Notes on Intuitionistic Fuzzy Sets
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Special types of morphisms in the category $\mathbf{C_{R\text{-}IFM}}$

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Abstract: The aim of this paper is to introduce two special type of morphisms, namely Retraction and Coretraction in the category ($\mathbf{C_{R\text{-}IFM}}$) of intuitionistic fuzzy modules. We obtain the condition under which a morphism in $\mathbf{C_{R\text{-}IFM}}$, that is an intuitionistic fuzzy R-homomorphism, to be a retraction or a coretraction. Then, we acquire some equivalent statements for these two morphisms. Further, we study free, projective and injective objects in $\mathbf{C_{R\text{-}IFM}}$ and establish their relation with morphism in $\mathbf{C_{R\text{-}IFM}}$ and retraction, coretraction.
Keywords: Intuitionistic fuzzy modules, Intuitionistic fuzzy R-homomorphism, Intuitionistic fuzzy coretraction, Intuitionistic fuzzy retraction, Intuitionistic fuzzy projective modules, Intuitionistic fuzzy injective modules.
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1 Introduction

The category theory is concerned with the mathematical entities and the relationships between them. Categories develop as a unifying concept in many domains of mathematics, particularly in computer technology and mathematical physics. The detailed study about category theory can be found in [14, 26]. In Zadeh's introductory paper [27], fundamental research is being carried out in the fuzzy set's context. Almost all of this mathematical development has been categorical. Several other researchers have developed and researched theories of fuzzy modules, fuzzy exact sequences of fuzzy complexes, and fuzzy homologies of fuzzy chain complexes [1, 16–19, 25, 28, 29].

One of the generalizations of fuzzy sets – the concept of intuitionistic fuzzy sets – was introduced by Atanassov in 1983 [2, 3]. Biswas was the first to introduce the intuitionistic fuzzification of the algebraic structures and developed the concept of an intuitionistic fuzzy subgroup of a group in [5]. Hur et al. in [9, 10] defined and studied intuitionistic fuzzy subrings and ideals of a ring. In [8], Davvaz et al. introduced the notion of intuitionistic fuzzy submodules, which was further studied by many mathematicians (see [4, 11, 15, 20–24]).

The concept of category on intuitionistic fuzzy sets was studied by Kim et al. in [12]. Lee and Chu in [13] applied the concept of category to intuitionistic fuzzy topological spaces. Cigdem and Davvaz in [6] introduced the concepts of inverse and direct systems in the category of intuitionistic fuzzy submodules. Authors in [24] studied the category of intuitionistic fuzzy modules $\mathbf{C_{R\text{-}IFM}}$ over the category of modules $\mathbf{C_{R\text{-}M}}$.

Our present study focuses on intuitionistic fuzzy modules over a commutative ring R with identity element, and in the course of our study, we have made an attempt to develop a parallel theory of category by applying intuitionistic fuzzy techniques. In this paper, we extend the notion of intuitionistic fuzzy modules and intuitionistic fuzzy R- homomomorphism to intuitionistic fuzzy coretracts (retracts) and intuitionistic fuzzy coretraction (retraction), and various properties are being investigated.

In section 4, we demonstrate that an intuitionistic fuzzy module A is a projective object in $\mathbf{C}_{R-IFM}$ if and only if, M is projective module and $A = \bar{0}$. One goal of the present paper is to initiate the concept of intuitionistic fuzzy coretraction and intuitionistic fuzzy retraction in a intuitionistic fuzzy contexts. Further investigations on this, we believe, will lead to the applications of these notions to categorical approach to pave the platform for future research.

2 Preliminaries

Definition 2.1 ([2, 3])**.** A mapping $A = (\mu_A, \nu_A) : X \to [0,1] \times [0,1]$ is called an intuitionistic fuzzy set on X if $\mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$, where the mappings $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denotes the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to A, respectively.

Remark 2.2.

(i) When $\mu_A(x) + \nu_A(x) = 1$, i.e., $\nu_A(x) = 1 - \mu_A(x) = \mu_{A^c}(x)$. Then A is called a fuzzy set.

(ii) We denote the IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ by $A = (\mu_A, \nu_A)$.

For all the basic terms like R-module, free module, projective module, injective module, R-homomorphism, etc. we refer the reader to [7].

