Notes on Intuitionistic Fuzzy Sets
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On intuitionistic fuzzy almost prime ideals and intuitionistic fuzzy almost prime submodules

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Abstract: The aim of this paper is to present some characterizations of almost prime ideals and almost prime submodules in the intuitionistic fuzzy environment. We investigate various properties of these concepts and achieve many results.
Keywords: Almost prime ideal, Almost prime submodule, Intuitionistic fuzzy (almost) prime ideal, Intuitionistic fuzzy (almost) prime submodule, Intuitionistic fuzzy multiplication module.
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1 Introduction

The concept of almost prime ideal, which is a generalisation of prime ideal, was introduced by Bhatwadekar and Sharma in [10], which was also studied in detail by Bataineh in [8]. Presently, studies on the different generalisations of prime ideals, namely weakly prime ideals, almost prime ideals, n-almost prime ideals, and ω-prime ideals are progressing rapidly. Anderson and Bataineh [1] unify these generalisations into ϕ_α-prime ideals and derive many results.

A proper ideal P of a commutative ring R is called an almost prime, if $ab \in P - P^2$ implies $a \in P$ or $b \in P$. A proper ideal P of a ring R is called a weakly prime, if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. A proper ideal P of R is said to be almost prime, if for any ideals A and B of R

such that $AB \subseteq P$ and $AB \nsubseteq P^2$, we have $A \subseteq P$ or $B \subseteq P$. As every prime ideal is a weakly prime, and a weakly prime ideal is almost prime.

One of the natural generalisations of prime ideals that has attracted the interest of several authors is the notion of prime submodules (see, for example, [13], [20]). These have led to more information on the structure of the R-module M. A proper submodule N of M is called prime, if for $r \in R$ and $x \in M$, $rx \in N$ implies that $r \in (N : M)$ or $x \in N$, where $(N : M) = \{r \in R : rM \subseteq N\}$ is called fraction of submodule N by its module M, which is clearly an ideal of R. Also, if N is a prime submodule of M, then $(N : M)$ is a prime ideal of R. Among various generalisations of the prime submodules, two generalisations, namely weakly prime submodules and almost prime submodules, are very progressing rapidly. A proper submodule N of an R-module M is called a weakly prime submodule, if for each submodule K of M and elements a, b of R, $abK \subseteq N$, it implies that $aK \subseteq N$ or $bK \subseteq N$ (see [5], [9]). Also, a proper submodule N of an R-module M is called almost prime, if for $r \in R$ and $x \in M$ such that $rx \in N$-$(N : M)N$, then either $x \in N$ or $r \in (N : M)$ (see [19]). Zamani [28] unifies different generalisations of prime submodules to ϕ-prime submodules and derives many results.

In his classical paper, published in 1965, Zadeh [27] coined the idea of a fuzzy set by generalizing the concept of a classical set by replacing the binary membership of an element in a set by a gradation of membership ranging between 0 and 1. Since then, plenty of research work has been carried out in the field of fuzzy set theory and its applications. By adding one more element of gradation of non-membership along with the gradation of membership with the condition that their sum is always less than or equal to one, was introduced by Atanassov in [2–4]. This is one of the most progressing and widely used generalization among other generalizations to the theory of fuzzy sets. Therefore, an intuitionistic fuzzy subset A of a nonempty set X is as an ordered function $(\mu_A, \nu_A) : X \to [0, 1] \times [0, 1]$. Research on the theory of intuitionistic fuzzy sets has been witnessing exponential growth both within mathematics and in its applications. This ranges from traditional mathematical logic, topology, algebra, analysis, etc. to pattern recognition, information theory, artificial intelligence, neural networks and planning (see [4], [15] and [16]). Consequently, intuitionistic fuzzy set theory has emerged. As a potential area of interdisciplinary research, intuitionistic fuzzy module theory is also of recent interest. In [11, 12], Biswas considered the intuitionistic fuzzification of algebraic structures and introduced the notion of intuitionistic fuzzy subgroup of a group. Hur et al. [17] introduced and examined the notion of an intuitionistic fuzzy ideal of a ring. Davvaz et al. in [14] introduced the notion of intuitionistic fuzzy submodules of a module. In the last few years, a considerable amount of work has been done on intuitionistic fuzzy ideals and intuitionistic fuzzy modules (see [6, 7, 14, 18, 21–26]).

The purpose of the present paper is to study the structural characteristics of the concept of intuitionistic fuzzy almost prime ideals and intuitionistic fuzzy almost prime submodules. In Section 3, we define intuitionistic fuzzy almost prime ideals as a new generalisation of intuitionistic fuzzy prime ideals in a commutative ring with unity. We gave a non-trivial example of an intuitionistic fuzzy almost prime ideal that is not an intuitionistic fuzzy prime ideal. Apart from these, we have investigated various properties of intuitionistic fuzzy almost prime ideals and obtained many results. In section 4, we define intuitionistic fuzzy almost prime submodules as a new generalisation of intuitionistic fuzzy prime submodules of unitary modules over a commutative ring with identity. We studied some basic properties of intuitionistic fuzzy almost prime submodules and gave some characterizations of them, especially for finitely generated faithful multiplication modules.

2 Preliminaries

Throughout this paper R is a commutative ring with identity.

Definition 2.1. ([2–4]) An intuitionistic fuzzy set (IFS) A in X can be represented as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where the functions $\mu_A, \nu_A : X \to [0,1]$ denote the degree of membership (namely, $\mu_A(x)$) and the degree of non-membership (namely, $\nu_A(x)$) of each element $x \in X$ to A respectively and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

Remark 2.2. ([3,4]) (i) When $\mu_A(x) + \nu_A(x) = 1$, for all $x \in X$, then A is called a fuzzy set.

