A variety of functions concerning intuitionistic fuzzy M-clopen sets in intuitionistic fuzzy topological spaces

G. Saravanakumar [1] and K. A. Venkatesan [2]

[1] Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala
R&D Institute of Science and Technology (Deemed to be University)
Avadi, Chennai-600062, India
e-mail: `saravananguru2612@gmail.com`

[2] Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala
R&D Institute of Science and Technology (Deemed to be University)
Avadi, Chennai-600062, India
e-mail: `venkimaths1975@gmail.com`

Abstract: This paper introduces a novel class of mappings: slightly intuitionistic fuzzy M-continuous functions, intuitionistic fuzzy totally M-continuous functions, and intuitionistic fuzzy M-totally continuous functions, utilizing intuitionistic fuzzy M clopen sets. We examine the topological properties and characterizations of these mappings, explore the relationships between these new sets and existing sets in intuitionistic fuzzy topological spaces, and provide examples to illustrate the concepts.
Keywords: Intuitionistic fuzzy topological spaces, Slightly intuitionistic fuzzy M-continuous functions, Intuitionistic fuzzy totally M-continuous functions, Intuitionistic fuzzy M-totally continuous functions.
2020 Mathematics Subject Classification: 54A40, 54A99, 03E72, 03E99.

1   Introduction

The concept of fuzzy sets was introduced by Zadeh [24] in his seminal paper. Fuzzy sets find applications in numerous fields such as Information [18] and Control [22]. Following their introduction, several authors have proposed generalizations of the fuzzy set concept. Atanassov [2, 4] extended fuzzy sets to intuitionistic fuzzy sets (abbreviated as IFS). Some fundamental results on IFS were published in [2, 3], with the book [3] offering a comprehensive coverage of nearly all aspects of the theory and applications of IFS. Çoker and his colleagues [6–8] introduced intuitionistic fuzzy topology (abbreviated as IFTS) following Chang's approach. Subsequently, Samanta and Mondal [16, 17] introduced and studied the definition of IFTS using the IF gradation of openness. In 2004, Caldas *et al.* [5] introduced properties of θ-open sets, while in 2011, Maghrabi and Johany [11] introduced M-open sets in topological spaces. In 2013 and 2014, Maghrabi and Johany [12–14] introduced several mappings utilizing M-open sets in topological spaces. Additionally, in 2017, Fora [9] discussed properties of fuzzy clopen sets in fuzzy topological spaces. In this paper, we introduce a new class of functions termed as slightly IF M-continuous functions, IF totally M-continuous functions, and IF M-totally continuous functions. We investigate the topological properties and characterizations of these mappings. Additionally, we explore the interrelations between these mappings and existing ones in the theory of IFTSs. Finally, we provide suitable examples to illustrate the theory.

Motivation and organization of the paper

Continuity in IFTSs has been examined from several perspectives, yet most existing approaches rely on classical IF-open or IF-closed structures. While these notions are fundamental, they often behave rigidly in situations where a more refined sensitivity to the underlying topology is required. The emergence of M-open and M-closed sets has shown that the IF environment admits a hierarchy of openness beyond the traditional framework. These M-structures capture subtle topological transitions that ordinary IF-open sets fail to detect, especially in contexts where closure and interior do not interact in a symmetric way.

This observation naturally raises a deeper question: *how do mappings behave when continuity is governed not by IF-open sets alone, but by M-open, M-closed, or M-clopen structures?*

The answer is far from immediate. Traditional continuity, total continuity, irresoluteness, and their IF counterparts provide only partial information because they depend on specific families of sets. Once M-clopen sets are brought into the picture, new phenomena appear for instance, mappings that are too weak to be totally continuous but too strong to be merely continuous. These intermediate behaviours have not been systematically explored in the literature.

To bridge this gap, the present work develops three levels of continuity derived from IF M-clopen structures. Each level isolates a different aspect of how inverse images interact with M-open and M-closed sets. The resulting concepts not only generalize known classes of IF-continuous mappings but also reorganize them into a clearer hierarchy, revealing new relationships and providing the groundwork for future extensions of M-based IFT.

Organization of the paper. The exposition is arranged so that each section introduces exactly the tools required for the next one:

- Section 2 gathers the necessary preliminaries: basic operations on IFS, closure and interior operators, and previously defined M-open and M-closed structures. These definitions serve as the algebraic backbone for the new continuity notions introduced later.
- Section 3 begins the development with slightly IF M-continuous functions. This class is the weakest of the three and captures the behaviour of functions that preserve M-clopen structure only in a minimal sense. Several equivalent formulations are established to clarify their role.
- Section 4 strengthens the framework by introducing IF totally M-continuous mappings. Here, continuity is examined through a pair of closure–interior relations involving M-closures and M-interiors. The section shows how these mappings occupy a middle position in the overall hierarchy.
- Section 5 completes the hierarchy with IF M-totally continuous functions. This is the strongest class among the three, requiring inverse images of M-open sets to be strictly IF-clopen. We also investigate their behaviour under composition and provide examples illustrating the distinctions from previous sections.

