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On λ-statistical convergence of order α in intuitionistic fuzzy normed spaces

Ekrem Savaş

Department of Mathematics, Istanbul Commerce University
Sutluce-Istanbul, Turkey
e-mail: ekremsavas@yahoo.com

Abstract: The purpose of this paper is to introduce the notion $[V, \lambda]\,(\mathcal{I})$-summability and $\mathcal{I}_\lambda$-statistical convergence of order α with respect to the intuitionistic fuzzy norm (μ, v), investigate their relationship, and make some observations about these classes. We also study the relation between $\mathcal{I}_\lambda$-statistical convergence of order α and $\mathcal{I}$-statistical convergence of order α in intuitionistic fuzzy normed space (μ, v).
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1 Introduction

The idea statistical convergence was given by Zygmund [32] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Fast [8] and Schoenberg [30] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from the sequence space point of view and linked with summability theory by Fridy [9], Šalát [20], Cakalli [3], Malkowsky and Savas [14], Di Maio and Kocinac [12], Maddox [13] and many others.

However, Mursaleen [15] defined the concept of λ-statistical convergence as a generalization of the statistical convergence and found its relation to statistical convergence, $(C, 1)$-summability and strong (V, λ)-summability. In [15], the relation between λ- statistical convergence and statistical convergence was established among other things.

Recently in [7, 21] we used ideals to introduce the concepts of $\mathcal{I}$-statistical convergence, $\mathcal{I}$-lacunary statistical convergence and $\mathcal{I}_\lambda$-statistical convergence and investigated their properties.

On the other hand, in ([4, 5]) a different direction was given to the study of these important summability methods where the notions of statistical convergence of order α and λ-statistical convergence of order α were introduced and studied.

The notion of statistical convergence depends on the density of subsets of $\mathbf{N}$, the set of the natural numbers. A subset E of $\mathbf{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ exists.}$$

Note that if $K \subset \mathbf{N}$ is a finite set, then $\delta(K) = 0$, and for any set $K \subset \mathbf{N}, \delta(K^C) = 1 - \delta(K)$.

Definition 1. *A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$*

$$\delta(\{k \in \mathbf{N} : |x_k - L| \geq \varepsilon\}) = 0.$$

In [11], P. Kostyrko et al. introduced the concept of $\mathcal{I}$-convergence of sequences in a metric space and studied some properties of such convergence. Note that $\mathcal{I}$-convergence is an interesting generalization of statistical convergence. More investigations in this direction and more applications of ideals can be found in [6, 7, 21–25].

Following the introduction of fuzzy set theory by Zadeh [31], there has been extensive research to find applications and fuzzy analogues of the classical theories. The theory of intuitionistic fuzzy sets was introduced by Atanassov [1]; it has been extensively used in decision-making problems [2]. The concept of an intuitionistic fuzzy metric space was introduced by Park [18]. Furthermore, Saadati and Park [19] gave the notion of an intuitionistic fuzzy normed space. Also, in [16] Mohiuddine and Lohani introduced the notion of the generalized statistical convergence in intuitionistic fuzzy normed spaces. Some works related to the convergence of sequences in several normed linear spaces in a fuzzy setting can be found in ([10, 17, 26–28]).

Quite recently, $\mathcal{I}$-statistical convergence has been established as a better study than statistical convergence. It is found very interesting that some results on sequences, series and summability can be proved by replacing the statistical convergence by $\mathcal{I}$-statistical convergence. Also, it should be noted that the results of $\mathcal{I}_\lambda$-statistical convergence in an intuitionistic fuzzy normed linear space happen to be stronger than those proved for λ-statistical convergence in an intuitionistic fuzzy normed linear space. Recently, Savas and Gurdal [28] introduced the concept of $\mathcal{I}_\lambda$-statistical convergence with respect to the intuitionistic fuzzy normed space (μ, v).

In this paper, we intend to introduce the concept of $\mathcal{I}_\lambda$-statistical convergence of order α, $0 < \alpha \leq 1$, with respect to the intuitionistic fuzzy normed space (μ, v), and investigate some of its consequences.

We now recall some notation and basic definitions used in the paper.