Definition 2.3 ([4,8,11])**.** An intuitionistic fuzzy set A of an R-module M is called an intuitionistic fuzzy submodule (IFSM) of M, if for every $x, y \in M, r \in R$, the following conditions are satisfied:

(i) $\mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x+y) \leq \nu_A(x) \vee \nu_A(y)$;

(ii)$\mu_A(rx) \geq \mu_A(x)$ and $\nu_A(rx) \leq \nu_A(x)$;

(iii) $\mu_A(\theta) = 1$ and $\nu_A(\theta) = 0$, where θ is a zero element of M.
Condition (i) and (ii) can be combined to a single condition $\mu_A(rx+sy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(rx+sy) \leq \nu_A(x) \vee \nu_A(y)$, where $r, s \in R$ and $x, y \in M$.

Remark 2.4.

(i) The set of intuitionistic fuzzy submodules of R-module M is denoted by IFSM(M).

(ii) We denote the IFSM A of an R-module M by $(\mu_A, \nu_A)_M$.

Definition 2.5 ([4, 23])**.** Let K be a submodule of an R-module M. The intuitionistic fuzzy characteristic function of K is defined by χ_K described by $\chi_K(a) = (\mu_{\chi_K}(a), \nu_{\chi_K}(a))$, where

$$\mu_{\chi_K}(a) = \begin{cases} 1, & \text{if } a \in K \\ 0, & \text{if } a \notin K \end{cases} \quad \text{and} \quad \nu_{\chi_K}(a) = \begin{cases} 0, & \text{if } a \in K \\ 1, & \text{if } a \notin K. \end{cases}$$

Clearly, χ_K is an IFSM of M. The IFSMs $\chi_{\{\theta\}}, \chi_M$ (also represented by $\overline{0}$ and $\overline{1}$, respectively) are called trivial IFSMs of module M. Any IFSM of the module M apart from this is called proper IFSM.

Definition 2.6 ([24])**.** Let $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ are IFSM of R-modules M and N, respectively. Then the map $f : A \to B$ is called an intuitionistic fuzzy R-homomorphism from A to B if

(i) $f : M \to N$ is R-homomorphism and

(ii) $\mu_B(f(a)) \geq \mu_A(a)$ and $\nu_B(f(a)) \leq \nu_A(a), \forall a \in M$.

To avoid confusion between an R-homomorphism $f : M \to N$ and an intuitionistic fuzzy R-homomorphism $f : A \to B$. We denote the latter by $\bar{f} : A \to B$. So, given an IF R-homomorphism $\bar{f} : A \to B$, $f : M \to N$ is the underlying R-homomorphism of $\bar{f}$. The set of all IF R-homomorphisms from A to B is denoted by $\mathrm{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A, B)$. From this, we can say that $\bar{f} \in \mathrm{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A, B)$.

Definition 2.7 ([24])**.** A category of R-modules denoted by $\mathbf{C_{R\text{-}M}}$ has R-modules as objects and R-homomorphisms as morphisms, with composition of morphisms defined as composition of mappings.

Definition 2.8 ([24]). The category $\mathbf{C_{R\text{-}M}} = (\mathrm{Ob}(\mathbf{C_{R\text{-}M}}), \mathbf{Hom}(\mathbf{C_{R\text{-}M}}), \circ)$ has R-modules as objects and R-homomorphisms as morphisms, with composition of morphisms defined as the composition of mappings.

As described in [24], an IF-module category $\mathbf{C_{R\text{-}IFM}}$ over the base category $\mathbf{C_{R\text{-}M}}$ is completely characterize by the two mappings:

$$\alpha : \mathrm{Ob}(\mathbf{C_{R\text{-}M}}) \to I \times I;$$
$$\beta : \mathrm{Hom}(\mathbf{C_{R\text{-}M}}) \to I \times I$$

Thus, an IF-module category $\mathbf{C_{R\text{-}IFM}}$ consists of:

(C1) $\mathrm{Ob}(\mathbf{C_{R\text{-}IFM}})$ the set of objects as IFSMs on $\mathrm{Ob}(\mathbf{C_{R\text{-}M}})$, i.e., the objects of the form $\alpha : \mathrm{Ob}(\mathbf{C_{R\text{-}M}}) \to I \times I$;

(C2) $\mathbf{Hom}(\mathbf{C_{R\text{-}IFM}})$ the set of IF R-homomorphisms corresponding to underlying R-homomorphisms from $\mathbf{Hom}(\mathbf{C_{R\text{-}M}})$, i.e., IF R-homomorphisms of the form $\beta : \mathrm{Hom}(\mathbf{C_{R\text{-}M}}) \to I \times I$, such that for $f \in \mathbf{Hom}_{\mathbf{C_{R\text{-}M}}}(M, N)$,

$$\beta(\bar{f}) = (\mu_{\beta(\bar{f})}, \nu_{\beta(\bar{f})})$$

as defined in [24, Theorem 1], a composition law associating to each pair of morphisms $f \in \mathrm{Hom}(M, N)$ and $g \in \mathrm{Hom}(N, P)$, a morphism $g \circ f \in \mathrm{Hom}(P, Q)$, such that the following axioms hold:

(M1) Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$, for all $f \in \mathrm{Hom}(M, N)$, $g \in \mathrm{Hom}(N, P)$ and $h \in \mathrm{Hom}(P, Q)$;

(M2) preservation of morphisms: $\beta(g \circ f) = \beta(g) \circ \beta(f)$;

(M3) existence of identity: $\forall M \in \mathrm{Ob}(\mathbf{C_{R\text{-}M}})$ there is an identity $i_M \in \mathbf{Hom}_{\mathbf{C_{R\text{-}M}}}(M, M)$ such that $\beta(i_M) = \alpha(M)$.

In other words, a category of IF R-modules can be constructed as

$$\mathbf{C_{R\text{-}IFM}} = (\mathrm{Ob}(\mathbf{C_{R\text{-}IFM}}), \mathrm{Hom}(\mathbf{C_{R\text{-}IFM}}), \circ)$$

Definition 2.9 ([21]). Let M be a direct product of a family of R-modules $\{M_i | i \in J\}$. Let $A_i = (\mu_{A_i}, \nu_{A_i})$ be an IFSMs of M_i. Then an IFS $A = (\mu_A, \nu_A)$ of M defined by

$$\mu_A(m) = Inf\{\mu_{A_i}(m(i)) | i \in J\}; \nu_A(m) = Inf\{\nu_{A_i}(m(i)) | i \in J\}, \forall\, m = \prod_{i \in J} m(i) \in M,$$

is an IFSM of M, called the direct product of of IFSMs A_i and is written as $A = \prod_{i \in J} A_i$.

Remark 2.10. The category of intuitionistic fuzzy modules has product.

Definition 2.11 ([21]). Let M be a direct product of a family of R-modules $\{M_i | i \in J\}$. Let $A_i = (\mu_{A_i}, \nu_{A_i})$ be an IFSMs of M_i. Then an IFS $A = (\mu_A, \nu_A)$ of M defined by

$$\mu_A(m) = \wedge\{\mu_{A_i}(m(i)) | i \in J\}; \nu_A(m) = \vee\{\nu_{A_i}(m(i)) | i \in J\}, \forall m = \coprod m(i) \in M,$$

is an IFSM of M, called the direct coproduct of of IFSMs A_i and is written as $A = \coprod_{i \in J} A_i$.

Remark 2.12. The category of intuitionistic fuzzy modules has coproduct.

Definition 2.13. Let $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ are IFSM of R-modules M and N, respectively and $\mathrm{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A, B)$ is the set of IF R-homomorphisms from A to B. An IF-R homomorphism $\bar{f} \in \mathrm{Hom}_{\mathbf{C_{R\text{-}IFM}}}(A, B)$ is said to be an Intuitionistic fuzzy split (IF-split), if there is an IF R-homomorphism $\bar{g} \in \mathrm{Hom}_{\mathbf{C_{R\text{-}IFM}}}(B, A)$ such that $\bar{g} \circ \bar{f} = I_A$.

Definition 2.14 ([21])**.** Let N and P be two R-modules. Then an IFSM A of an R-module M is said to be an intuitionistic fuzzy projective submodule (IF-projective), if for any IFSMs B of N, C of P, any IF-epimorphism $\bar{p}$ from C to B and IF R-homomorphism $\bar{\phi}$ from A to B, there exists an IF R-homomorphism $\bar{\psi}$ from A to C such that $\bar{p} \circ \bar{\psi} = \bar{\phi}$, i.e., the following diagram commute

$$\begin{tikzcd} & C \arrow[dd, "\bar{p}"] \\ A \arrow[ur, "\bar{\psi}"] \arrow[dr, "\bar{\phi}"'] & \\ & B \end{tikzcd}$$

Definition 2.15 ([21])**.** Let N and P be two R-modules. Then an IFSM A of an R-module M is said to be an intuitionistic fuzzy injective submodule (IF-injective), if for any IFSM B of N, C of P, and IF-monomorphism $\bar{k}$ from B to C and IF R-homomorphism $\bar{\phi}$ from B to A, there exists an IF R-homomorphism $\bar{\psi}$ from C to A such that $\bar{\psi} \circ \bar{k} = \bar{\phi}$, i.e., the following diagram commute

$$\begin{tikzcd} B \arrow[r, "\bar{k}"] \arrow[dr, "\bar{\phi}"] & C \arrow[d, "\bar{\psi}"] \\ & A \end{tikzcd}$$

