About OWA operators of dimension 2 and Atanassov's operators. Generalization.

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Outline

- Introduction
- Motivation
- OWA operators and Atanassov’s operators
- Generalized Atanassov’s operators
- Conclusions

Introduction

A-IFSs

$$A = \{(u, \mu_A(u), \nu_A(u)) | u \in U\} \text{ where}$$
$$\mu, \nu : U \to [0,1]$$
$$\mu_A(u) + \nu_A(u) \leq 1$$

- L([0,1])= $\{\mathbf{x} = [\underline{x}, \overline{x}] | (\underline{x}, \overline{x}) \in [0,1]^2 \text{ and } \underline{x} \leq \overline{x}\}$
- L([0,1], ≤L) is a partial ordered set, with

$$\mathbf{x} \leq_L \mathbf{y} \text{ if and only if } \underline{x} \leq \underline{y} \text{ and } \overline{x} \leq \overline{y}.$$

Introduction

IVFSs $A = \{(u, M_A(u)) | u \in U\}$ where

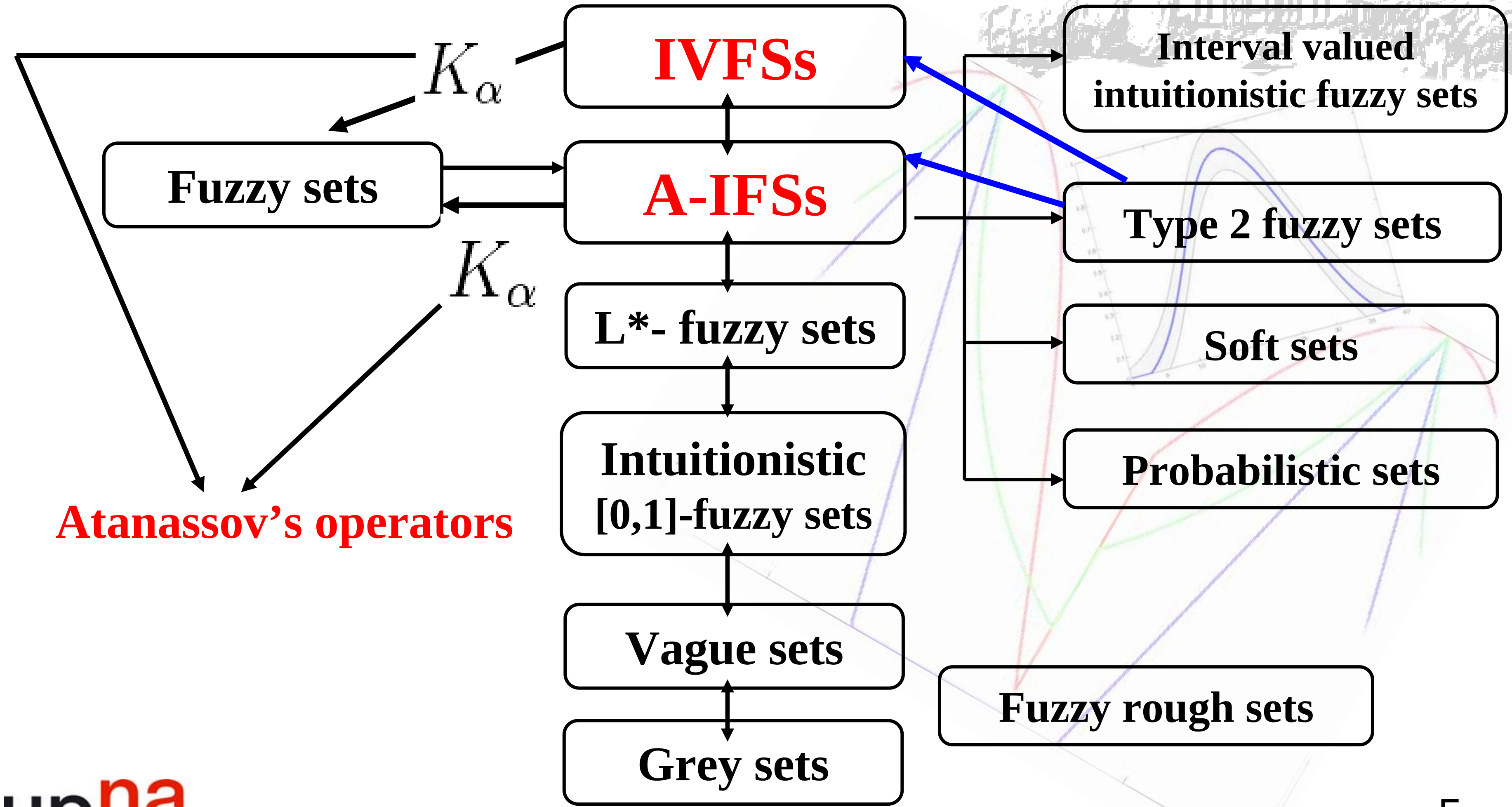
$$M_A : U \to L([0,1]); \text{ that is}$$