(ii) An IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ is briefly written as $A(x) = (\mu_A(x), \nu_A(x))$, for all $x \in X$. We denote by $IFS(X)$ the set of all IFSs of X.

(iii) If $p, q \in [0,1]$ such that $p+q \leq 1$. Then $A \in IFS(X)$ defined by $\mu_A(x) = p$ and $\nu_A(x) = q$, for all $x \in X$, is called a constant intuitionistic fuzzy set of X. Any IFS of X defined other than this is referred to as a non-constant intuitionistic fuzzy set.

If $A, B \in IFS(X)$, then $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, for all $x \in X$ and $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$. For any subset Y of X, the intuitionistic fuzzy characteristic function χ_Y is an intuitionistic fuzzy set of X, defined as $\chi_Y(x) = (1,0)$, for all $x \in Y$ and $\chi_Y(x) = (0,1)$, for all $x \in X\backslash Y$. Let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. Then the crisp set $A_{(\alpha,\beta)} = \{x \in X : \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta\}$ is called the (α,β)-level cut subset of A. Also the IFS $x_{(\alpha,\beta)}$ of X defined as $x_{(\alpha,\beta)}(y) = (\alpha,\beta)$, if $y = x$, otherwise $(0,1)$ is called the intuitionistic fuzzy point (IFP) in X with support x. By $x_{(\alpha,\beta)} \in A$ we mean $\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta$. Further, if $f : X \to Y$ is a mapping and A, B be respectively IFS of X and Y. Then the image $f(A)$ is an IFS of Y is defined as $\mu_{f(A)}(y) = \sup\{\mu_A(x) : f(x) = y\}$, $\nu_{f(A)}(y) = \inf\{\nu_A(x) : f(x) = y\}$, for all $y \in Y$ and the inverse image $f^{-1}(B)$ is an IFS of X is defined as $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$, $\nu_{f^{-1}(B)}(x) = \nu_B(f(x))$, for all $x \in X$, i.e., $f^{-1}(B)(x) = B(f(x))$, for all $x \in X$.

Definition 2.3. ([6,7,17]) Let $A \in IFS(R)$. Then A is called an intuitionistic fuzzy ideal (IFI) of ring R, if for all $x, y \in R$, the followings are satisfied

(i) $\mu_A(x-y) \geq \mu_A(x) \wedge \mu_A(y)$;

(ii) $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$;

(iii) $\nu_A(x-y) \leq \nu_A(x) \vee \nu_A(y)$;

(iv)$\nu_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$.

Note that $\mu_A(0_R) \geq \mu_A(x) \geq \mu_A(1_R), \mu_A(0_R) \leq \mu_A(x) \leq \nu_A(1_R)$, for all $x \in R$. The set of all intuitionistic fuzzy ideals of R is denoted by $IFI(R)$.

Definition 2.4. ([6, 7, 17]) Let $A, B \in IFI(R)$. Then the intuitionistic fuzzy product AB of A and B are defined as: For all $x \in R$

$$(\mu_{AB}(x), \nu_{AB}(x)) = \begin{cases} (\sup_{x=yz}(\mu_A(y) \wedge \mu_B(z)), \inf_{x=yz}(\nu_A(y) \vee \nu_B(z)), & \text{if } x = yz \\ (0, 1), & \text{otherwise,} \end{cases}$$

where as usual supremum and infimum of an empty set are taken to be 0 and 1 respectively.

Remark 2.5. ([17]) Let R be a commutative ring. Then for any $x_{(p,q)}, y_{(t,s)} \in IFP(R)$

(i) $x_{(p,q)} + y_{(t,s)} = (x + y)_{(p \wedge t, q \vee s)}$;

(ii) $x_{(p,q)} y_{(t,s)} = (xy)_{(p \wedge t, q \vee s)}$.

Theorem 2.6. ([7]) *Let $A \in IFS(R)$. Then A is an intuitionistic fuzzy ideal if and only if $A_{(\alpha,\beta)}$ is an ideal of R, for all $\alpha \leq \mu_A(0), \beta \geq \nu_A(0)$ with $\alpha + \beta \leq 1$. In particular, if A is an IFI of R, then $A_* = \{x \in R : \mu_A(x) = \mu_A(0), \nu_A(x) = \nu_A(0)\}$ is always an ideal of R.*

Definition 2.7. ([6, 25]) Let P be a non-constant IFI of a ring R. Then P is said to be an intuitionistic fuzzy prime ideal (IFPI) of R, if for any two IFIs A, B of R such that $AB \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$.

Theorem 2.8. ([6]) *Let P be an IFI of a ring R. Then for any $x_{(p,q)}, y_{(t,s)} \in IFP(R)$ the following are equivalent:*

(i) P is an intuitionistic fuzzy prime ideal of R

(ii) $x_{(p,q)} y_{(t,s)} \subseteq P$ implies $x_{(p,q)} \subseteq P$ or $y_{(t,s)} \subseteq P$.

Theorem 2.9. ([17]) *If P is an intuitionistic fuzzy prime ideal of a ring R, then the following conditions hold:*

(i) $P(0_R) = (1, 0)$,

(ii) P_* is a prime ideal of R,

(iii) $Img(P) = \{(1, 0), (t, s)\}$, where $t, s \in [0, 1)$ such that $t + s \leq 1$.