Definition 2 ([30])**.** *A binary operation $\Diamond : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-conorm if it satisfies the following conditions:*

(i) $\Diamond$ is associate and commutative,

(ii) $\Diamond$ is continuous,

(iii) $a\Diamond 0 = a$ *for all* $a \in [0,1]$,
(iv) $a\Diamond b \leq c\Diamond d$ *whenever* $a \leq c$ *and* $b \leq d$ *for each* $a,b,c,d \in [0,1]$.

For example, we can give $a * b = ab$, $a * b = \min\{a,b\}$, $a\Diamond b = \min\{a+b,1\}$ and $a\Diamond b = \max\{a,b\}$ for all $a,b \in [0,1]$.

Using the continuous t-norm and t-conorm, Saadati and Park [19] has recently introduced the concept of intuitionistic fuzzy normed space as follows.

Definition 3 ([19])**.** *The five-tuple* $(X,\mu,v,*,\Diamond)$ *is said to be an intuitionistic fuzzy normed space (for short, IFNS) if* X *is a vector space,* $*$ *is a continuous* t*-norm,* $\Diamond$ *is a continuous* t*-conorm, and* μ, v *are fuzzy sets on* $X \times (0,\infty)$ *satisfying the following conditions for every* $x,y \in X$*, and* $s,t > 0$*:*

(a) $\mu(x,t) + v(x,t) \leq 1$,

(b) $\mu(x,t) > 0$,

(c) $\mu(x,t) = 1$ *if and only if* $x = 0$,

(d) $\mu(\alpha x,t) = \mu\left(x,\frac{t}{|\alpha|}\right)$ *for each* $\alpha \neq 0$,

(e) $\mu(x,t) * \mu(y,s) \leq \mu(x+y,t+s)$,

(f) $\mu(x,.) : (0,\infty) \to [0,1]$ *is continuous,*

(g) $\lim_{t\to\infty} \mu(x,t) = 1$ *and* $\lim_{t\to 0} \mu(x,t) = 0$,

(h) $v(x,t) < 1$,

(i) $v(x,t) = 0$ *if and only if* $x = 0$,

(j) $v(\alpha x,t) = \mu\left(x,\frac{t}{|\alpha|}\right)$ *for each* $\alpha \neq 0$,

(k) $v(x,t)\Diamond v(y,s) \geq v(x+y,t+s)$,

(l) $v(x,.) : (0,\infty) \to [0,1]$ *is continuous,*

(m) $\lim_{t\to\infty} v(x,t) = 0$ *and* $\lim_{t\to 0} v(x,t) = 1$.

In this case (μ,v) is called an intuitionistic fuzzy norm. As a standard example, we can give the following:

Let $(X,\|.\|)$ be a normed space, and let $a * b = ab$ and $a\Diamond b = \min\{a+b,1\}$ for all $a,b \in [0,1]$. For all $x \in X$ and every $t > 0$, consider

$$\mu(x,t) = \frac{t}{t+\|x\|} \text{ and } v_0(x,t) = \frac{\|x\|}{t+\|x\|}.$$

Then observe that $(X,\mu,v,*,\Diamond)$ is an intuitionistic fuzzy normed space.

We also recall that the concept of convergence in an intuitionistic fuzzy normed space is studied in [19].

Definition 4 ([19]). *Let* $(X, \mu, v, *, \Diamond)$ *be an IFNS. Then, a sequence* $x = (x_k)$ *is said to be convergent to* $L \in X$ *with respect to the intuitionistic fuzzy norm* (μ, v) *if, for every* $\varepsilon > 0$ *and* $t > 0$*, there exists* $k_0 \in \mathbb{N}$ *such that* $\mu(x_k - L, t) > 1 - \varepsilon$ *and* $v(x_k - L, t) < \varepsilon$ *for all* $k \geq k_0$*. It is denoted by* (μ, v)-$\lim x = L$ *or* $x_k \overset{(\mu,v)}{\to} L$ *as* $k \to \infty$.

2 $\mathcal{I}_\lambda$-statistical convergence on IFNS

Before proceeding further, we should recall some notation on the $\mathcal{I}$-statistical convergence and ideal convergence.