$$M_A(u) = [M_{AL}(u), M_{AU}(u)] = [\underline{M}_A(u), \overline{M}_A(u)]$$

Equivalence IVFSs and A-IFSs

$$\mu \longleftrightarrow M_L$$
$$1 - \nu \longleftrightarrow M_U$$

Introduction



Atannassov's operators

Let $\alpha \in [0,1]$. The Atanassov's operator K_α is a mapping

$$K_\alpha : L([0,1]) \to [0,1] \text{ defined by}$$

(i) $K_0(\mathbf{x}) = \underline{x}$ for all $\mathbf{x} \in L([0,1])$.

(ii) $K_1(\mathbf{x}) = \overline{x}$ for all $\mathbf{x} \in L([0,1])$.

(iii) $K_\alpha(\mathbf{x}) = K_\alpha([K_0(\mathbf{x}), K_1(\mathbf{x})]) = \alpha K_1(\mathbf{x}) + (1-\alpha)K_0(\mathbf{x})$

$$K_\alpha(\mu,\nu) = \alpha + (1-\alpha)\mu - \alpha\nu$$

Motivation

- 1983: Atanassov's operators

  K. Atanassov, Intuitionistic fuzzy sets, In: VIIth ITKR Session,Deposited in the Central Science and Technology Library of the Bulgarian Academy of Sciences, Sofia, Bulgaria, 1983, pp. 1684-1697.

- 1988: OWA operators

  R.R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision making, IEEE Trans. Syst. Man Cybern, 18 (1988) 183-190.

Can we relate both concepts?

Aggregation functions

An aggregation function is a function

$$A : \bigcup_{n \in \mathbb{N}} [0,1]^n \to [0,1] \text{ such that}$$

$$(i)\ A(x_1, \cdots, x_n) \leq A(y_1, \cdots, y_n) \text{ whenever } x_i \leq y_i \text{ for all } n \in \mathbb{N} \text{ and } i \in \{1, \cdots, n\}.$$

$$(ii)\ A(0, \cdots, 0) = 0 \text{ and } A(1, \cdots, 1) = 1.$$

OWA Operators

A mapping $F : [0,1]^n \to [0,1]$ is called an OWA operator of dimension n if there exists a weighting vector W, $W = (w_1, w_2, \cdots, w_n) \in [0,1]^n$ with $\sum_i w_i = 1$,and such that

$$F(a_1, a_2, \cdots, a_n) = \sum_{j=1}^{n} w_j b_j \text{ with } b_j \text{ the j-th largest of the } a_i$$

for any $(a_1, \cdots, a_n) \in [0,1]^n$.

$F : [0,1]^2 \to [0,1]$ is called an OWA operator of dimension 2 if there exists a weighting vector W, $W = (w_1, w_2) \in [0,1]^2$ with $w_1 + w_2 = 1$ and such that

$$F(a_1, a_2) = w_1 b_1 + w_2 b_0$$

with $b_1 = \max(a_1, a_2)$ and $b_0 = \min(a_1, a_2)$ for any $(a_1, a_2) \in [0,1]^2$

Relation between Atanassov's and OWA operators I

We want to relate Atanassov' operators and OWA operators in dimension 2.

$$F(a_1, a_2) = w_1 b_1 \quad + (1 - w_1) b_0 \quad \text{with } b_1 \geq b_0$$

$$K_\alpha(\mathbf{x}) = \alpha K_1(\mathbf{x}) + (1 - \alpha) K_0(\mathbf{x}) \text{ with } K_1(\mathbf{x}) \geq K_0(\mathbf{x})$$

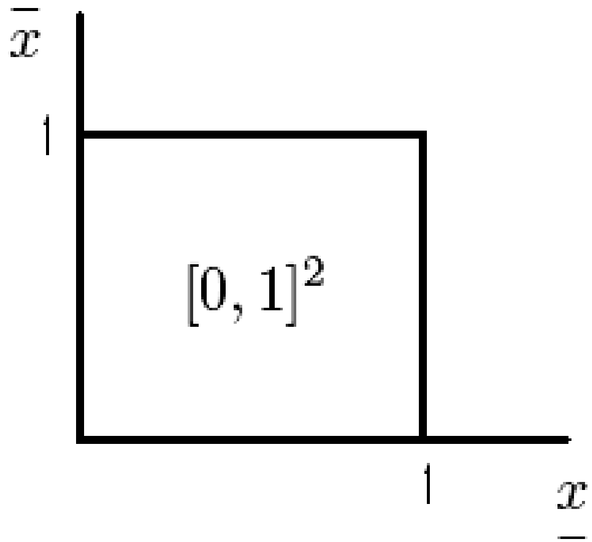
$$K_\alpha(\mu, \nu) = \alpha + (1 - \alpha)\mu - \alpha\nu \text{ with } \mu + \nu \leq 1$$