Proposition 2.10. *Let A be an IFI of ring R. Then*

(1)A is a non-constant IFPI if for every IFIs B, C of R whenever $BC \subseteq A$ implies either $B \subseteq A$ or $C \subseteq A$.

(2) If A is an IFPI, then for all $x, y \in R$, either $\mu_A(xy) = \mu_A(x)$ and $\nu_A(xy) = \nu_A(x)$ or $\mu_A(xy) = \mu_A(y)$ and $\nu_A(xy) = \nu_A(y)$ [In other words $\mu_A(xy) = \mu_A(x) \vee \mu_A(y)$ and $\nu_A(xy) = \nu_A(x) \wedge \nu_A(y)$]

(3) If A is an IFPI of R, then for all $x, y \in R$, $\mu_A(xy) = \mu_A(0_R)$ and $\nu_A(xy) = \nu_A(0_R)$ implies either $\mu_A(x) = \mu_A(0_R)$ and $\nu_A(x) = \nu_A(0_R)$ or $\mu_A(y) = \mu_A(0_R)$ and $\nu_A(y) = \nu_A(0_R)$.

Proof. (1) Let B, C be two IFIs of R such that $BC \subseteq A$. Suppose that $B \nsubseteq A$ and $C \nsubseteq A$, then there exists $x, y \in R$ such that $\mu_B(x) > \mu_A(x)$ or $\nu_B(x) < \nu_A(x)$ and $\mu_C(y) > \mu_A(y)$ or $\nu_C(y) < \nu_A(y)$. Hence $x_{(\mu_B(x),\nu_B(x))} \notin A$ and $y_{(\mu_C(y),\nu_C(y))} \notin A$. But

$$\begin{aligned} \mu_A(xy) &\geq \mu_{BC}(xy) \\ &\geq \mu_B(x) \wedge \mu_C(y) \\ &= \mu_{(xy)_{\mu_B(x)\wedge\mu_C(y)}}(xy) \\ &= \mu_{x_{(\mu_B(x),\nu_B(x))}y_{(\mu_C(y),\nu_C(y))}}(xy) \end{aligned}$$

Similarly, we can show that $\nu_A(xy) \leq \nu_{x_{(\mu_B(x),\nu_B(x))}y_{(\mu_C(y),\nu_C(y))}}(xy)$, i.e., $x_{(\mu_B(x),\nu_B(x))}y_{(\mu_C(y),\nu_C(y))} \in A$ so either $x_{(\mu_B(x),\nu_B(x))} \in A$ or $y_{(\mu_C(y),\nu_C(y))} \in A$, which is a contradiction. Hence either $B \subseteq A$ or $C \subseteq A$.

(2) Suppose A is an IFPI of R. Then by Theorem (2.9), A is of the form

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in I \\ t, & \text{if } x \notin I \end{cases}, \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in I \\ s, & \text{if } x \notin I \end{cases},$$

where I is prime ideal of R and $t, s \in (0, 1)$ such that $t + s \leq 1$. Let $x, y \in R$. If $xy \in I$. Then as I is prime ideal, therefore, either $x \in I$ or $y \in I$.
This further implies, either $\mu_A(x) = 1$ and $\nu_A(x) = 0$ or $\mu_A(y) = 1$ and $\nu_A(y) = 0$. Thus, either $\mu_A(xy) = \mu_A(x)$ and $\nu_A(xy) = \nu_A(x)$ or $\mu_A(xy) = \mu_A(y)$ and $\nu_A(xy) = \nu_A(y)$.

If $xy \notin I$, then $\mu_A(xy) = t$ and $\nu_A(xy) = s$. But $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \wedge \mu_A(y)$.

Now, if either $\mu_A(x) = 1$ or $\mu_A(y) = 1$, then $t \geq 1$ which is not possible. Alternatively, if either $\nu_A(x) = 0$ or $\nu_A(y) = 0$, then $s \leq 0$ which is not possible. Therefore, at least one of the values of $\mu_A(x)$ or $\mu_A(y)$ must be t. Similarly, at least one of the values of $\nu_A(x)$ or $\nu_A(y)$ must be s. This implies that either $\mu_A(xy) = \mu_A(x)$ and $\nu_A(xy) = \nu_A(x)$, or $\mu_A(xy) = \mu_A(y)$ and $\nu_A(xy) = \nu_A(y)$.

(3) Let A be an IFPI of R. Suppose that $\mu_A(xy) = \mu_A(0_R)$ and $\nu_A(xy) = \nu_A(0_R)$ for some $x, y \in R$. Then we have $\mu_A(x) \vee \mu_A(y) = \mu_A(0_R)$ and $\nu_A(x) \wedge \nu_A(y) = \nu_A(0_R)$. So, we infer that either $\mu_A(x) = \mu_A(0_R)$ and $\nu_A(x) = \nu_A(0_R)$, or $\mu_A(y) = \mu_A(0_R)$ and $\nu_A(y) = \nu_A(0_R)$. □

3 Intuitionistic fuzzy almost prime ideal

In this section, some definitions and examples for the intuitionistic fuzzy almost prime ideals are constructed. Also, it has been shown that intuitionistic fuzzy almost prime ideals are generalisations of intuitionistic fuzzy prime ideals. Moreover, its relationships between intuitionistic fuzzy prime ideals have been established, and the rings were classified in such a way that every proper intuitionistic fuzzy almost prime ideal is an intuitionistic fuzzy prime ideal.