The family $\mathcal{I} \subset 2^Y$ of subsets a nonempty set Y is said to be an ideal in Y if (i) $\emptyset \notin \mathcal{I}$; (ii) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$; (iii) $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$. If $\mathcal{I}$ is an ideal in Y then the collection $F(\mathcal{I}) = \{M \subset Y : M^c \in \mathcal{I}\}$ forms a filter in Y which is called the filter associated with $\mathcal{I}$. Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. Then a sequence $x = (x_n)$ in X is said to be $\mathcal{I}$-convergent to $x \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\}$ belongs to $\mathcal{I}$ (see [11]).

Definition 5. *A sequence* $x = (x_k)$ *is said to be* $\mathcal{I}$*-statistically convergent of order* α *to* L *or* $S(\mathcal{I})^\alpha$*-convergent to* L*, where* $0 < \alpha \leq 1$*, if for each* $\epsilon > 0$ *and* $\delta > 0$

$$\{n \in \mathbb{N} : \frac{1}{n^\alpha}|\{k \leq n : |x_k - L| \geq \epsilon\}| \geq \delta\} \in \mathcal{I}.$$

In this case we write $x_k \to L(S(\mathcal{I})^\alpha)$. The class of all $\mathcal{I}$-statistically convergent of order α sequences will be denoted by simply $S(\mathcal{I})^\alpha$.

Remark 1. *For* $\mathcal{I} = \mathcal{I}_f in = \{A \subseteq \mathbf{N} : A \text{ is a finite subset }\}$*,* $S^\alpha(\mathcal{I})$*-convergence coincides with statistical convergence of order* α*, [4]. For an arbitrary ideal* $\mathcal{I}$ *and for* $\alpha = 1$ *it coincides with* $\mathcal{I}$*-statistical convergence,(see, [21]). When* $\mathcal{I} = \mathcal{I}_{fin}$ *and* $\alpha = 1$ *it becomes only statistical convergence.*

Definition 6. *Let* $(X, \mu, v, *, \Diamond)$ *be an IFNS. Then, a sequence* $x = (x_k)$ *is said to be* $\mathcal{I}$*-statistically convergent of order* α *to* $L \in X$*, where* $0 < \alpha \leq 1$*, with respect to the intuitionistic fuzzy normed space* (μ, v)*, if for every* $\varepsilon > 0$*, and every* $\delta > 0$ *and* $t > 0$*,*

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha}\left|\{k \leq n : \mu(x_k - L, t) \leq 1 - \varepsilon \text{ or } v(x_k - L, t) \geq \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

In this case we write $x_k \overset{(\mu,v)}{\to} L\left(S^\alpha(I)^{(\mu,v)}\right)$.

Remark 2. *For* $\mathcal{I} = \mathcal{I}_{fin}, S^\alpha(\mathcal{I})$*-convergence coincides with statistical convergence of order* α*, with respect to the intuitionistic fuzzy normed space* (μ, v)*. For an arbitrary ideal* $\mathcal{I}$ *and for* $\alpha = 1$*, it coincides with* $\mathcal{I}$*-statistical convergence, with respect to the intuitionistic fuzzy normed space* (μ, v) *. When* $\mathcal{I} = \mathcal{I}_{fin}$ *and* $\alpha = 1$ *it becomes only statistical convergence with respect to the intuitionistic fuzzy normed space* (μ, v)*, [10].*

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$. The collection of such a sequence λ will be denoted by Δ.

The generalized de Valée-Pousin mean of order α, where $0 < \alpha \leq 1$, is defined by

$$t_n(x) = \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

Throughout $\mathcal{I}$ will stand for a proper admissible ideal of $\mathbb{N}$.

We are now ready to obtain our main results.

Definition 7. *Let $(X, \mu, v, *, \Diamond)$ be an IFNS. We say that a sequence $x = (x_k)$ is said to be $[V, \lambda](\mathcal{I})$-summable of order α to $L \in X$ with respect to the intuitionistic fuzzy normed space (μ, v), if for any $\delta > 0$ and $t > 0$,*

$$\{n \in \mathbb{N} : \mu(t_n(x) - L, t) \leq 1 - \delta \text{ or } v(t_n(x) - L, t) \geq \delta\} \in \mathcal{I}.$$

In this case we write $[V, \lambda]^\alpha (\mathcal{I})^{(\mu,v)} - \lim x = L$.