Problem:

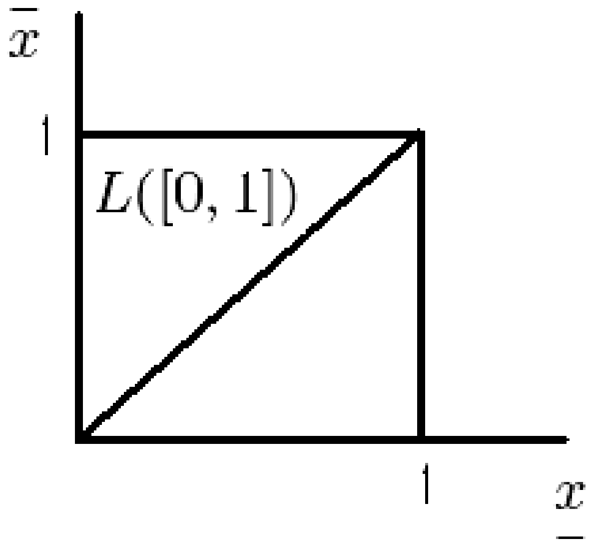
They are defined over different spaces

$$[0, 1]^2 \longleftrightarrow L([0, 1])$$

Relation between Atanassov’s and OWA operators I



OWA operators



Atanassov’s operators

Relation between Atanassov's and OWA operators I

A Solution:

i: [0,1]x[0,1] —> L([0,1])

(x,y) —> [min(x,y), max(x,y)]

Relation between Atanassov's and OWA operators II

Theorem 1 *Let the function:*

$$i : [0,1]^2 \longrightarrow L([0,1]) \text{ given by}$$
$$i(x,y) = [\min(x,y), \max(x,y)].$$

Let $\alpha \in [0,1]$ *and*

$$\mathbb{K}_\alpha : [0,1]^2 \longrightarrow [0,1] \text{ given by}$$
$$\mathbb{K}_\alpha(x,y) = (K_\alpha \circ i)(x,y)$$

where K_α *is the Atanassov's operator.*
Under these conditions the following items hold:

(a) $\mathbb{K}_\alpha$ *is commutative e idempotent;*

(b) $\mathbb{K}_0(x,y) = \min(x,y)$ *and* $\mathbb{K}_1(x,y) = \max(x,y)$*;*

(c) $\mathbb{K}_\alpha$ *is monotonic,* $\mathbb{K}_\alpha(0,0) = 0$, $\mathbb{K}_\alpha(1,1) = 1$*;*

(d) Let $\beta \in [0,1]$*. If* $\alpha \leq \beta$*, then* $\mathbb{K}_\alpha(x,y) \leq \mathbb{K}_\beta(x,y)$ *for all* $(x,y) \in [0,1]^2$*;*

(e) $\mathbb{K}_\alpha(0,1) = \alpha$.

Relation between Atanassov's and OWA operators III

Theorem 2 ($\mathbb{K}_\alpha$ operators are OWA operators of dimension 2) *Let $\alpha \in [0,1]$ and $\mathbb{K}_\alpha = K_\alpha \circ i$. Then, if $F(x,y)$ is the OWA operator (of dimension 2) defined by the weighting vector $W = (\alpha, 1-\alpha)$, we have that*

$$\mathbb{K}_\alpha(x,y) = F(x,y) \text{ for all } x,y \in [0,1]$$

Relation between Atanassov's and OWA operators IV

Theorem 3 (OWA operators of dimension 2 are $\mathbb{K}_\alpha$ operators) *Let F an OWA operator of dimension 2 with weighting vector $W = (w_1, w_2)$. Then for any $(x, y) \in [0, 1]^2$ we have that*

$$F(x, y) = \mathbb{K}_\alpha(x, y)$$

with $\alpha = w_1$.