Definition 3.1. An intuitionistic fuzzy ideal P of a ring R is called an intuitionistic fuzzy almost prime ideal (IFAPI) if for any $x, y \in R$ with $\mu_P(xy) > \mu_P(1_R)$ and $\nu_P(xy) < \nu_P(1_R)$ whenever $\mu_{P^2}(xy) = \mu_P(1_R)$ and $\nu_{P^2}(xy) = \nu_P(1_R) \Rightarrow \mu_P(xy) = \mu_P(x) \vee \mu_P(y)$ and $\nu_P(xy) = \nu_P(x) \wedge \nu_P(y)$.

Example 3.2. Let $R = \mathbb{Z}$ be the ring of integers. Define an IFS A on $\mathbb{Z}$ as follows:

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in \langle 6 \rangle \\ 0, & \text{otherwise} \end{cases} ; \quad \nu_P(x) = \begin{cases} 0, & \text{if } x \in \langle 6 \rangle \\ 1, & \text{otherwise} . \end{cases}$$

It is easy to verify that A is an IFI of R. However, A is not an IFAPI, for $\mu_A(2.3) = \mu_A(6) = 1 > 0 = \mu_A(1)$ and $\nu_A(2.3) = \nu_A(6) = 0 < 1 = \nu_A(1)$, also, $\mu_{A^2}(6) = \mu_A(2) \wedge \mu_A(3) = 0 \wedge 0 = 0 = \mu_A(1)$ and $\nu_{A^2}(6) = \nu_A(2) \vee \nu_A(3) = 1 \vee 1 = 1 = \mu_A(1)$. But $\mu_A(2.3) = 1 \neq 0 = \mu_A(2) \vee \mu_A(3)$ and $\nu_A(2.3) = 0 \neq 1 = \nu_A(2) \wedge \nu_A(3)$.

Proposition 3.3. *Every intuitionistic fuzzy prime ideal in a ring R is an intuitionistic fuzzy almost prime ideal.*

Proof. Let P be a non-constant IFPI in R and $x, y \in R$ such that $\mu_P(xy) > \mu_P(1_R)$ and $\nu_P(xy) < \nu_P(1_R)$. Also, $\mu_{P^2}(xy) = \mu_P(1_R)$ and $\nu_{P^2}(xy) = \nu_P(1_R)$. Since P is an IFPI therefore, either $\mu_P(xy) = \mu_P(x)$ and $\nu_P(xy) = \nu_P(x)$ or $\mu_P(xy) = \mu_P(y)$ and $\nu_P(xy) = \nu_P(y)$. Without loss of generality let $\mu_P(x) = \mu_P(x) \wedge \mu_P(y)$ and $\nu_P(x) = \nu_P(x) \vee \nu_P(y)$. But then $\mu_{P^2}(xy) \geq \mu_P(x) \wedge \mu_P(y) = \mu_P(x)$ and $\nu_{P^2}(xy) \leq \nu_P(x) \vee \nu_P(y) = \nu_P(x)$ implies that $\mu_P(1_R) \geq \mu_P(x)$ but $\mu_P(1_R) \leq \mu_P(x)$ always implies that $\mu_P(1_R) = \mu_P(x)$. Similarly we get $\nu_P(1_R) = \nu_P(x)$. This implies that P is a constant IFPI of R, a contradiction. So, $\mu_P(xy) = \mu_P(y) = \mu_P(x) \vee \mu_P(y)$ and $\nu_P(xy) = \nu_P(y) = \nu_P(x) \wedge \nu_P(y)$. Hence P is an IFAPI in R. □

Remark 3.4. The converse of the above proposition need not be true, as seen in the following:

Example 3.5. Let $R = (\mathbb{Z}, +, .)$ be the ring of integers and $I = \langle 3 \rangle$ be the prime ideal of $\mathbb{Z}$. Define the IFS P on $\mathbb{Z}$ as follows:

$$\mu_P(x) = \begin{cases} 1, & \text{if } x \in I^2 \\ \alpha, & \text{if } x \in I - I^2 \\ 0, & \text{otherwise} \end{cases}, \quad \nu_P(x) = \begin{cases} 0, & \text{if } x \in I^2 \\ \beta, & \text{if } x \in I - I^2 \\ 1, & \text{otherwise} \end{cases},$$

where $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta \leq 1$. First we show that P is an IFI of $\mathbb{Z}$.
Consider $x, y \in \mathbb{Z}$, then we have the following cases.

Case (1) When $x \in I$ but $y \notin I$, then $x - y \in I$ but $xy \notin I$. Therefore we have $\mu_P(x - y) = 0$ and $\nu_P(x - y) = 1$ also $\mu_P(y) = 0$ and $\nu_P(y) = 1$ implies that

$$\mu_P(x - y) = \mu_P(y) = \mu_P(x) \wedge \mu_P(y) \text{ and } \nu_P(x - y) = \nu_P(y) = \nu_P(x) \vee \nu_P(y).$$

Thus we have

$$\mu_P(x) = \begin{cases} 1, & \text{if } x \in I^2 \\ \alpha, & \text{if } x \in I - I^2 \end{cases}; \quad \nu_P(x) = \begin{cases} 0, & \text{if } x \in I^2 \\ \beta, & \text{if } x \in I - I^2 \end{cases}.$$

If $\mu_P(xy) = 1$ and $\nu_P(xy) = 0$, then $\mu_P(xy) \geq \mu_P(x) \vee \mu_P(y)$ and $\nu_P(xy) \leq \nu_P(x) \wedge \nu_P(y)$.
If $\mu_P(xy) = \alpha$ and $\nu_P(xy) = \beta$, then $xy \in I - I^2$. So, $\mu_P(x) \vee \mu_P(y) = \alpha$ and $\nu_P(x) \wedge \nu_P(y) = \beta$.
Thus $\mu_P(xy) \geq \mu_P(x) \vee \mu_P(y)$ and $\nu_P(xy) \leq \nu_P(x) \wedge \nu_P(y)$.