Definition 8. *Let $(X, \mu, v, *, \Diamond)$ be an IFNS. A sequence $x = (x_k)$ is said to be $\mathcal{I}_\lambda$-statistically convergent of order α or $S_\lambda^\alpha(\mathcal{I})$- convergent to $L \in X$ with respect to the intuitionistic fuzzy normed space (μ, v), if for every $\varepsilon > 0$, $\delta > 0$ and $t > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} |\{k \in I_n : \mu(x_k - L, t) \leq 1 - \varepsilon \text{ or } v(x_k - L, t) \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case we write $S_\lambda^\alpha(\mathcal{I})^{(\mu,v)} - \lim x = L$ or $x_k \to L(S_\lambda^\alpha(\mathcal{I})^{(\mu,v)})$.

Remark 3. *For $\mathcal{I} = \mathcal{I}_{fin}, S_\lambda^\alpha(\mathcal{I})$-convergence with respect to the intuitionistic fuzzy normed space (μ, v), again coincides with $\lambda-$ statistical convergence of order α, with respect to the intuitionistic fuzzy normed space (μ, v). For an arbitrary ideal $\mathcal{I}$ and for $\alpha = 1$, it coincides with $\mathcal{I}_\lambda$-statistical convergence with respect to the intuitionistic fuzzy normed space (μ, v), [28]. Finally for $\mathcal{I} = \mathcal{I}_{fin}$ and $\alpha = 1$, it becomes λ-statistical convergence with respect to the intuitionistic fuzzy normed space (μ, v), [16]. Also note that taking $\lambda_n = n$ we get definition 6 from definition 8.*

We shall denote by $S_\lambda^\alpha(\mathcal{I})^{(\mu,v)}$ and $[V, \lambda]^\alpha (\mathcal{I})^{(\mu,v)}$ the collections of all $S_\lambda(\mathcal{I})$-convergent of order α and $[V, \lambda](\mathcal{I})$ -convergent of order α sequences respectively.

We now have,

Theorem 1. *Let $(X, \mu, v, *, \Diamond)$ be an IFNS. Let $\lambda = (\lambda_n) \in \Delta$. Then $x_k \to L([V, \lambda]^\alpha (\mathcal{I})^{(\mu,v)}) \Rightarrow x_k \to L(S_\lambda^\alpha(\mathcal{I})^{(\mu,v)})$.*

Proof. By hypothesis, for every $\varepsilon > 0$, $\delta > 0$ and $t > 0$, let $x_k \to L [V, \lambda]^\alpha (I)^{(\mu,v)}$. We have

$$\begin{aligned} & \sum_{k \in I_n} (\mu(x_k - L, t) \text{ or } v(x_k - L, t)) \\ \geq & \sum_{\substack{k \in I_n \ \& \ \mu(x_k - L, t) < 1 - \varepsilon \\ \text{or } v(x_k - L, t) > \varepsilon}} (\mu(x_k - L, t) \text{ or } v(x_k - L, t)) \\ \geq & \ \varepsilon |\{k \in I_r : \mu(x_k - L, t) \leq 1 - \varepsilon \text{ or } v(x_k - L, t) \geq \varepsilon\}|. \end{aligned}$$

Then observe that

$$\frac{1}{\lambda_n^\alpha}\left|\left\{k\in I_n:\mu\left(x_k-L,t\right)\le 1-\varepsilon \text{ or } v\left(x_k-L,t\right)\ge\varepsilon\right\}\right|\ge\delta$$

$$\Rightarrow \frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}\mu\left(x_k-L,t\right)\le\left(1-\varepsilon\right)\delta \text{ or } \frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}v\left(x_k-L,t\right)\ge\varepsilon\delta$$

which implies

$$\begin{aligned} &\left\{n\in\mathbb{N}:\frac{1}{\lambda_n}\left|\left\{k\in I_n:\mu\left(x_k-L,t\right)\le 1-\varepsilon \text{ or } v\left(x_k-L,t\right)\ge\varepsilon\right\}\right|\ge\delta\right\} \\ \subset\quad &\left\{n\in\mathbb{N}:\frac{1}{\lambda_n^\alpha}\left\{\sum_{k\in I_n}\mu\left(x_k-L,t\right)\le 1-\varepsilon \text{ or } \sum_{k\in I_n}v\left(x_k-L,t\right)\ge\varepsilon\right\}\ge\varepsilon\delta\right\}. \end{aligned}$$