Overall, the paper builds a coherent progression from foundational concepts to the most structured form of M-based continuity, offering new insights into how IF mappings behave when governed by refined topological operators.

2 Preliminaries

This section collects the basic definitions and operations of intuitionistic fuzzy sets, intuitionistic fuzzy topological spaces in Çoker's sense, and the various closure and interior operators that will be used throughout the paper. Since the continuity notions introduced in later sections rely heavily on IF-closures, IF-interiors, θ-closures, δ-closures, and especially M-open / M-closed sets, we recall them here for completeness and to fix notation.

Definition 2.1. [2] *Let X be a nonempty predetermined set and I the closed interval $[0,\ 1]$. An $\mathcal{IFS}$ A assumes the appearance of $A = \{\langle x,\ \mu_A(x),\ \nu_A(x)\rangle : x \in X\}$, where the mapping $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely, $\mu_A(x)$) and the degree of nonmembership (namely, $\nu_A(x)$) for each element $x \in X$ to the set A, respectively, and $0 \le \mu_A(x) + \nu_A(x) \le 1$ for each $x \in X$.*

Definition 2.2. [1, 2] *Let X not be empty set, and the $\mathcal{IFS}$'s A and γ in X is structured as $A = \{\langle x,\ \mu_A(x),\ \nu_A(x)\rangle : x \in X\}, B = \{\langle x,\ \mu_B(x),\ \nu_B(x)\rangle : x \in X\}$ Furthermore, let $\{A_i : i \in J\}$ (J be an index set) be an arbitrary family of $\mathcal{IFS}$'s in X. Then*

(i) $A \le B$ iff $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$, for all $x \in X$.
(ii) $A = B$ iff $A \le B$ and $B \le A$.

(iii) $A \wedge B = \{\langle x, \ \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x) \rangle : x \in X \}.$

(iv) $A \vee B = \{\langle x, \ \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x) \rangle : x \in X \}.$

(v) $\overline{A} = \{\langle x, \nu_A(x), \mu_A(x) \rangle : x \in \ X \}.$

(vi) $A - B = A \wedge \overline{B}.$

(vii) $\bigwedge A_i = \{\langle x, \bigwedge \mu_{A_i}(x), \bigvee \nu_{A_i}(x) \rangle : x \in X \}.$

(viii) $\bigvee A_i = \{\langle x, \bigvee \mu_{A_i}(x), \bigwedge \nu_{A_i}(x) \rangle : x \in X \}.$

(ix) $\underline{0} = \{\langle x, 0, 1 \rangle : x \in X \} \ and \ \underline{1} = \{\langle x, 1, 0 \rangle : x \in X \}.$

Definition 2.3. [7] *A family τ of $\mathcal{IFS}'s$ on X satisfying the following axioms is termed an $\mathcal{IFT}$ in Çoker's sense*

(i) $\underline{0}, \ \underline{1} \in \tau.$

(ii) $H_1 \wedge H_2 \in \tau$, *for any* $H_1, \ H_2 \in \tau.$

(iii) $\vee H_i \in \tau$ *for any arbitrary family* $\{H_i : i \in J \} \subseteq \tau.$

Each $\mathcal{IFS}$ A which belongs to τ is called an $\mathcal{IF}$ open ($\mathcal{IF}o$) set in X. The complement $\overline{A}$ of an $\mathcal{IF}o$ set A in X is called an $\mathcal{IF}$ closed ($\mathcal{IF}c$) set in X.

Definition 2.4. [10] *A subset of an $\mathcal{IFTS}$ is said to be $\mathcal{IF}$clopen (in brief, $\mathcal{IF}$clo) if it is $\mathcal{IF}o$ set and $\mathcal{IF}c$ set in X.*

Definition 2.5. [15] *Let $f : (X, \tau) \rightarrow (W, \sigma)$, then f is said to be $\mathcal{IF}$ totally continuous if inverse image of every $\mathcal{IF}o$ set in W is an $\mathcal{IF}$clo set in X*

Definition 2.6. [7] *Let $(X, \ \tau)$ be an $\mathcal{IFTS}$ and $A = \{\langle x, \ \mu_A, \ \nu_A \rangle : x \in X\}$ be an $\mathcal{IFS}$ in X. Then the $\mathcal{IF}$ closure (in brief, $\mathcal{IFC}$) and $\mathcal{IF}$ interior (in brief, $\mathcal{IFI}$) of A are defined by*

(i) $\mathcal{IFC}(A) = \bigwedge \{C : C$ *is an $\mathcal{IF}cs$ in X and $C \geq A \}.$*

(ii) $\mathcal{IFI}(A) = \bigvee \{D : D$ *is an $\mathcal{IF}os$ in X and $D \leq A \}.$*

Definition 2.7. *[23] Let A be $\mathcal{IFS}$ in an $\mathcal{IFTS}$ $(X, \ \tau)$. A is called an $\mathcal{IF}$*

(i) regular open (in brief, $\mathcal{IF}ro$) set if $A = \mathcal{IFI}(\mathcal{IFC}(A))$.