Case (2) When $x \in I$ and $y \in I - I^2$, then $x - y \in I$ and $xy \in I^2$. Therefore we have $\mu_P(y) = \alpha$ and $\nu_P(y) = \beta$ implies $\mu_P(x) \wedge \mu_P(y) = \alpha$ and $\nu_P(x) \vee \nu_P(y) = \beta$. Therefore, $\mu_P(x - y) = \alpha = \mu_P(y) = \mu_P(x) \wedge \mu_P(y)$ and $\nu_P(x - y) = \beta = \nu_P(y) = \nu_P(x) \vee \nu_P(y)$.
Also, $\mu_P(xy) = 1 \geq \mu_P(x) \vee \mu_P(y)$ and $\nu_P(xy) = 0 \leq \nu_P(x) \wedge \nu_P(y)$.

Case (3) When $x \notin I$ and $y \notin I$, then $xy \notin I$ [Since I is prime ideal]. Therefore we have $\mu_P(x) = 0 = \mu_P(y)$ and $\nu_P(x) = 1 = \nu_P(y)$. So, $\mu_P(x) \wedge \mu_P(y) = 0$ and $\nu_P(x) \vee \nu_P(y) = 1$.
Also, $\mu_P(x - y) \geq 0$ and $\nu_P(x - y) \leq 1$ always implies that $\mu_P(x - y) \geq \mu_P(x) \wedge \mu_P(y)$ and $\nu_P(x - y) \leq \nu_P(x) \vee \nu_P(y)$.
Also, $\mu_P(xy) = 0$ and $\nu_P(xy) = 1 \Rightarrow \mu_P(xy) = \mu_P(x) \vee \mu_P(y)$ and $\nu_P(xy) = \nu_P(x) \wedge \nu_P(y)$.
Combining all the cases we see that P is an intuitionistic fuzzy ideal of $\mathbb{Z}$.

Next, we show that P is an IFAPI of $\mathbb{Z}$. Let $x, y \in \mathbb{Z}$ such that $\mu_P(xy) = \alpha$ and $\nu_P(xy) = \beta$ and $\mu_{P^2}(xy) = 0$ and $\nu_{P^2}(xy) = 1$ such x, y exists in $\mathbb{Z}$.
For example: take $x = 2, y = 3 \in \mathbb{Z}$ then
$\mu_{P^2}(2.3) = \mu_{P^2}(6) = \vee\{\mu_P(1) \wedge \mu_P(6), \mu_P(2) \wedge \mu_P(3)\} = \vee\{0 \wedge \alpha, 0 \wedge \alpha\} = \vee\{0, 0\} = 0$ and
$\nu_{P^2}(2.3) = \nu_{P^2}(6) = \wedge\{\nu_P(1) \vee \nu_P(6), \nu_P(2) \vee \nu_P(3)\} = \wedge\{1 \vee \beta, 1 \vee \beta\} = \wedge\{1, 1\} = 1$.
So, $\mu_P(x) \wedge \mu_P(y) = 0$ and $\nu_P(x) \vee \nu_P(y) = 1$.
Consider $\mu_P(x) \wedge \mu_P(y) = \mu_P(x)$ and $\nu_P(x) \vee \nu_P(y) = \nu_P(x)$, then $\mu_P(x) = 0$ and $\nu_P(x) = 1$.
This implies $x \notin I$. But $\mu_P(xy) = \alpha$ and $\nu_P(xy) = \beta$ implies that $xy \in I$, as I is prime ideal of $\mathbb{Z}$ therefore, $y \in I$.

Assume that $y \in I^2$, then there exist $a, b \in \mathbb{Z}$ with $y = ab$ and $\mu_{P^2}(xy) = 0$ and $\nu_{P^2}(xy) = 1$. This contradict the choice of x and y. Thus, $y \in I - I^2$ and so $xy \in I - I^2$.

Therefore, $\mu_P(xy) = \mu_P(y) = \alpha = \mu_P(x) \vee \mu_P(y)$ and $\nu_P(xy) = \nu_P(y) = \beta = \nu_P(x) \wedge \nu_P(y)$.
So, P is an IFAPI of $\mathbb{Z}$. As P has three membership and non-membership values. Hence P is not an IFPI of $\mathbb{Z}$ (By Theorem (2.9)).

Proposition 3.6. *Let R be a ring. If P be an IFI in R satisfies that for any $x_{(p,q)}, y_{(t,s)} \in IFP(R)$ with $x_{(p,q)}y_{(t,s)} \in P$ and $x_{(p,q)}y_{(t,s)} \notin P^2$ implies that either $x_{(p,q)} \in P$ or $y_{(t,s)} \in P$, then P is an IFAPI in R.*

Proof. Let $x, y \in R$ such that $xy = 0_R$, $\mu_{P^2}(xy) = 0, \nu_{P^2}(xy) = 1$ with $\mu_P(xy) = p$, $\nu_P(xy) = q$, where $p, t, q, s \in (0, 1]$ such that $p = t, q = s$, then $x_{(p,q)}y_{(t,s)} \in P$.