Since $x_k\to L\left(\left[V,\lambda\right]^\alpha\left(\mathcal{I}\right)^{(\mu,v)}\right)$, we immediately see that $x_k\to L\left(S_\lambda^\alpha\left(\mathcal{I}\right)^{(\mu,v)}\right)$, this completed the proof. □

Theorem 2. *Let $(X,\mu,v,*,\Diamond)$ be an IFNS. $S^\alpha\left(\mathcal{I}\right)^{(\mu,v)}\subset S_\lambda^\alpha\left(\mathcal{I}\right)^{(\mu,v)}$ if* $\lim\inf_{n\to\infty}\frac{\lambda_n^\alpha}{n^\alpha}>0.$

Proof. For given $\varepsilon>0$ and every $t>0$, we have

$$\begin{aligned} &\frac{1}{n^\alpha}\left|\left\{k\le n:\mu\left(x_k-L,t\right)\le 1-\varepsilon \text{ or } v\left(x_k-L,t\right)\ge\varepsilon\right\}\right| \\ \ge\quad &\frac{1}{n^\alpha}\left|\left\{k\in I_n:\mu\left(x_k-L,t\right)\le 1-\varepsilon \text{ or } v\left(x_k-L,t\right)\ge\varepsilon\right\}\right| \\ =\quad &\frac{\lambda_n^\alpha}{n^\alpha}\frac{1}{\lambda_n^\alpha}\left|\left\{k\in I_n:\mu\left(x_k-L,t\right)\le 1-\varepsilon \text{ or } v\left(x_k-L,t\right)\ge\varepsilon\right\}\right|. \end{aligned}$$

If $\lim\inf_{n\to\infty}\frac{\lambda_n^\alpha}{n^\alpha}=\alpha$ then from definition $\left\{n\in\mathbb{N}:\frac{\lambda_n^\alpha}{n^\alpha}<\frac{\alpha}{2}\right\}$ is finite. For every $\delta>0$,

$$\begin{aligned} &\left\{n\in\mathbb{N}:\frac{1}{\lambda_n^\alpha}\left|\left\{k\in I_n:\mu\left(x_k-L,t\right)\le 1-\varepsilon \text{ or } v\left(x_k-L,t\right)\ge\varepsilon\right\}\right|\ge\delta\right\} \\ \subset\quad &\left\{n\in\mathbb{N}:\frac{1}{n^\alpha}\left|\left\{k\in I_n:\mu\left(x_k-L,t\right)\le 1-\varepsilon \text{ or } v\left(x_k-L,t\right)\ge\varepsilon\right\}\right|\ge\frac{\alpha}{2}\delta\right\} \\ &\cup\left\{n\in\mathbb{N}:\frac{\lambda_{n^\alpha}^\alpha}{n^\alpha}<\frac{\alpha}{2}\right\}. \end{aligned}$$

Since $\mathcal{I}$ is admissible, the set on the right-hand side belongs to $\mathcal{I}$ and this completed the proof of the theorem. □

Theorem 3. *Let $(X,\mu,v,*,\Diamond)$ be an IFNS. If $\lambda\in\triangle$ be such that for a particular $\alpha, 0<\alpha<1$, $\lim_n\frac{n-\lambda_n}{n^\alpha}=0$, then $S_\lambda^\alpha\left(\mathcal{I}\right)^{(\mu,v)}\subset S^\alpha\left(\mathcal{I}\right)^{(\mu,v)}$.*

Proof. Let $\delta>0$ be given. Since $\lim_n\frac{n-\lambda_n}{n^\alpha}=0$, we can choose $m\in\mathbb{N}$ such that $\mu\left(\frac{n-\lambda_n}{n^\alpha},t\right)>$