(ii) regular closed (in brief, $\mathcal{IF}rc$) set if $A = \mathcal{IFC}(\mathcal{IFI}(A))$.

Definition 2.8. *[19] Let $(X, \ \tau)$ be an $\mathcal{IFTS}$ and $A = \langle x, \mu_A(x), \nu_A(x) \rangle$ be a $\mathcal{IFS}$ in X. Then the $\mathcal{IF}$ δ closure of A are denoted and defined by $\mathcal{IF}\delta C(A) = \bigwedge \{C : C$ is an $\mathcal{IF}rc$ set in X and $A \leq C \}$ and $\mathcal{IF}\delta I(A) = \bigvee \{D : D$ is an $\mathcal{IF}ro$ set in X and $D \leq A \}.$*

Definition 2.9. *[21] Let (X, τ) be a $\mathcal{IFTS}$, $\forall$ $\mathcal{IFS}$ $D, \ B$ the operators $\mathcal{IF}$- θ interior and $\mathcal{IF}$- θ closure denoted by $(\mathcal{IF})\theta I$ and $\mathcal{IF}\theta C$ are defined as*

$$\mathcal{IF}\theta I(D) = \bigvee \{B \mid B \in \tau \ \& \ \mathcal{IFC}(D) \leq B \}$$

and

$$\mathcal{IF}\theta C(D) = \bigvee \{B \mid B \in \tau \ \& \ \mathcal{IFI}(D) \geq B \}.$$

Definition 2.10. [21] *In an $\mathcal{IFTS}$ (X, τ) and $\mathcal{IFS}$ D is called an*

(i) *$\mathcal{IF}$ -θ open (respectively, $\mathcal{IF}$ -θ semi open) (briefly $\mathcal{IF}\theta o$ (respectively, $\mathcal{IF}\theta so$)) set if $D = \mathcal{IF}\theta I(D)$. (respectively, $D \leq \mathcal{IF}C(\mathcal{IF}\theta I(D))$).*

(ii) *$\mathcal{IF}$ -θ closed (respectively, $\mathcal{IF}$-θ semi closed) (briefly $\mathcal{IF}\theta c$ (respectively, $\mathcal{IF}\theta sc$)) set if $\overline{D}$ is an $\mathcal{IF}$ θo (respectively, $\mathcal{IF}\theta so$) set.*

Definition 2.11. [21] *In an $\mathcal{IFTS}$ $(X, \tau),$ and $\mathcal{IFS}$ D is called an*

1. *$\mathcal{IF}$-M closed (briefly $\mathcal{IF}Mc$) set if $D \geq \mathcal{IF}I(\mathcal{IF}\theta C(D)) \wedge \mathcal{IF}C(\mathcal{IF}\delta I(D))$.*
2. *$\mathcal{IF}$-M open (briefly $\mathcal{IF}Mo$) set if $\overline{D}$ is an $\mathcal{IF}Mc$ set.*

Definition 2.12. [21] *Let (X, τ) be a $\mathcal{IFTS}$, then the*

(i) *union of all $\mathcal{IF}Mo$ (respectively, $\mathcal{IF}\theta so$) sets contained in D is called the $\mathcal{IF}M$ (respectively, $\mathcal{IF}\theta$ semi) interior of D and is denoted by $\mathcal{IF}MI(D)$ (respectively, $\mathcal{IF}\theta sI(D)$).*

(ii) *intersection of all $\mathcal{IF}Mc$ (respectively, $\mathcal{IF}\theta sc$) sets containing D is called the $\mathcal{IF}M$ (respectively, $\mathcal{IF}\theta$ semi) closure of D and is denoted by $\mathcal{IF}MC(D)$ (respectively, $\mathcal{IF}\theta sC(D)$).*

Definition 2.13. [20] *A function f from a $\mathcal{IFTS}$ (X, τ) to a $\mathcal{IFTS}$ (W, σ) is called as $\mathcal{IF}$ M continuous (briefly, $\mathcal{IF}MCts$) function if $f^{-1}(A)$ is an $\mathcal{IF}Mc$ set in τ for every $\mathcal{IF}c$ set $A \in \sigma$.*

3 Intuitionistic fuzzy slightly M-continuous functions

This section introduces the notion of slightly $\mathcal{IF}$ M-continuous functions. The motivation arises from the observation that classical $\mathcal{IF}$-continuity and $\mathcal{IF}$ M-continuity do not capture intermediate behaviour between M-open and M-closed structures. The concept defined here provides a weaker form of M-continuity based on $\mathcal{IF}$ M-open sets, and the characterizations obtained in this section establish links with inverse images of $\mathcal{IF}$ M-clopen sets. These results form the foundational layer for the stronger forms of the M-continuity studied in Sections 4 and 5.