Theorem 2.16. *Every intuitionistic fuzzy free submodule of a module is an IF-projective.*

Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are IFSM of R-modules M and N, respectively and $f : M \to N$ is a R-homomorphism. With the help of A and f, we can dispense an IF module structure on N by

$$\mu_{f(A)}(b) = \sup\{\mu_A(a) : f(a) = b\} \text{ and } \nu_{f(A)}(b) = \inf\{\nu(a) : f(a) = b\}.$$

It is clear that $f(A) = (\mu_{f(A)}, \nu_{f(A)})$ is an IFSM of and $\bar{f} : A \to f(A)$ is an IF R-hom.

With the help of B and f, we can dispense an IF module structure on M by

$$\mu_{f^{-1}(B)}(a) = \mu_B(f(a)) \text{ and } \nu_{f^{-1}(B)}(a) = \nu_B(f(a)).$$

Hence, $f^{-1}(B) = (\mu_{f^{-1}(B)}, \nu_{f^{-1}(B)})$ is an IFSM of M and $\bar{f} : f^{-1}(B) \to B$ is an IF R-homomorphism.

Lemma 2.17. *Let M and N are R-modules and $f : M \to N$ be a R-homomorphism. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are IFSM of R-modules M and N, respectively, and $\bar{f} : A \to B$ is an IF R-homomorphism. Then*

(i) $A \subseteq f^{-1}(f(A))$.

(ii) $A = f^{-1}(f(A))$ if and only if both f and $\bar{f}$ are one-one functions.

(iii)$f(f^{-1}(B)) \subseteq B$.

(iv)$f(f^{-1}(B)) = B$ if and only if both f and $\bar{f}$ are onto functions.

3 Retractions and coretractions in the category of intuitionistic fuzzy modules

In this section, we define and study some special morphisms like coretraction, retraction, monomorphism, isomorphism, etc., in the category of intuitionistic fuzzy modules.

Definition 3.1. An IF R-homomorphism $\bar{f} : A \to B$ is said to be an intuitionistic fuzzy coretraction (IF-coretraction) if there exists an IF R-homomorphism $\bar{g} : B \to A$ such that

$$\bar{g} \circ \bar{f} = I_A.$$

In other words, $\bar{f}$ is an IF-coretraction, if it is a left invertible. In this case, the IF R-homomorphism $\bar{g}$ is called a left inverse of $\bar{f}$.

Lemma 3.2. *Composite of two intuitionistic fuzzy coretractions is also an intuitionistic fuzzy coretraction in $\mathbf{C}_{R-IFM}$.*

Proof. Let $\bar{f} : A \to B$ and $\bar{g} : B \to C$ be two intuitionistic fuzzy coretractions in $\mathbf{C}_{R-IFM}$. Then there exist IF R-homomorphisms $\bar{u} : B \to A$ and $\bar{v} : C \to B$ such that

$$\bar{u} \circ \bar{f} = I_A \text{ and } \bar{v} \circ \bar{g} = I_B$$

Now

$$\begin{aligned}(\bar{u} \circ \bar{v}) \circ (\bar{g} \circ \bar{f}) &= \bar{u} \circ (\bar{v} \circ \bar{g}) \circ \bar{f} \qquad \text{[Using associativity of composition]}\\ &= \bar{u} \circ I_B \circ \bar{f}\\ &= \bar{u} \circ \bar{f}\\ &= I_A.\end{aligned}$$

Thus $\bar{u} \circ \bar{v} : A \to C$ is left inverse of $\bar{g} \circ \bar{f}$. Hence $\bar{g} \circ \bar{f}$ is an intuitionistic fuzzy coretraction in $\mathbf{C}_{R-IFM}$. □

Proposition 3.3. *Let A and B are IFSM of R-modules M and N, respectively, and $f : M \to N$ be a R-homomorphism. If an IF R-homomorphism $\bar{f} : A \to B$ is an intuitionistic fuzzy coretraction in $\mathbf{C}_{R\text{-}IFM}$, then both f and $\bar{f}$ are one-one functions.*

Proof. Since $\bar{f} : A \to B$ is an intuitionistic fuzzy coretraction in $\mathbf{C_{R\text{-}IFM}}$. Therefore, there exists an IF R-homomorphism $\bar{g} : B \to A$ such that $\bar{g} \circ \bar{f} = I_A$. By Lemma 2.17, both f and $\bar{f}$ are one-one functions. □

The converse of the Proposition 3.3 is not true. See the following "counterexample".