Now, since $x_{(p,q)}y_{(t,s)} \notin 0_{(0,1)}$, then $x_{(p,q)}y_{(t,s)} \notin P^2$ so either $x_{(p,q)} \in P$ or $y_{(t,s)} \in P$, but $\mu_{P^2}(xy) = 0, \nu_{P^2}(xy) = 1$ implies that $\mu_P(x) \wedge \mu_P(y) = 0, \nu_P(x) \vee \nu_P(y) = 1$.

Without loss of generality, consider that $\mu_P(x) \vee \mu_P(y) = \mu_P(x), \nu_P(x) \wedge \nu_P(y) = \nu_P(x)$. So, $\mu_{x_{(p,q)}}(x) = p > \mu_P(x), \nu_{x_{(p,q)}}(x) = q < \nu_P(x)$, then $x_{(p,q)} \notin P$ and $y_{(t,s)} \notin P$ implies that $\mu_{y_{(t,s)}}(y) = t \leq \mu_P(y), \nu_{y_{(t,s)}}(y) = s \geq \nu_P(y)$. Therefore $\mu_P(y) \geq t, \nu_P(y) \leq s$ and $\mu_P(xy) \leq \mu_P(y), \nu_P(xy) \geq \nu_P(y)$ implies that $\mu_P(xy) = \mu_P(y), \nu_P(xy) = \nu_P(y)$. Hence, P is an IFAPI of R. $\square$

Proposition 3.7. *Let I be a proper ideal in R such that $I^2 \subseteq I$. Define the IFS P of R by*

$$\mu_P(x) = \begin{cases} 1, & \text{if } x \in I^2 \\ \alpha, & \text{if } x \in I - I^2 \\ 0, & \text{otherwise} \end{cases}; \quad \nu_P(x) = \begin{cases} 0, & \text{if } x \in I^2 \\ \beta, & \text{if } x \in I - I^2 \\ 1, & \text{otherwise} \end{cases};$$

where $\alpha, \beta \in (0,1)$ such that $\alpha + \beta \leq 1$. Then

(I) If P is an IFAPI in R, then I is an almost prime ideal in R.

(II) If R is UFD, then I is an almost prime ideal in R if and only if P is an IFAPI in R.

Proof. One can easily show that P is an IFI of R.

(I) Let P be an IFAPI in R. For any $x, y \in R$ with $xy \in I - I^2$ we have $\mu_P(xy) = \alpha, \nu_P(xy) = \beta$ and $\mu_{P^2}(xy) = 0, \nu_{P^2}(xy) = 1$ implies that either $\mu_P(xy) = \mu_P(x), \nu_P(xy) = \nu_P(x)$ or $\mu_P(xy) = \mu_P(y), \nu_P(xy) = \nu_P(y)$, i.e., either $\mu_P(x) = \alpha, \nu_P(x) = \beta$ or $\mu_P(y) = \alpha, \nu_P(y) = \beta$. This implies that either $x \in I$ or $y \in I$. Therefore, I is almost prime ideal of R.

(II) Let I be an almost prime ideal in R. Let $x, y \in R$ with $\mu_P(xy) > 0, \nu_P(xy) < 1$ and $\mu_{P^2}(xy) = 0, \nu_{P^2}(xy) = 1$. Then there exists $d \in R$ with $I = \langle d \rangle$ and $xy \in \langle d \rangle$. Assume that $xy \in \langle d \rangle^2 = \langle d^2 \rangle$ so we have $\mu_{P^2}(xy) = 0, \nu_{P^2}(xy) = 1$ which contradict the choice of x, y. Therefore, $xy \in \langle d \rangle - \langle d^2 \rangle$ which implies that $\mu_P(xy) = \alpha, \nu_P(xy) = \beta$ also either $x \in \langle d \rangle - \langle d^2 \rangle$ or $y \in \langle d \rangle - \langle d^2 \rangle$ implies that $\mu_P(x) \vee \mu_P(y) = \alpha = \mu_P(xy)$ or $\nu_P(x) \wedge \nu_P(y) = \beta = \nu_P(xy)$. Therefore, P is an IFAPI in R.

The converse part follows as in (I). $\square$

Proposition 3.8. *Let R be a PID and P be an IFAPI in R. If $\mu_P(0_R) = 1, \nu_P(0_R) = 0$ and there exists a non-zero element $x_o \in R$ with $\mu_P(x_o) = 1$, $\nu_P(x_o) = 0$ and $\mu_{P^2}(x_o) = \mu_P(1_R)$, $\nu_{P^2}(x_o) = \nu_P(1_R)$, then P is an IFPI in R.*

Proof. Let P be an IFAPI of the ring R, where R is PID. Let $P_* = \{r \in R : \mu_P(r) = 1$ and $\nu_P(r) = 0\}$ be a $(1,0)$-cut set with respect to P.

Since $0_R \in P_*$ then $P_* \neq \varnothing$. So for any $x, y \in P_*$ we have $\mu_P(x - y) \geq \mu_P(x) \wedge \mu_P(y)$ and $\nu_P(x - y) \leq \nu_P(x) \vee \nu_P(y)$. Also, $\mu_P(x) = 1 = \mu_P(y)$ and $\nu_P(x) = 0 = \nu_P(y)$, then $\mu_P(x - y) = 1$ and $\nu_P(x - y) = 0$. Hence $x - y \in P_*$. Also for every $r \in R$ and every $x \in P_*$, we have $\mu_P(rx) \geq \mu_P(r) \vee \mu_P(x)$ and $\nu_P(rx) \leq \nu_P(r) \wedge \nu_P(x)$. Also, $\mu_P(x) = 1$ and $\nu_P(x) = 0$ implies that $\mu_P(rx) = 1$ and $\nu_P(rx) = 0$. Therefore, $rx \in P_*$ and so P_* is an ideal in R. As R is PID, so there exists $d \in R$ with $P_* = \langle d \rangle$. We claim that d is a prime element in R.