Definition 3.1. *A function $f : (X, \tau) \rightarrow (W, \sigma)$ is said to be intuitionistic fuzzy slightly M-continuous (briefly, $\mathcal{IF}sMCts$) if for every $B \in \tau$ and for every $\mathcal{IF}$-clopen set $C \in \sigma$ with $f(B) \leq C$, there exists an $\mathcal{IF}M$-open set $A \in \tau$ such that $B \leq A$ and $f(A) \leq C$.*

Definition 3.2. *A function $f : (X, \tau) \rightarrow (W, \sigma)$ is called $\mathcal{IF}$ M irresolute (briefly, $\mathcal{IF}MIrr$) if $f^{-1}(B)$ is a $\mathcal{IF}Mo$ set in τ for every $\mathcal{IF}Mo$ set B in σ.*

Proposition 3.1. *For any mapping $f : (X, \tau) \rightarrow (W, \sigma)$*

(i) *f is a $\mathcal{IF}sMCts$ function.*
(ii) *$f^{-1}(C)$ is $\mathcal{IF}Mo$ set for each $C \in \sigma$ such that A is a $\mathcal{IF}clo$ set.*
(iiii) *$f^{-1}(C)$ is a $\mathcal{IF}Mc$ set for each $C \in \sigma$ such that A is a $\mathcal{IF}clo$ set.*
(iv) *$f^{-1}(C)$ is a $\mathcal{IF}Mclo$ set for each $C \in \sigma$ such that A is a $\mathcal{IF}clo$ set*

are equivalent.

Proof. (i) $\Rightarrow$ (ii): Let $C \in \sigma$ such that C is an $\mathcal{IF}clo$ set and let $B \in \tau$ such that $B \leq f^{-1}(C)$. Since C is an $\mathcal{IF}clo$ set with $f(B) \leq C$. By (i), there exists $A \in \tau$ such that A is an $\mathcal{IF}Mo$, $B \leq A$ and $f(A) \leq C$. Hence $f^{-1}(C)$ is an $\mathcal{IF}Mo$ set.

(ii) $\Rightarrow$ (iii) & (iii)$\Rightarrow$ (iv) : These are direct from the fact that if A is $\mathcal{IF}clo$ so is $\overline{A}$.

(iv) $\Rightarrow$ (i): Let $C \in \sigma$ be an $\mathcal{IF}clo$ and $B \in \tau$ such that $f(B) \leq C$. By (iv), $f^{-1}(C)$ is $\mathcal{IF}Mclo$. Take $B = f^{-1}(C)$. Then, $f(B) \leq C$. Hence, f is $\mathcal{IF}sMCts$. $\qquad\square$

Proposition 3.2. *Let $f : (X, \tau) \to (W, \sigma)$ and $g : (W, \sigma) \to (V, \omega)$ be any two functions.*

(i) If f is $\mathcal{IF}MIrr$ and g is $\mathcal{IF}sMCts$ (respectively, $\mathcal{IF}sCts$ and $\mathcal{IF}MCts$)

(ii) If f is $\mathcal{IF}sMCts$ and g is $\mathcal{IF}sCts$, then $g \circ f$ is $\mathcal{IF}sMCts$.

Proof. (i) Let $B \in \omega$ be a $\mathcal{IF}clo$. Since g is $\mathcal{IF}sMCts$ (respectively, $\mathcal{IF}sCts$ and $\mathcal{IF}MCts$), is an $\mathcal{IF}Mo$ (respectively, $\mathcal{IF}o$ and $\mathcal{IF}Mo$). Then $f^{-1}(g^{-1}(B)) = (g \circ f)^{-1}(B)$ is $\mathcal{IF}Mo$, since f is $\mathcal{IF}MIrr$.