Example 3.4. Assume $M = \mathbb{Z}_2$ and $N = \mathbb{Z}_4$. Clearly, M, N are Z-modules. Consider $A = \chi_M$ and $B = \chi_N$. Then A and B are IFSMs of the Z-modules M and N, respectively. Define the mapping $f : M \to N$ by $f(0) = 0, f(1) = 2$. Clearly, f is one one Z-homomorphism. Also, $\mu_B(f(a)) \geq \mu_A(a)$ and $\nu_B(f(a)) \leq \nu_A(a), \forall a \in M$. Note that $\bar{f}$ is one one IF Z-homomorphism. However there exists no IF Z-homomorphism $\bar{g} : B \to A$ such that $\bar{g} \circ \bar{f} = I_A$. That is, $\bar{f} : A \to B$ is not an intuitionistic fuzzy coretraction.

Definition 3.5. An IF R-homomorphism $\bar{f} : A \to B$ is said to be an intuitionistic fuzzy retraction (IF-retraction), if there exists an IF R-homomorphism $\bar{g} : B \to A$ such that

$$\bar{f} \circ \bar{g} = I_B.$$

An IFSM B is said to be retract of an IFSM A. In other words, an IF R-homomorphism $\bar{f}$ is an intuitionistic fuzzy retraction if it is right invertible. An IF R-homomorphism $\bar{g}$ in the above definition is called a right inverse of $\bar{f}$.

Lemma 3.6. *Composite of two intuitionistic fuzzy retractions is also an intuitionistic fuzzy retraction in* $\boldsymbol{C}_{R-IFM}$.

Proof. It can be proved similarly to Lemma 3.2. □

Proposition 3.7. *Suppose A and B are IFSM of an R-modules M and N, respectively and $f : M \to N$ is R-homomorphism. If an IF R-homomorphism $\bar{f} : A \to B$ is an intuitionistic fuzzy retraction in* $\boldsymbol{C_{R\text{-}IFM}}$, *then both f and $\bar{f}$ are onto functions.*

Proof. It can be easily proved by using Lemma 2.17. □

The converse of the Proposition 3.7. is not true. See the following "counterexample".

Example 3.8. Assume $M = \mathbb{Z}_2$ and $N = \mathbb{Z}_2$. Clearly, M, N are Z-modules. Define IFS A and B on M and N, respectively as

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.5, & \text{if } x = 1 \end{cases}, \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.4, & \text{if } x = 1 \end{cases},$$

and $\mu_B(y) = 1, \nu_B(y) = 0, \forall y \in N$. Clearly, A and B are IFSM of M and N, respectively. Define the mapping $f : M \to N$ by $f(0) = 0, f(1) = 1$. Clearly, f is an onto Z-homomorphism. Also, $\mu_B(f(a)) \geq \mu_A(a)$ and $\nu_B(f(a)) \leq \nu_A(a), \forall a \in M$. Note that $\bar{f}$ is onto IF Z-homomorphism. However, there exists no IF Z-homomorphism $\bar{g} : B \to A$ such that $\bar{f} \circ \bar{g} = I_B$. That is, $\bar{f} : A \to B$ is not an intuitionistic fuzzy retraction.

4 Intuitionistic fuzzy projective and injective modules

In this section, we study the notions of a free, a projective and an injective objects in $\mathbf{C}_{\textbf{R-IFM}}$ and establish their relation with morphism in $\mathbf{C}_{\textbf{R-IFM}}$ and retraction (coretraction).

Lemma 4.1. *In* $\boldsymbol{C}_{R-IFM}$

(i) *(underlying maps of) epimorphisms are onto, and*

(ii) *(underlying maps of) monomorphisms are one to one.*

Proof. (i) Let A and B be IFSMs of R-modules M and N, respectively. Suppose $\bar{f} : A \to B$ is an IF-epimorphism in $\mathbf{C}_{R-IFM}$, and let $g, h : N \to T$ be R-homomorphisms such that $g \circ f = h \circ f$. Let us denote $\bar{g}, \bar{h} : B \to \bar{1_T}$ [i.e., the IF R-homomorphisms obtained by trivially intuitionistic fuzzifying g and h relative to B] following that, $\bar{g} \circ \bar{f} = \bar{h} \circ \bar{f}$. But after that, $\bar{g} = \bar{h}$ as a result of which $g = h$. Thus f is an epimorphism in $\mathbf{C}_{R-M}$. But epimorphisms in abelian categories are onto and category of R-modules ($\mathbf{C}_{R-M}$) is an abelian category. Consequently, f is onto.
(ii) The proof is similar to the one above. □

Theorem 4.2. $A \in \mathrm{Ob}(\boldsymbol{C}_{R-IFM})$ *is an IF-projective object if and only if* $M \in \mathrm{Ob}(\boldsymbol{C}_{R-M})$ *is a projective module and* $A = \bar{0}$.