$1-\frac{\delta}{2}$ or $v\left(\frac{n-\lambda_n}{n^\alpha},t\right)<\frac{\delta}{2}$, for all $n\geq m$. Now observe that, for $\varepsilon>0$, every $t>0$ and $n\geq m$

$$\begin{aligned}
& \frac{1}{n^\alpha}\left|\left\{k\leq n:\mu\left(x_k-L,t\right)\leq 1-\varepsilon \text{ or } v\left(x_k-L,t\right)\geq\varepsilon\right\}\right| \\
= & \frac{1}{n^\alpha}\left|\left\{k\leq n-\lambda_n:\mu\left(x_k-L,t\right)\leq 1-\varepsilon \text{ or } v\left(x_k-L,t\right)\geq\varepsilon\right\}\right| \\
& +\frac{1}{n^\alpha}\left|\left\{k\in I_n:\mu\left(x_k-L,t\right)\leq 1-\varepsilon \text{ or } v\left(x_k-L,t\right)\geq\varepsilon\right\}\right| \\
\leq & \frac{n-\lambda_n}{n^\alpha}+\frac{1}{n^\alpha}\left|\left\{k\in I_n:\mu\left(x_k-L,t\right)\leq 1-\varepsilon \text{ or } v\left(x_k-L,t\right)\geq\varepsilon\right\}\right| \\
\leq & \frac{\delta}{2}+\frac{1}{\lambda_n^\alpha}\left|\left\{k\in I_n:\mu\left(x_k-L,t\right)\leq 1-\varepsilon \text{ or } v\left(x_k-L,t\right)\geq\varepsilon\right\}\right|.
\end{aligned}$$

for all $n\geq m$. Hence

$$\begin{aligned}
& \left\{n\in\mathbb{N}:\frac{1}{n^\alpha}\left|\left\{k\leq n:\mu\left(x_k-L,t\right)\leq 1-\varepsilon \text{ or } v\left(x_k-L,t\right)\geq\varepsilon\right\}\right|\geq\delta\right\} \\
\subset & \left\{n\in\mathbb{N}:\frac{1}{\lambda_n^\alpha}\left|\left\{k\in I_n:\mu\left(x_k-L,t\right)\leq 1-\varepsilon \text{ or } v\left(x_k-L,t\right)\geq\varepsilon\right\}\right|\geq\frac{\delta}{2}\right\} \\
& \cup\left\{1,2,3,...,m\right\}.
\end{aligned}$$

If $S_\lambda^\alpha\left(\mathcal{I}\right)^{(\mu,v)}$-$\lim x=L$ then the set on the right-hand side belongs to $\mathcal{I}$ and so the set on the left-hand side also belongs to $\mathcal{I}$. This shows that $x=(x_k)$ is $\mathcal{I}$-statistically convergent of order to L with respect to the intuitionistic fuzzy normed space (μ,v). □

Remark 4. *We do not know whether the conditions in the above theorem are necessary and leave them as open problems.*

It can be checked as in the case of statistically and λ- statistically convergent sequences that both $S^\alpha\left(\mathcal{I}\right)^{(\mu,v)}$ and $S_\lambda^\alpha\left(\mathcal{I}\right)^{(\mu,v)}$ are linear subspaces of the space of real sequences. As the proofs for both the assertions are similar, we present the proof for $S_\lambda^\alpha\left(\mathcal{I}\right)^{(\mu,v)}$ only.

Theorem 4. *Let $(X,\mu,v,*,\lozenge)$ be an IFNS such that $\frac{\varepsilon_n}{4}\lozenge\frac{\varepsilon_n}{4}<\frac{\varepsilon_n}{2}$ and $\left(1-\frac{\varepsilon_n}{4}\right)*\left(1-\frac{\varepsilon_n}{4}\right)>1-\frac{\varepsilon_n}{2}$. If X is a Banach space then $S_\lambda^\alpha\left(\mathcal{I}\right)^{(\mu,v)}\cap\ell_\infty^{(\mu,v)}$ is a closed subset of $\ell_\infty^{(\mu,v)}$,where $\ell_\infty^{(\mu,v)}$ stands for the space of all bounded sequences of intuitionistic fuzzy norm (μ,ν).*

Proof. We first assume that $(x^n)\subset S_\lambda^\alpha\left(\mathcal{I}\right)^{(\mu,v)}\cap\ell_\infty^{(\mu,v)}$, $0<\alpha\leq 1$, is a convergent sequence and it converges to $x\in\ell_\infty^{(\mu,v)}$. We need to show that $x\in S_\lambda^\alpha\left(\mathcal{I}\right)^{(\mu,v)}\cap\ell_\infty^{(\mu,v)}$. Suppose that $x^n\to L_n\left(S_\lambda^\alpha\left(\mathcal{I}\right)^{(\mu,v)}\right)$ for all $n\in\mathbb{N}$. Take a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of strictly decreasing positive numbers converging to zero. We can find an $n\in\mathbb{N}$ such that $\sup_j v\left(x-x^j,t\right)<\frac{\varepsilon_n}{4}$ for all $j\geq n$. Choose $0<\delta<\frac{1}{5}$.