(ii) Let $B \in \omega$ be an $\mathcal{IF}clo$. Since g is $\mathcal{IF}sCts$, $g^{-1}(B)$ is an $\mathcal{IF}clo$. Then $f^{-1}(g^{-1}(B)) = (g \circ f)^{-1}(B)$ is $\mathcal{IF}Mo$, since f is $\mathcal{IF}sMCts$. $\qquad\square$

Proposition 3.3. *If $f : (X, \tau) \to (W, \sigma)$ is a surjective $\mathcal{IF}MIrr$ and $\mathcal{IF}MO$ function and $g : (W, \sigma) \to (V, \omega)$ be any function, then $g \circ f$ is a $\mathcal{IF}sMCts$ function iff g is $\mathcal{IF}sMCts$.*

Proof. Suppose that $g \circ f$ is a $\mathcal{IF}sMCts$ function, $B \in \omega$ is an $\mathcal{IF}Mclo$ set. By using Proposition 3.2 (ii), $f^{-1}(g^{-1}(B)) = (g \circ f)^{-1}(B)$ is an $\mathcal{IF}Mo$ set in τ. Since f is $\mathcal{IF}MO$, $g^{-1}(B) = f(f^{-1}(g^{-1}(B)))$ is an $\mathcal{IF}Mo$ set. Therefore by Proposition 3.2 g is $\mathcal{IF}sMCts$ function.

Conversely, let $A \in \omega$ be an $\mathcal{IF}Mclo$ set. Since g is a $\mathcal{IF}sMCts$ function, $g^{-1}(A)$ is an $\mathcal{IF}Mo$ set $\in \sigma$ and f is a $\mathcal{IF}MIrr$ function, $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$ is an $\mathcal{IF}Mo$ set $\in \tau$. Therefore by Proposition 3.2 $(g \circ f)$ is a $\mathcal{IF}sMCts$ function. $\qquad\square$

4 Intuitionistic fuzzy totally M-continuous functions

In this section we study $\mathcal{IF}$ totally M-continuous functions, which extend the idea of totally continuous mappings by replacing open sets with M-open / M-closed structures. The aim is to understand how total continuity behaves under M-closures and M-interiors. We show that several natural conditions involving the operators $\mathcal{IF}I, \mathcal{IF}C, \mathcal{IF}MI,$ and $\mathcal{IF}MC$ are equivalent to this new form of continuity. This prepares the ground for the stronger notion introduced in Section 5.

Definition 4.1. *The map $f : (X, \tau) \to (W, \sigma)$ is called $\mathcal{IF}$ totally M continuous (briefly, $\mathcal{IF}tMCts$) if $f^{-1}(C)$ is $\mathcal{IF}Mco$, for each $C \in \sigma$.*

Theorem 4.1. *For any $f : (X, \tau) \to (W, \sigma)$,*

(i) *f is a $\mathcal{IF}tMCts$ function,*

(ii) *$f^{-1}(C)$ is an $\mathcal{IF}Mclo$ set of τ for each $C \in \sigma$,*

(iii) $\mathcal{IFMC}(f^{-1}(C)) \leq f^{-1}(\mathcal{IFC}(C))$ *and* $f^{-1}(C) \leq \mathcal{IFMI}(f^{-1}(\mathcal{IFC}(C)))$ *for each* $C \in \sigma$.

(iv) $f^{-1}(\mathcal{IFI}(C)) \leq \mathcal{IFMI}(f^{-1}(C))$ *and* $\mathcal{IFMC}(f^{-1}(\mathcal{IFI}(C))) \leq f^{-1}(C)$ *for each* $C \in \sigma$

are equivalent.

Proof. (i) $\Rightarrow$ (ii): Let $\overline{C} \in W$. By using (i), $f^{-1}(\overline{C})$ is an $\mathcal{IFMclo}$ set in τ. Since $f^{-1}(\overline{W})$ is both $\mathcal{IFMo}$ and $\mathcal{IFMc}$ set. Which implies that $f^{-1}(C)$ is both $\mathcal{IFMo}$ and $\mathcal{IFMc}$ set. Thus $f^{-1}(C)$ is an $\mathcal{IFMclo}$ set in τ.

(ii) $\Rightarrow$ (iii): Let $C \in \sigma$. Then, $\mathcal{IFC}(C)$ is an $\mathcal{IFc}$ set in σ. By (ii), $f^{-1}(\mathcal{IFC}(C))$ is $\mathcal{IFMclo}$ in τ. Hence, $\mathcal{IFMC}(f^{-1}(C)) \leq \mathcal{IFMC}(f^{-1}(C(C))) = f^{-1}(\mathcal{IFC}(C))$. Again by using (ii) $f^{-1}(C) \leq f^{-1}(\mathcal{IFC}(C)) = \mathcal{IFMI}(f^{-1}(C(C)))$.

(iii) $\Rightarrow$ (iv): Let $C \in \sigma$. By using (iii), $f^{-1}(\mathcal{IFC}(\overline{C})) \geq \mathcal{IFMC}(f^{-1}(\overline{C}))$. This implies that $f^{-1}(\mathcal{IFC}(C)) \geq \mathcal{IFMC}(f^{-1}(\overline{C}))$, that is $f^{-1}(\mathcal{IFC}(C)) \geq \mathcal{IFMC}(f^{-1}(C))$. Hence, $f^{-1}(\mathcal{IFI}(C)) \leq \mathcal{IFMI}(f^{-1}(C))$.