Proof. Let A be a projective object in $\mathbf{C}_{R-IFM}$, then A is an IFSM on R-module M. Let N and P be two R-modules and $f : M \to N$ be a R-homomorphism and $\phi : P \to N$ be an R-epimorphism.
Take $B = \chi_N$ and $C = \chi_P$. Then B, C are IFSMs of R-modules N and P, respectively. Then, $\bar{f} : A \to B$ is an IF R-homomorphism and $\bar{\phi} : C \to B$ is an IF-epimorphism obtained by trivially intuitionistic fuzzifying f and ϕ in $\mathbf{C}_{R-IFM}$. As A is a projective object in $\mathbf{C}_{R-IFM}$, then there exists an IF R-homomorphism $\bar{\psi} : A \to C$ such that $\bar{\phi} \circ \bar{\psi} = \bar{f}$. Thus, there exists a R-homomorphism $\psi : M \to P$ such that $\phi \circ \psi = f$. Hence M is a projective object in $\mathbf{C}_{R-M}$.

$$\begin{tikzcd}
A \arrow[r, "\bar{\psi}"] \arrow[dr, "\bar{f}"'] & \chi_P \arrow[d, "\bar{\phi}"] \\
 & \chi_M
\end{tikzcd}$$

If $A \neq \bar{0}$, then there exist no IF R-homomorphism $\bar{\psi} : A \to \chi_P$ i.e., A will no longer be a projective object in $\mathbf{C}_{R-IFM}$ as the above diagram is not commutative. This completes the proof of the theorem. □

Theorem 4.3. *Retraction of a projective object in* $\boldsymbol{C}_{R-IFM}$ *is a projective object.*

Proof. Let A be a projective object in $\mathbf{C}_{R-IFM}$ and let B be a retract of A.
For an IF R-homomorphism $\bar{f} : A \to B$, there exists an IF R-homomorphism $\bar{g} : B \to A$ such that $\bar{f} \circ \bar{g} = 1_B$. We claim that B is an IF-projective object in $\mathbf{C}_{R-IFM}$.
Let $\bar{h} : B \to C$ be any IF R-homomorphism and $\bar{p} : P \to C$ be an IF-epimorphism in $\mathbf{C}_{R-IFM}$.

Now, $\mu_C \circ (\bar{h} \circ \bar{f})$ = $(\mu_C \circ \bar{h}) \circ \bar{f}$ = $\mu_B \circ \bar{f}$ = μ_A. Similarly, we have $\nu_C \circ (\bar{h} \circ \bar{f})$ = ν_A.
As a result, $\bar{h} \circ \bar{f} : A \to C$ is an IF R-homomorphism. Since A is an IF-projective object, then there exists an IF R-homomorphism $\bar{q} : A \to P$ such that $\bar{p} \circ \bar{q} = \bar{h} \circ \bar{f}$
$\Rightarrow$ $(\bar{p} \circ \bar{q}) \circ \bar{g} = (\bar{h} \circ \bar{f}) \circ \bar{g} = \bar{h} \circ (\bar{f} \circ \bar{g}) = \bar{h} \circ I_B = \bar{h}$. This implies that $\bar{p} \circ \bar{r} = \bar{h}$, where $\bar{r} = \bar{q} \circ \bar{g}$.

$$\begin{tikzcd}
B \arrow[dr, "\bar{g}"] \arrow[ddr, bend right, "\bar{r}=\bar{q}\circ\bar{g}"'] & & \\
& A \arrow[r, "\bar{p}"] \arrow[d, "\bar{q}"] & B \arrow[d, "\bar{h}"] \\
& P \arrow[r, "\bar{g}"] & C
\end{tikzcd}$$

Consequently, $\mu_P \circ \bar{r}$ = $\mu_P \circ (\bar{q} \circ \bar{g})$ = $(\mu_P \circ \bar{q}) \circ \bar{g}$ = $\mu_A \circ \bar{g}$ = μ_B.
Thus, $\bar{r} : B \to P$ is an IF R-homomorphism such that $\bar{p} \circ \bar{r} = \bar{h}$.
Hence, B is an IF-projective object in $\mathbf{C}_{R-IFM}$. $\square$

