4] a finite set S of states is considered. Further there is given an ordered set X of consequences with a neutral element 0. An uncertainty prospect is a function $f : S \to X$.

A capacity is a function $W : 2^S \to [0,1]$ assigning to each set $A \subset S$ a real number $W(A)$ such that $W(\emptyset) = 0, W(S) = 1$ and such that $A \subset B$ implies $W(A) \leq W(B)$. If $x_{-m} < x_{-m+1} < ... < x_{-1} < x_0 < x_1 < x_2 < ... < x_n$ are outcomes and

$$A_i = f^{-1}(\{x_i\}),$$

then

$$V(f) = \sum_{i=-m}^{n} \pi_i \nu(x_i),$$

where

$$\begin{aligned} \pi_n &= W(A_n), \\ \pi_i = W(A_i \cup ... \cup A_n) &- W(A_{i+1} \cup ... \cup A_n), 0 \leq i \leq n-1, \\ \pi_j = W(A_{-m} \cup ... \cup A_j) &- W(A_{-m} \cup ... \cup A_{j-1}), -m+1 \leq j \leq 0, \\ \pi_{-m} &= W(A_{-m}) \end{aligned} \tag{1}$$

is called a utility function.

3 Atanassov space

Let us look for the Atanassov generalization of the notion of fuzzy set so-called IF-set (see [1]) what is a pair

$$A = (\mu_A, \nu_A) : \Omega \to [0,1] \times [0,1]$$

of fuzzy sets $\mu_A, \nu_A : \Omega \to [0,1]$, where $\mu_A + \mu_A \leq 1$.

Evidently a fuzzy set $\varphi_A : \Omega \to [0,1]$ can be considered as an IF-set, where

$$\mu_A = \varphi_A : \Omega \to [0,1], \nu_A = 1 - \varphi_A : \Omega \to [0,1].$$

Here we have

$$\mu_A + \nu_A = 1,$$

while generally it can be $\mu_A(\omega) + \nu_A(\omega) < 1$ for some $\omega \in \Omega$. Geometrically an IF-set can be regarded as a function $A : \Omega \to \Delta$ to the triangle

$$\Delta = \{(u,v) \in R^2 : 0 \leq u, 0 \leq v, u + v \leq 1\}.$$

Fuzzy set can be considered as a mapping $\varphi_A : \Omega \to D$ to the segment

$$D = \{(u,v) \in R^2; u + v = 1, 0 \leq u \leq 1\}$$

and the classical set as a mapping $\psi : \Omega \to D_0$ from Ω to two-point set

$$D_0 = \{(0,1), (1,0)\}.$$

In the next definition we again use the Łukasiewicz operations.

Definition. By an IF subset of a set Ω a pair $A = (\mu_A, \nu_A)$ of functions

$$\mu_A : \Omega \to [0,1], \nu_A; \Omega \to [0,1]$$

is considered such that $\mu_A + \nu_A \leq 1$. We call μ_A the membership function, ν_A the non-membership function and

$$A \leq B \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

If $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ are two IF-sets, then we define

$$A \oplus B = ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0),$$

$$A \odot B = ((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B) \wedge 1),$$

$$\neg A = (1 - \mu_A, 1 - \nu_A).$$

Denote by $\mathcal{F}$ a family of IF sets such that

$$A, B \in \mathcal{F} \Longrightarrow A \oplus B \in \mathcal{F}, A \odot B \in \mathcal{F}, \neg A \in \mathcal{F}.$$

Example 1. Let $\mathcal{F}$ be the set of all fuzzy subsets of a set Ω. If $f : \Omega \to [0,1]$ then we define

$$A = (f, 1 - f),$$

i.e. $\nu_A = 1 - \mu_A$.

Example 2. Let $(\Omega, \mathcal{S})$ be a measurable space, $\mathcal{S}$ a σ-algebra, $\mathcal{F}$ the family of all pairs such that $\mu_A : \Omega \to [0,1], \nu_A : \Omega \to [0,1]$ are measurable. Then $\mathcal{F}$ is closed under the operations $\oplus, \odot, \neg$.