If possible, let $d = ab$ for some $a, b \in R$. Now, assume that there exists $x_o \in R$ with $x_o \neq 0_R$ such that $\mu_P(x_o) = 1, \nu_P(x_o) = 0$ and $\mu_{P^2}(x_o) = \mu_P(1_R), \nu_{P^2}(x_o) = \nu_P(1_R)$, then

$x_o \in P_*$ and there exists $r \in R$ with $x_o = rd$. Now, $\mu_{P^2}(x_o) = \mu_{P^2}(rab) = \mu_P(1_R)$ and $\nu_{P^2}(x) = \nu_{P^2}(rab) = \nu_P(1_R)$. So either $\mu_P(ra) = \mu_P(1_R), \nu_P(ra) = \nu_P(1_R)$ which implies that $\mu_P(a) = \mu_P(1_R)$ and $\nu_P(a) = \nu_P(1_R)$ or $\mu_P(b) = \mu_P(1_R)$ and $\nu_P(b) = \nu_P(1_R)$ which implies $\mu_P(d) = 1, \nu_P(d) = 0$ and $\mu_{P^2}(d) = \mu_P(1_R)$ and $\nu_{P^2}(d) = \nu_P(1_R)$, then $\mu_P(d) = \mu_P(a) \vee \mu_P(b)$ and $\nu_P(d) = \nu_P(a) \wedge \nu_P(b)$.

Now, assume that $\mu_P(a) \vee \mu_P(b) = \mu_P(b)$ and $\nu_P(a) \wedge \nu_P(b) = \nu_P(b)$, then $\mu_P(b) = 1$ and $\nu_P(b) = 0$ which implies that $b \in \langle d \rangle$. Therefore, $d|b$, but $b|d$ and so $b = ud$ for some unit $u \in R$ and d is prime in R, then $P_* = \langle d \rangle$ is a prime ideal in R.

Now, let $y \in R$ with $\mu_P(y) < 1$ and $\nu_P(y) > 0$. Define

$$P_{(\mu_P(y),\nu_P(y))} = \{r \in R : \mu_P(r) > \mu_P(y) \text{ and } \nu_P(r) < \nu_P(y)\}.$$

Clearly, $P_{(\mu_P(y),\nu_P(y))}$ is an ideal in R as did with P_*. So it is $\mu_P(d) = 1 > \mu_P(y)$ and $\nu_P(d) = 0 < \nu_P(y)$, then $d \in P_{(\mu_P(y),\nu_P(y))}$ implies that $\langle d \rangle \subseteq P_{(\mu_P(y),\nu_P(y))}$, but it has obtained that $\langle d \rangle$ is a prime ideal in R and R is PID. So $\langle d \rangle$ is a maximal ideal in R, then $P_{(\mu_P(y),\nu_P(y))} = R$, $\mu_P(1_R) \geq \mu_P(y)$ and $\nu_P(1_R) \leq \nu_P(y)$ implies that $\mu_P(1_R) = \mu_P(y)$ and $\nu_P(1_R) = \nu_P(y)$. Hence P takes only two values

$$\mu_P(x) = \begin{cases} 1, & \text{if } x \in \langle d \rangle \\ \alpha, & \text{otherwise} \end{cases}, \quad \nu_P(x) = \begin{cases} 0, & \text{if } x \in \langle d \rangle \\ \beta, & \text{otherwise} \end{cases},$$

where $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta \leq 1$. Hence P is IFPI in R. □

Remark 3.9. If P is an IFAPI in a ring R, then P_* needs not be an almost prime ideal of R, for see Example (3.5). Here $P_* = \langle 9 \rangle$, which is not an almost prime ideal in $\mathbb{Z}$, for $3.3 = 9 \in P_* - P_*^2$, but $3 \notin P_*$.

Proposition 3.10. *Let $f : R \to R'$ be a ring isomorphism. If P is an IFAPI of R, then $f(P)$ is an IFAPI of R'.*

Proof. As P is a non-constant IFI of R, one can get that $f(P)$ is also a non-constant IFI of R'. Now, consider $x, y \in R'$ such that $\mu_{f(P)}(xy) > \mu_{f(P)}(1_{R'})$ and $\nu_{f(P)}(xy) < \nu_{f(P)}(1_{R'})$ and that $\mu_{(f(P))^2}(xy) = \mu_{f(P)}(1_{R'})$ and $\nu_{(f(P))^2}(xy) = \nu_{f(P)}(1_{R'})$, then $x = f(a), y = f(b)$, for unique $a, b \in R$ so that $f(ab) = f(a)f(b) = xy$.