Again by using (iii), we have $f^{-1}(\overline{C}) \leq \mathcal{IFMI}(f^{-1}(\mathcal{IFC}(\overline{C})))$. This implies that $f^{-1}(\overline{C}) \leq \mathcal{IFMI}(f^{-1}(\mathcal{IFC}(C)))$, that is $f^{-1}(\overline{C}) \leq \mathcal{IFMI}(f^{-1}(\mathcal{IFC}(C)))$. Hence $f^{-1}(\overline{C}) \leq \mathcal{IFMI}(f^{-1}(\mathcal{IFC}(C)))$. Therefore $f^{-1}(C) \geq \mathcal{IFMC}(f^{-1}(\mathcal{IFI}(C)))$.

(iv) $\Rightarrow$ (i): Let $C \in \sigma$ such that $C = \mathcal{IFI}(C)$. By using (iv), we have $f^{-1}(C) \leq \mathcal{IFMI}(f^{-1}(C))$. Hence, $f^{-1}(C) = \mathcal{IFMI}(f^{-1}(C))$, that is, $f^{-1}(C)$ is $\mathcal{IFMo}$. By using (iv), we have $f^{-1}(C) \geq \mathcal{IFMC}(f^{-1}(C))$. Hence, $f^{-1}(C) = \mathcal{IFMC}(f^{-1}(C))$, that is, $f^{-1}(C)$ is $\mathcal{IFMc}$. Therefore $f^{-1}(C)$ is an $\mathcal{IFMclo}$ set in τ. Thus, $\mathcal{IF}$ is a $\mathcal{IFtMCts}$ function. $\square$

Theorem 4.2. *Let* $f : (X, \tau) \to (W, \sigma)$ *be a function. Then every*

(i) $\mathcal{IFtCts}$ *is* $\mathcal{IFtMCts}$.

(ii) $\mathcal{IFtMCts}$ *is* $\mathcal{IFMCts}$,

but not conversely.

Proof. (i) Let f be $\mathcal{IFtCts}$ function and $C \in \sigma$. By hypothesis, $f^{-1}(C)$ is $\mathcal{IFclo}$ and hence $\mathcal{IFMclo}$ set in τ. Therefore f is a $\mathcal{IFtMCts}$ function.

(ii) Let $C \in \sigma$. By the hypothesis, $f^{-1}(C)$ is $\mathcal{IFMclo}$ and hence $\mathcal{IFMo}$ set in τ. Hence f is a $\mathcal{IFMCts}$ function. $\square$

Example 4.1. Let $X = W = \{a, b, c, d\}$. Define the $\mathcal{IFS}$s

$$A = \Big\langle (a,\, 1,\, 0),\ (b,\, 0,\, 1),\ (c,\, 0.2,\, 0.7),\ (d,\, 0,\, 1) \Big\rangle,$$

$$B = \Big\langle (a,\, 0,\, 1),\ (b,\, 1,\, 0),\ (c,\, 0,\, 1),\ (d,\, 0,\, 0.1) \Big\rangle,$$

$$C = \Big\langle (a,\, 1,\, 0),\ (b,\, 0,\, 0.2),\ (c,\, 0,\, 0),\ (d,\, 1,\, 0) \Big\rangle,$$

$$D = \Big\langle (a,\, 0,\, 0),\ (b,\, 0.8,\, 0.2),\ (c,\, 0,\, 0.9),\ (d,\, 1,\, 0) \Big\rangle.$$

Then the families $\tau = \{\underline{0}, \underline{1}, A, B, C\}$ is an $\mathcal{IFT}$ on X and $\sigma = \{\underline{0}, \underline{1}, A, D\}$ is an $\mathcal{IFT}$ on W. Let us consider the identity functions f_1 and f_2:

1. $f_1 : (X, \tau) \to (W, \sigma)$ is $\mathcal{IF}tMCts$ function but not $\mathcal{IF}tCts$, since, $f_1^{-1}(D)$, is - $\mathcal{IF}Mclo$ but not an - $\mathcal{IF}clo$.

2. $f_2 : (X, \tau) \to (W, \sigma)$ is $\mathcal{IF}MCts$ function but not $\mathcal{IF}tMCts$, since, $f_2^{-1}(A)$, is - $\mathcal{IF}Mo$ but not an- $\mathcal{IF}Mclo$.

5 Intuitionistic fuzzy M-totally continuous functions

This section introduces the strongest form among the three new mappings: $\mathcal{IF}$ M-totally continuous functions. Here the inverse image of every $\mathcal{IF}$ M-open set is required to be an $\mathcal{IF}$-clopen set, a condition more restrictive than in the previous sections. We provide characterizations similar in structure to those in Section 4, and we investigate the behaviour of this class under composition. Finally, examples demonstrate that this class is strictly stronger than both $\mathcal{IF}$ totally M-continuous and $\mathcal{IF}$ M-continuous functions.