Now let

$$A_{\mu,v}\left(\varepsilon_n,t\right)=\left\{m\in\mathbb{N}:\frac{1}{\lambda_m^\alpha}\left|\left\{k\in I_m:\mu\left(x_k^n-L_n,t\right)\leq 1-\frac{\varepsilon_n}{4} \text{ or } v\left(x_k^n-L_n,t\right)\geq\frac{\varepsilon_n}{4}\right\}\right|<\delta\right\}$$

belongs to $F\left(\mathcal{I}\right)$ and

$$B_{\mu,v}\left(\varepsilon_n,t\right)=\left\{m\in\mathbb{N}:\frac{1}{\lambda_m^\alpha}\left|\left\{\begin{array}{c} k\in I_m:\mu\left(x_k^{n+1}-L_{n+1},t\right)\leq 1-\frac{\varepsilon_n}{4} \text{ or} \\ v\left(x_k^{n+1}-L_{n+1},t\right)\geq\frac{\varepsilon_n}{4}\end{array}\right\}\right|<\delta\right\}$$

belongs to $F(\mathcal{I})$. Since $A_{\mu,v}(\varepsilon_n,t)\cap B_{\mu,v}(\varepsilon_n,t)\in F(\mathcal{I})$ and $\emptyset\notin F(\mathcal{I})$, we can choose $m\in A_{\mu,v}(\varepsilon_n,t)\cap B_{\mu,v}(\varepsilon_n,t)$. Then

$$\frac{1}{\lambda_m^\alpha}\left|\left\{\begin{array}{c} k\in I_m:\mu(x_k^n-L_n,t)\leq 1-\frac{\varepsilon_n}{4}\text{ or } v(x_k^n-L_n,t)\geq\frac{\varepsilon_n}{4}\vee \\ \mu\left(x_k^{n+1}-L_{n+1},t\right)\leq 1-\frac{\varepsilon_n}{4}\text{ or } v\left(x_k^{n+1}-L_{n+1},t\right)\geq\frac{\varepsilon_n}{4}\end{array}\right\}\right|\leq 2\delta<1.$$

Since $\lambda_m\to\infty$ and $A_{\mu,v}(\varepsilon_n,t)\cap B_{\mu,v}(\varepsilon_n,t)\in F(\mathcal{I})$ is infinite, we can actually choose the above m so that $\lambda_m>5$. Hence there must exist a $k\in I_m$ for which we have simultaneously, $\mu(x_k^n-L_n,t)>1-\frac{\varepsilon_n}{4}$ or $v(x_k^n-L_n,t)<\frac{\varepsilon_n}{4}$ and $\mu\left(x_k^{n+1}-L_{n+1},t\right)>1-\frac{\varepsilon_n}{4}$ or $v\left(x_k^{n+1}-L_{n+1},t\right)<\frac{\varepsilon_n}{4}$. For a given $\varepsilon_n>0$ choose $\frac{\varepsilon_n}{2}$ such that $\left(1-\frac{\varepsilon_n}{2}\right)*\left(1-\frac{\varepsilon_n}{2}\right)>1-\varepsilon_n$ and $\frac{\varepsilon_n}{2}\Diamond\frac{\varepsilon_n}{2}<\varepsilon_n$. Then it follows that

$$v\left(L_n-x_k^n,\frac{t}{2}\right)\Diamond v\left(L_{n+1}-x_k^{n+1},\frac{t}{2}\right)\leq\frac{\varepsilon_n}{4}\Diamond\frac{\varepsilon_n}{4}<\frac{\varepsilon_n}{2}$$

and

$$\begin{aligned} v\left(x_k^n-x_k^{n+1},t\right) &\leq \sup_n v\left(x-x^n,\frac{t}{2}\right)\Diamond\sup_n v\left(x-x^{n+1},\frac{t}{2}\right)\\ &\leq \frac{\varepsilon_n}{4}\Diamond\frac{\varepsilon_n}{4}<\frac{\varepsilon_n}{2}.\end{aligned}$$