4 Riesz space

A prototype of a Riesz space is the space R of real numbers. Generally a Riesz space is a linear space $(Y, +, \leq)$ with an ordering $\leq$ such that

$$a \leq b, a, b \in Y, \alpha \in R, \alpha \geq 0 \Longrightarrow a + c \leq b + c, \alpha a \leq \alpha b.$$

We write

$$| a |= a \vee 0 + (-a) \vee 0.$$

We say that a net $(r_t)_{t \in T}$ in Y converges to an element $r \in Y$, if there exists a net $(p_t)_{t \in T}$ in Y such that

$$| r_t - r | \leq p_t$$

and

$$p_t \searrow 0.$$

We write

$$\lim_{t \in T} r_t = r.$$

Example 3. Let R^n be the space of all n-tuples $a = (a_1, ...a_n)$ of real numbers with natural operations

$$a + b = (a_1 + b_1, ..., a_n + b_n),$$

$$\alpha a = (\alpha a_1, ..., \alpha a_n),$$

and ordering

$$a \leq b \Longleftrightarrow a_1 \leq b_1, ..., a_n \leq b_n.$$

Evidently

$$| a |= (| a_1 |, ..., | a_n |).$$

Example 4. Let Y be the set of all real functions defined on an interval $[a, b]$. Here we have for $f, g \in Y$

$$(f + g)(x) = f(x) + g(x), x \in [a, b],$$

$$(\alpha f)(x) = \alpha f(x), x \in [a, b],$$

$$f \leq g \Longleftrightarrow f(x) \leq g(x)$$

for every $x \in [a, b]$,

$$| f | (x) =| f(x) | .$$

If Y is a Riesz space, then an Y-valued capacity is a function

$$W : 2^S \to Y$$

such that $W(\emptyset) = 0$, and $W(A) \leq W(B)$, whenever $A \subset B \subset S$.

5 Šipoš integral

Consider a set S, a Riesz space Y, a Y-valued capacity $W : 2^S \to Y$ and a real function $f : S \to R$. Let $\mathcal{F}$ be the family of all finite subsets of R containing 0. Let $F \in \mathcal{F}$,

$$F = \{b_k, b_{k-1}, ..., b_0, a_0, ..., a_n\},$$

where

$$b_k < b_{k-1} < ... < b_0 = 0 = a_0 < ... < a_n.$$

Put

$$S_{\mathcal{F}}(f) = \sum_{i=1}^{n}(a_i - a_{i-1})W(A_i) + \sum_{j-1}^{k}(b_j - b_{j-1})W(B_j)$$

where

$$A_i = f^{-1}([a_i, \infty)), i = 0, 1, ..., n,$$

$$B_j = f^{-1}((-\infty, b_j]), j = 0, 1, ..., k.$$

Šipoš integral is defined as the limit

$$(S)\int_S f dW = \lim_{F \in \mathcal{F}} S_F(f).$$

6 Generalized theory

An uncertainty prospect is a mapping $f : S \to R$. where S is an arbitrary nonempty set. Further a capacity W with values in a Riesz space Y is defined as a mapping assigning a vector

$$W(A) \in Y$$

to any set $A \subset S$.

Expected utility function assigned to any prospect $f : S \to R$ a member $V(f) \in Y$ by the formula

$$V(f) = (S) \int_S f dW.$$

Proposition 1. If f, g are two nonnegative prospects, $f \leq g$, then $V(f) \leq V(g)$.
Proof: By [2] Theorem 8.7

$$(S) \int_S f dW = \int_0^\infty W(\{s \in W; f(s) \geq t\}) dt.$$

Similarly

$$(S) \int_S g dW = \int_0^\infty W(\{s \in W; g(s) \geq t\}) dt.$$

Of course

$$\{s \in W; f(s) \geq t\} \subset \{s \in W; g(s) \geq t\}.$$

Therefore

$$V(f) = (S) \int_S f dW = \int_0^\infty W(\{s \in W; f(s) \geq t\}) dt \leq$$

$$\leq \int_0^\infty W(\{s \in W; g(s) \geq t\}) dt = V(g).$$

This completes the proof. □

Proposition 2. Let $f : S \to R, f^+ = f \vee 0, f^- = (-f) \vee 0$. Then

$$V(f) = V(f^+) - V^(f^-).$$

Proof: Use Theorem 8.10 of [2]:

$$V(f) = (S) \int_S f dW = (S) \int_S f^+ dW - (S) \int_S f^- dW =$$

$$= V(f^+) - V(f^-).$$

□

Proposition 3. If S is finite set, and $f : S \to R$ has n values $x_1, ..., x_n$, then

$$V(f) = \sum_{i=1`}^{n} \pi_i \nu(x_i),$$

where π_i is defined by formulas (1) in Section 2.
Proof: It follows by Definition of $S_{\mathcal{F}}(f)$ in Section 4. □

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