Relation between Atanassov's and OWA operators IV

$$K_\alpha(\mathbf{x}) = \alpha K_1(\mathbf{x}) + (1-\alpha)K_0(\mathbf{x})$$

Proposition 1

If F is an OWA operator of dimension 2 defined by the weighting vector (w_1, w_2), then, for any $x, y \in [0,1]$

$$F(x,y) = w_1\mathbb{K}_1(x,y) + (1-w_1)\mathbb{K}_0(x,y).$$

Generalized Atanassov's operators

Definition 1 *Let $\alpha \in [0,1]$. An operator GK_α is a mapping $GK_\alpha : L([0,1]) \to [0,1]$ such that it satisfies the following conditions:*

(i) If $\underline{x} = \overline{x}$, then $GK_\alpha(\mathbf{x}) = \underline{x}$.

(ii) $GK_0(\mathbf{x}) = \underline{x}$, $GK_1(\mathbf{x}) = \overline{x}$ for all $\mathbf{x} \in L([0,1])$.

(iii) If $\mathbf{x} \leq_L \mathbf{y}$, with $\mathbf{x}, \mathbf{y} \in L([0,1])$, then $GK_\alpha(\mathbf{x}) \leq GK_\alpha(\mathbf{y})$.

(iv) Let $\beta \in [0,1]$ if $\alpha \leq \beta$, then $GK_\alpha(\mathbf{x}) \leq GK_\beta(\mathbf{x})$ for all $\mathbf{x} \in L([0,1])$.

(v) $GK_\alpha([0,1]) = \alpha$ for any $\alpha \in [0,1]$.

H. Bustince, J. Montero, M. Pagola, E. Barrenechea, D. Gómez: A Survey On Interval-Valued Fuzzy Sets. In W. Pedrycz, A. Skowron, V. Kreinovich (Eds.): Handbook of Granular Computing John Wiley and sons, NewYork,(2008), Chapter 22.

Generalized Atanassov's operators II

Example 1

$$GK_\alpha([\underline{x}, \overline{x}]) = \begin{cases} \overline{x} \ \ if \ \overline{x} \leq \alpha \\ \underline{x} \ \ if \ \underline{x} \geq \alpha \\ \alpha \ \ otherwise \end{cases}$$

Proposition 1

For any $\alpha \in [0,1]$, Atanassov's operator K_α is a GK_α operator.

Generalized Atanassov's operators II

On the other hand, the K_α operators can be considered as the "simplest" GK_α operators, in the sense of the following proposition.

Proposition 2 *Let $\mathcal{H}_\alpha$ be a family of GK_α operators such that, for any $\alpha \in [0,1]$, $\mathcal{H}_\alpha$ is a linear mapping of the extremes of the interval, i.e.*

$$\mathcal{H}_\alpha(\mathbf{x}) = a(\alpha)\underline{x} + b(\alpha)\overline{x}$$

for some mappings $a, b : [0,1] \to [0,1]$. Then

1. *$a(\alpha) = 1 - b(\alpha)$ for any $\alpha \in [0,1]$;*
2. *$b(\alpha)$ is a monotone, increasing function such that $b(0) = 0$ and $b(1) = 1$.*

Generalized Atanassov's operators III. Construction.

Theorem 4 *Let φ an automorphism on the unit interval. Then, for any $\alpha \in [0,1]$ the operator*

$GK_\alpha : L([0,1]) \to [0,1]$ *defined by*

$$GK_\alpha(\mathbf{x}) = \varphi^{-1}\left(\varphi(\alpha)\varphi(\overline{x}) + (1-\varphi(\alpha))\varphi(\underline{x})\right)$$

is a generalized Atanassov's operator.

Example 2

For each $q > 0$ the function $\varphi(x) = x^q$ defines the family of GK_α operators given by

$$GK_\alpha(\mathbf{x}) = \left(\alpha^q(\overline{x})^q + (1-\alpha^q)(\underline{x})^q\right)^{\frac{1}{q}} .$$

Fuzzy sets associated to IVFSs.

Let $A \in \mathcal{IVFSs}(U)$ we can associate a fuzzy set $A_\alpha \in FSs(U)$ to A in the following way:

$$A_\alpha = \{(u, \mu_{A_\alpha}(u)) | u \in U\} \text{ with } \mu_{A_\alpha}(u) = GK_\alpha(\mathbf{M_A}(u))$$

where $\mathbf{M_A}$ denotes the membership function of A.

Theorem 5
Let $A \in \mathcal{IVFSs}(U)$, let $\beta \in [0,1]$ and let GK_α.
Then $\{A_\alpha\}_{\alpha\in[0,1]}$ is a totally ordered family of fuzzy sets with respect to the order

$$A_\alpha \leq A_\beta \text{ if and only if } \alpha \leq \beta$$

Conclusions

- In dimension 2, Atanassov's operators and OWA operators provide the same numerical results.
- OWA operators of dimension 2 can be obtained from Atanassov's operators.
- In particular, OWA operators can be obtained from the "extremal"Atanassov's operators K_0 and K_1.

Thanks for your attention!!

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