Hence we have

$$\begin{aligned} v(L_n-L_{n+1},t) &\leq \left[v\left(L_n-x_k^n,\frac{t}{3}\right)\Diamond v\left(x_k^{n+1}-L_{n+1},\frac{t}{3}\right)\right]\Diamond v\left(x_k^n-x_k^{n+1},\frac{t}{3}\right)\\ &\leq \frac{\varepsilon_n}{2}\Diamond\frac{\varepsilon_n}{2}<\varepsilon_n\end{aligned}$$

and similarly $\mu(L_n-L_{n+1},t)>1-\varepsilon_n$. This implies that $\{L_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X and let $L_n\to L\in X$ as $n\to\infty$. We shall prove that $x\to L\left(S_\lambda^\alpha(\mathcal{I})^{(\mu,v)}\right)$. For any $\varepsilon>0$ and $t>0$, choose $n\in\mathbb{N}$ such that $\varepsilon_n<\frac{\varepsilon}{4}$, $\sup_n v(x-x^n,t)<\frac{\varepsilon}{4}$, $\mu(L_n-L,t)>1-\frac{\varepsilon}{4}$ or $v(L_n-L,t)<\frac{\varepsilon}{4}$. Now since

$$\begin{aligned} &\frac{1}{\lambda_n^\alpha}\left|\{k\in I_n:v(x_k-L,t)\geq\varepsilon\}\right|\\ \leq\;&\frac{1}{\lambda_n^\alpha}\left|\left\{k\in I_n:v\left(x_k-x_k^n,\frac{t}{3}\right)\Diamond\left[v\left(x_k^n-L_n,\frac{t}{3}\right)\Diamond v\left(L_n-L,\frac{t}{3}\right)\right]\geq\varepsilon\right\}\right|\\ \leq\;&\frac{1}{\lambda_n^\alpha}\left|\left\{k\in I_n:v\left(x_k^n-L_n,\frac{t}{3}\right)\geq\frac{\varepsilon}{2}\right\}\right|\end{aligned}$$

and similarly

$$\frac{1}{\lambda_n^\alpha}\left|\{k\in I_n:\mu(x_k-L,t)\leq 1-\varepsilon\}\right|>\frac{1}{\lambda_n}\left|\left\{k\in I_n:\mu\left(x_k^n-L,\frac{t}{3}\right)\leq 1-\frac{\varepsilon}{2}\right\}\right|,$$

it follows that

$$\begin{aligned} &\left\{n\in\mathbb{N}:\frac{1}{\lambda_n^\alpha}\left|\{k\in I_n:\mu(x_k-L,t)\leq 1-\varepsilon\text{ or } v(x_k-L,t)\geq\varepsilon\}\right|\geq\delta\right\}\\ \subset\;&\left\{n\in\mathbb{N}:\frac{1}{\lambda_n^\alpha}\left|\left\{k\in I_n:\mu\left(x_k^n-L,\frac{t}{3}\right)\leq 1-\frac{\varepsilon}{2}\text{ or } v\left(x_k^n-L,\frac{t}{3}\right)\geq\frac{\varepsilon}{2}\right\}\right|\geq\delta\right\}\end{aligned}$$

for any given $\delta>0$. Hence we have $x\to L\left(S_\lambda^{(\mu,v)}(\mathcal{I})\right)$. This completes the proof of the theorem. □

3 Conclusion

In this paper the notions of $\mathcal{I}$-statistical convergence and $\mathcal{I}_\lambda$-statistical convergence of order α are introduced in an intuitionistic fuzzy normed linear space and some important results are established. We use ideals to introduce the concept of $\mathcal{I}$-statistical convergence and $\mathcal{I}_\lambda$-statistical convergence of order α with respect to the intuitionistic fuzzy norm (μ, v), which naturally extend the notions of statistical convergence and λ -statistical convergence. Our study of $\mathcal{I}$-statistical and $\mathcal{I}_\lambda$-statistical convergence of order α in intuitionistic fuzzy normed spaces also provides a tool to deal with convergence problems of sequences of fuzzy real numbers.

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