Definition 5.1. *A function $f : (X, \tau) \to (W, \sigma)$ is called $\mathcal{IF}$ M-totally continuous function (briefly, $\mathcal{IF}MtCts$) if $f^{-1}(C)$ is -$\mathcal{IF}clo$, for each -$\mathcal{IF}Mo$ set $C \in \sigma$.*

Theorem 5.1. *A map $f : (X, \tau) \to (W, \sigma)$ is a $\mathcal{IF}MtCts$ function iff for each $x_t \in \tau$ and each $\mathcal{IF}Mo$ set $C \in \sigma$ and $f(x_t) \le C$, there exists an $\mathcal{IF}clo$ set $B \in \tau$ such that $x_t \in B$ and $f(B) \le C$.*

Proof. Suppose f is $\mathcal{IF}MtCts$ and C is any $\mathcal{IF}Mo$ set in σ containing $f(x_t)$ so that $x_t \in f^{-1}(C)$. Take $B = f^{-1}(C)$, then B is an $\mathcal{IF}Mclo$ set in τ and $x_t \in B$, since f is $\mathcal{IF}MtCts$ and $f^{-1}(C)$ is an $\mathcal{IF}clo$ in τ. Also $f(B) = f(f^{-1}(C)) \le C$. This implies $f(B) \le C$.

Conversely, suppose C is an $\mathcal{IF}Mo$ set in σ and let $x_t \in f^{-1}(C)$ be any fuzzy point, then $f(x_t) \le C$. Therefore, by hypothesis, there is an $\mathcal{IF}clo$ set $f(Bx_t)$ containing x_t such that $f(x_t) \le C$, so $x_t \in Bx_t \le f^{-1}(C)$ which implies that $f^{-1}(C)$ is an $\mathcal{IF}clo$ of x_t and $x_tqf^{-1}(C)$. Hence it is an $\mathcal{IF}clo$ in τ. Therefore, f is a $\mathcal{IF}MtCts$ function. $\qquad\square$

Theorem 5.2. *For any map $f : (X, \tau) \to (W, \sigma)$,*

(i) *f is a $\mathcal{IF}MtCts$ function,*

(ii) *$f^{-1}(C)$ is an $\mathcal{IF}clo$ set of τ for each $\mathcal{IF}Mc$ set $C \in \sigma$,*

(iii) *$\mathcal{IF}C(f^{-1}(C)) \le \mathcal{IF}I(f^{-1}(\mathcal{IF}MC(C)))$ for each $C \in \sigma$,*

(iv) *$\mathcal{IF}C(f^{-1}(\mathcal{IF}MI(C))) \le \mathcal{IF}I(f^{-1}(C))$ for each $C \in \sigma$*

are equivalent.

Proof.   (i) $\Rightarrow$ (ii): Let $C \in \sigma$ be $\mathcal{IF}Mc$ set. Then $\overline{C}$ is a $\mathcal{IF}Mo$ set. By definition, $f^{-1}(\overline{C})$ is a $\mathcal{IF}clo$ set in τ. This implies that $f^{-1}(C)$ is -$\mathcal{IF}clo$ set.

(ii) $\Rightarrow$ (iii): Let $C \in \sigma$. Then $\mathcal{IF}MC(B)$ is an $\mathcal{IF}Mc$ set, by (ii), $f^{-1}(C)$ is an $\mathcal{IF}clo$ set in τ. Since $C \le \mathcal{IF}MC(C)$, $f^{-1}(C) \le f^{-1}(\mathcal{IF}MC(C))$. This implies that $\mathcal{IF}C(f^{-1}(C)) \le \mathcal{IF}C(f^{-1}(\mathcal{IF}MC(C)))$. But

$$\mathcal{IF}C(f^{-1}(\mathcal{IF}MC(C))) = f^{-1}(\mathcal{IF}MC(C)) = \mathcal{IF}I(f^{-1}(\mathcal{IF}MC(C))).$$

Therefore $\mathcal{IF}C(f^{-1}(C)) \le \mathcal{IF}I(f^{-1}(\mathcal{IF}MC(C)))$.

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(iii) $\Rightarrow$ (iv): Let $C \in \sigma$. by (iii), $\mathcal{IF}C(f^{-1}(\overline{C})) \leq \mathcal{IF}I(f^{-1}(\mathcal{IF}MC(\overline{C})))$. That is, $\mathcal{IF}C(f^{-1}(\overline{C})) \leq \mathcal{IF}I(f^{-1}(\mathcal{IF}MC(C)))$. This implies that $\mathcal{IF}C(f^{-1}(C)) \leq \mathcal{IF}I(f^{-1}(\mathcal{IF}MC(C)))$, i.e., $\mathcal{IF}C(f^{-1}(C)) \leq \mathcal{IF}I(f^{-1}(\mathcal{IF}MC(C)))$.