Theorem 4.4. *If A is a projective object in $\boldsymbol{C}_{R-IFM}$, then every IF-epimorphism $\bar{f} : B \to A$ is an intuitionistic fuzzy retraction, where $B \in \boldsymbol{C}_{R-IFM}$.*

Proof. Since A is a projective object in $\mathbf{C}_{R-IFM}$ and $\bar{f} : B \to A$ is an IF-epimorphism, then the following diagram commutes

$$\begin{tikzcd}
& A \arrow[dl, "\bar{g}"'] \arrow[dr, "I_A"] & \\
B \arrow[rr, "\bar{f}"] & & A
\end{tikzcd}$$

Consequently, there exists an IF R-homomorphism $\bar{g} : A \to B$ such that

$$\bar{f} \circ \bar{g} = I_A$$

Hence $\bar{f}$ is an IF-retraction. $\square$

Theorem 4.5. $A \in \mathrm{Ob}(\boldsymbol{C}_{R-IFM})$ *is a free object if and only if* $M \in \mathrm{Ob}(\boldsymbol{C}_{R-M})$ *is a free object and* $A = \bar{0}$.

Proof. Let A be a free object in $\mathbf{C}_{R-IFM}$. Then A is an IF free submodule of a module R-module M. As every IF free submodule of a module is an IF projective object, so A is an IF projective object. Further, according to Theorem 4.2., M is a projective module and $A = \bar{0}$. So, it is sufficient to prove that M is a free module.

Let X be a non-empty set such that there exists a R-homomorphism $i : X \to M$. Let $f : X \to N$ be a R-homomorphism from X to N for any R-module N. Let $B = \bar{0}_N$ be an IFSM of N and $P = \bar{0}_X$ be an IFSM of X such that P is a basis of an IFSM A. So, $\bar{i} : P \to A$ and $\bar{f} : P \to B$ are IF R-homomorphisms obtained by trivially intuitionistic fuzzifying i and f.

As A is a free IFSM, then there exist a unique IF R-homomorphism $\bar{g} : A \to B$ such that $\bar{g} \circ \bar{i} = \bar{f}$.

$$\begin{tikzcd} P \arrow[r, "\bar{i}"] \arrow[dr, "\bar{f}"'] & A \arrow[d, "\bar{g}"] \\ & B \end{tikzcd}$$

Thus, there exist a R-homomorphism $g : M \to N$ such that $g \circ i = f$.
Hence, M is a free module. $\square$

Theorem 4.6. *$(\mu_0, \nu_0)_M$ is an IF-projective object if and only if $(\mu_0, \nu_0)_M$ is a direct summand of a free object in $\boldsymbol{C}_{R-IFM}$.*

Proof. Firstly, let $(\mu_0, \nu_0)_M$ be an IF-projective object in $\mathbf{C}_{R-IFM}$. Then, by Theorem 4.2., M is a projective module. Also, we know that a projective object is a direct summand of free module. Therefore, there exist a free module F and an R-module K such that $F = K \oplus M$. Then $(\mu_0, \nu_0)_F = (\mu_0, \nu_0)_K \oplus (\mu_0, \nu_0)_M$.

Conversely, if $(\mu_0, \nu_0)_F = (\mu', \nu')_K \oplus (\mu'', \nu'')_M$ with the inclusion maps $i_K : K \to F$ and $i_M : M \to F$, then $\mu_0(i_K(x)) \geq \mu'(x)$, $\nu_0(i_K(x)) \leq \nu'(x)$ hence $\mu' = \mu_0$ and $\nu' = \nu_0$ and similarly, we can have $\mu'' = \mu_0$ and $\nu'' = \nu_0$. Thus, $(\mu_0, \nu_0)_M$ is an IF-projective object. $\square$

Theorem 4.7. *$A \in \mathrm{Ob}(\boldsymbol{C}_{R-IFM})$ is an IF-injective object if and only if $M \in \mathrm{Ob}(\boldsymbol{C}_{R-M})$ is an injective module and $A = \bar{1}$.*

Proof. Let A be an injective object in $\mathbf{C}_{R-IFM}$, then A is an IFSM on R-module M. Let N and P be two R-modules and $f : M \to N$ be R-homomorphism and $g : N \to P$ be monomorphism. Take $B = \chi_N$ and $C = \chi_P$. Then B, C are IFSMs of R-modules N and P, respectively. Then, $\bar{f} : B \to A$ is an IF R-homomorphism and $\bar{g} : B \to C$ is an IF-monomorphism obtained by trivially intuitionistic fuzzifying f and g in $\mathbf{C}_{R-IFM}$ (i.e., obtaining the mapping $\bar{f}$ and $\bar{g}$ as in Definition 2.6.) As A is an injective object in $\mathbf{C}_{R-IFM}$, then there exists an IF R-homomorphism $\bar{\psi} : C \to A$ such that $\bar{\psi} \circ \bar{g} = \bar{f}$.

$$\begin{tikzcd} B \arrow[r, "\bar{g}"] \arrow[dr, "\bar{f}"'] & C \arrow[d, "\bar{\psi}"] \\ & A \end{tikzcd}$$

Thus, there exists a R-homomorphism $\psi : P \to M$ such that $\psi \circ g = f$. Hence M is an injective object.