Hence $\mathcal{IF}I(f^{-1}(C)) \geq \mathcal{IF}C(f^{-1}(\mathcal{IF}MI(C)))$.

(iv) $\Rightarrow$ (i): Let $C \in \sigma$ be an $\mathcal{IF}Mo$ set, then $\mathcal{IF}MI(C) = C$. Therefore

$$\mathcal{IF}C(f^{-1}(\mathcal{IF}MI(C))) = \mathcal{IF}C(f^{-1}(C)) = \mathcal{IF}I(f^{-1}(C)).$$

Therefore, f is a $\mathcal{IF}MtCts$ function. $\qquad\square$

Theorem 5.3. *Let $f : (X, \tau) \to (W, \sigma)$, then every*

(i) $\mathcal{IF}MtCts$ is a $\mathcal{IF}tCts$ function;

(ii) $\mathcal{IF}MCts$ is a $\mathcal{IF}tMCts$ function.

But not conversely.

Proof. (i) Suppose f is a $\mathcal{IF}MtCts$ function and $C \in \sigma$. Thus C is a $\mathcal{IF}Mo$ set in σ. Since f is a $\mathcal{IF}MtCts$ function, it follows that $f^{-1}(C)$ is a $\mathcal{IF}clo$ in τ. Thus inverse image of every $\mathcal{IF}o$ set in σ is $\mathcal{IF}clo$ in τ. Therefore f is a $\mathcal{IF}tCts$ function.

(ii) Suppose f is a $\mathcal{IF}MtCts$ function and $C \in \sigma$. Then C is $\mathcal{IF}Mo$ in σ. Since f is a $\mathcal{IF}MtCts$ function, $f^{-1}(C)$ is $\mathcal{IF}clo$ set and hence $\mathcal{IF}Mclo$ in τ. Thus f is $\mathcal{IF}tMCts$ function. $\qquad\square$

Example 5.1. Let $X = W = \{a, b\}$. Define $\mathcal{IF}S$s

$$A = \Big\langle (a,\ 0.5,\ 0.3),\ (b,\ 0.5,\ 0.5) \Big\rangle,$$
$$B = \Big\langle (a,\ 0.7,\ 0.3),\ (b,\ 0.2,\ 0.2) \Big\rangle,$$
$$C = \Big\langle (a,\ 0.3,\ 0.5),\ (b,\ 0.4,\ 0.6) \Big\rangle,$$
$$D = \Big\langle (a,\ 0.5,\ 0.3),\ (b,\ 0.7,\ 0.2) \Big\rangle.$$

Then the families $\tau = \{\underline{0}, \underline{1}, A\}$ is an $\mathcal{IFT}$ on X and $\sigma = \{\underline{0}, \underline{1}, B\}$ is an $\mathcal{IFT}$ on W. Here C is an $\mathcal{IF}Mo$ set and the map $f : (X, \tau) \to (W, \sigma)$ is $\mathcal{IF}tCts$ but not $\mathcal{IF}MtCts$, since $f^{-1}(C)$ is not an $\mathcal{IF}clo$ set.

Theorem 5.4. *Let $(X, \tau), (W, \sigma)$ and (V, ω) be $\mathcal{IFTS}$s and let $f : (X, \tau) \to (W, \sigma), g : (W, \sigma) \to (V, \omega)$ and $g \circ f : (X, \tau) \to (V, \omega)$ be functions. If f is $\mathcal{IF}MtCts$ and*

(i) g is $\mathcal{IF}MtCts$ (respectively, $\mathcal{IF}MIrr$), then $g \circ f$ is $\mathcal{IF}MtCts$.

(ii) $g \circ f$ is $\mathcal{IF}MtCts$, then g is $\mathcal{IF}MIrr$.

Proof. (i) Let C be an $\mathcal{IF}Mo$ set in ω, and g is $\mathcal{IF}MtCts$ (respectively, $\mathcal{IF}MIrr$), so $g^{-1}(C)$ is a $\mathcal{IF}Mo$ set in σ. Also, since f is a $\mathcal{IF}MtCts$ function, then $f^{-1}(g^{-1}(C)) = (g \circ f)^{-1}(C)$ is a $\mathcal{IF}clo$ set in τ. Hence $g \circ f$ is a $\mathcal{IF}MtCts$ function.

(ii) Let $g \circ f$ be $\mathcal{IF}MtCts$ and let C be an $\mathcal{IF}Mo$ set in ω. Now, by hypothesis, $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$ is -$\mathcal{IF}clo$ in τ. But f is $\mathcal{IF}MtCts$, then $g^{-1}(C)$ is -$\mathcal{IF}Mo$ in σ. Hence g is $\mathcal{IF}MIrr$. $\qquad\square$

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