If $A \neq \bar{1}$, then there exists no IF R-homomorphism $\bar{\psi} : \chi_M \to A$, i.e., A will no longer be an injective object $\mathbf{C}_{R-IFM}$ as the following diagram is not commutative.

$$\begin{tikzcd}
\chi_M \arrow[r, "\bar{g}"] \arrow[dr, "\bar{\psi}"'] & \chi_M \arrow[d, "\bar{f}"] \\
& A
\end{tikzcd}$$

□

Theorem 4.8. *Let $\bar{f} : A \to B$ be an IF-coretraction. If B is an IF-injective object, then so is A.*

Proof. Since $\bar{f} : A \to B$ is an intuitionistic fuzzy coretraction. Therefore, there exists an R-homomorphism $\bar{g} : B \to A$ such that $\bar{g} \circ \bar{f} = 1_A$. Now, we show that A is an IF-injective object. Let $\bar{h} : C \to D$ be a IF-monomorphism and $\bar{p} : C \to A$ be any IF R-homomorphism in $\mathbf{C}_{R-IFM}$. Then $\bar{f} \circ \bar{p} : C \to B$ is an IF R-homomorphism. Since B is an IF-injective implies that there exists an IF R-homomorphism $\bar{q} : D \to B$ such that $\bar{q} \circ \bar{h} = \bar{f} \circ \bar{p}$ which implies that $\bar{g} \circ \bar{q} \circ \bar{h} = \bar{g} \circ \bar{f} \circ \bar{p} = \bar{p}$. This gives us $\bar{r} \circ \bar{h} = \bar{p}$, where $\bar{r} = \bar{g} \circ \bar{q}$.

$$\begin{tikzcd}
& D \arrow[dr, "\bar{q}"] \arrow[drr, bend left, "\bar{r}=\bar{g}\circ\bar{q}"] & & \\
C \arrow[ur, "\bar{h}"] \arrow[dr, "\bar{p}"'] & & B \arrow[r, "\bar{g}"] & A \\
& A \arrow[ur, "\bar{f}"] & &
\end{tikzcd}$$

Hence, A is an IF-injective object. □

Theorem 4.9. *If $A \in \mathrm{Ob}(\boldsymbol{C}_{R-IFM})$ is an IF-injective object, then every IF-injective R-homomorphism $\bar{f} : A \to B$ is an intuitionistic fuzzy coretraction, where $B \in \mathrm{Ob}(\boldsymbol{C}_{R-IFM})$.*

Proof. Since A is an injective object in $\mathbf{C}_{R-IFM}$ and $\bar{f} : A \to B$ is an IF-injective R-homomorphism, then we have the following diagram

$$\begin{tikzcd}
A \arrow[r, "\bar{f}"] \arrow[dr, "I_A"'] & B \arrow[d, "\bar{g}"] \\
& A
\end{tikzcd}$$

By the commutativity of the above diagram, there exists an IF R-homomorphism $\bar{g} : B \to A$ such that $\bar{g} \circ \bar{f} = I_A$. Hence, $\bar{f}$ is an intuitionistic fuzzy coretraction. □

5 Conclusion

This paper summarized the basic concepts of an intuitionistic fuzzy coretraction and intuitionistic fuzzy retraction in the category of intuitionistic fuzzy modules $\mathbf{C}_{R-IFM}$. We proved that if, an IF R-homomorphism $\bar{f} : A \to B$ is a coretraction (respectively, retraction), then both f and $\bar{f}$ are one-one (respectively, onto) functions but the converse of this result is not true in general. We have also proved that:

(i) A is a projective object in $\mathbf{C}_{R-IFM}$ if and only if M is a projective module and $A = \bar{0}$.

(ii) A is a free object in $\mathbf{C}_{R-IFM}$ if and only if M is a free module and $A = \bar{0}$.

(iii) A is an IF-projective object if and only if A is a direct summand of a free object.

(iv) A is an injective object in $\mathbf{C}_{R-IFM}$ if and only if M is an injective module and $A = \bar{1}$.

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