Thus, $\mu_P(ab) = \mu_P(f^{-1}(xy)) = \mu_{f(P)}(xy) > \mu_{f(P)}(1_{R'}) = \mu_P(f^{-1}(1_{R'})) = \mu_P(1_R)$ and $\nu_P(ab) = \nu_P(f^{-1}(xy)) = \nu_{f(P)}(xy) < \nu_{f(P)}(1_{R'}) = \nu_P(f^{-1}(1_{R'})) = \nu_P(1_R)$. Also, $\mu_{P^2}(ab) = \mu_{f(P^2)}(f(ab)) = \mu_{(f(P))^2}(f(ab)) = \mu_{(f(P))^2}(xy) = \mu_P(f^{-1}(1_{R'})) = \mu_P(1_R)$ and $\nu_{P^2}(ab) = \nu_{f(P^2)}(f(ab)) = \nu_{(f(P))^2}(f(ab)) = \nu_{(f(P))^2}(xy) = \nu_P(f^{-1}(1_{R'})) = \nu_P(1_R)$. As P is an IFAPI of R implies that $\mu_P(ab) = \mu_P(a) \vee \mu_P(b)$ and $\nu_P(ab) = \nu_P(a) \wedge \nu_P(b)$, i.e., $\mu_P(f^{-1}(xy)) = \mu_P(f^{-1}(x)) \vee \mu_P(f^{-1}(y))$ and $\nu_P(f^{-1}(xy)) = \nu_P(f^{-1}(x)) \wedge \nu_P(f^{-1}(y))$, i.e., $\mu_{f(P)}(xy) = \mu_{f(P)}(x) \vee \mu_{f(P)}(y)$ and $\nu_{f(P)}(xy) = \nu_{f(P)}(x) \wedge \nu_{f(P)}(y)$. Hence $f(P)$ is an IFAPI of R'. □

Proposition 3.11. *Let $f : R \to R'$ be a surjective ring homomorphism. If P' is an IFAPI of R', then $f^{-1}(P')$ is an IFAPI of R.*

Proof. Let $P^{'}$ be an IFAPI of $R^{'}$. Then it is easy to see that $f^{-1}(P^{'})$ is an IFI of R. Now consider $x, y \in R$ such that $\mu_{f^{-1}(P')}(xy) > \mu_{f^{-1}(P')}(1_R)$ and $\nu_{f^{-1}(P')}(xy) < \nu_{f^{-1}(P')}(1_R)$ and that $\mu_{(f^{-1}(P'))^2}(xy) = \mu_{f^{-1}(P')}(1_R)$ and $\nu_{(f^{-1}(P'))^2}(xy) = \nu_{f^{-1}(P')}(1_R)$.

Now, $\mu_{(f^{-1}(P'))^2}(xy) = \mu_{f^{-1}((P')^2)}(xy) = \mu_{(P')^2}(f(xy)) = \mu_{(P')^2}(f(x)f(y))$. Therefore, $\mu_{(P')^2}(f(x)f(y)) = \mu_{P'}(f(1_R)) = \mu_{P'}(1_{R'})$.

Similarly, we get $\nu_{(P')^2}(f(x)f(y)) = \nu_{P'}(1_{R'})$. As $P^{'}$ is an IFAPI of $R^{'}$, we have $\mu_{P'}(f(x)f(y)) = \mu_{P'}(f(x)) \vee \mu_{P'}(f(y))$ and $\nu_{P'}(f(x)f(y)) = \nu_{P'}(f(x)) \wedge \nu_{P'}(f(y))$ which implies that $\mu_{f^{-1}(P')}(xy) = \mu_{f^{-1}(P')}(x) \vee \mu_{f^{-1}(P')}(y)$ and $\nu_{f^{-1}(P')}(xy) = \nu_{f^{-1}(P')}(x) \wedge \nu_{f^{-1}(P')}(y)$. Hence $f^{-1}(P^{'})$ is an IFAPI of R. □

4 Intuitionistic fuzzy almost prime submodule

In this section, the definition and examples of the intuitionistic fuzzy almost prime submodule have been introduced. Also, a relationship between intuitionistic fuzzy almost prime submodules and almost prime submodules is established, as well as between intuitionistic fuzzy almost prime submodules and intuitionistic fuzzy almost prime ideals. So, it is necessary to recall some definitions.

Definition 4.1. ([14]) Let $A \in IFS(M)$. Then A is called an intuitionistic fuzzy submodule $(IFSM)$ of M if for all $x, y \in M, r \in R$, the followings are satisfied

(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$,

(ii) $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$,

(iii) $\mu_A(rx) \geq \mu_A(x)$,

(iv) $\nu_A(rx) \leq \nu_A(x)$,

(v) $\mu_A(\theta) = 1$,

(vi) $\nu_A(\theta) = 0$.

Clearly, $\chi_{\{\theta\}}$, χ_M are IFSMs of M and these are called trivial IFSMs of M. Any IFSM of M other than these is called non-trivial proper IFSM of M. Let $IFSM(M)$ denote the set of all intuitionistic fuzzy R-submodules of M and $IFI(R)$ denote the set of all intuitionistic fuzzy ideals of R. We note that when $R = M$, then $A \in IFSM(M)$ if and only if $\mu_A(\theta) = 1$, $\nu_A(\theta) = 0$ and $A \in IFI(R)$.

Definition 4.2. ([26]) A non-constant intuitionistic fuzzy submodule A of an R-module is said to be an intuitionistic fuzzy prime submodule (IFPSM) if for any $C \in IFI(R)$ and $D \in IFSM(M)$ such that $C \cdot D \subseteq A$ then either $D \subseteq A$ or $C \subseteq (A : \chi_M)$.

In terms of intuitionistic fuzzy point, a non-constant intuitionistic fuzzy submodule A of an R-module M is called an intuitionistic fuzzy prime submodule if for $r_{(s,t)} \in IFP(R)$, $x_{(p,q)} \in IFP(M)$ such that $r_{(s,t)}x_{(p,q)} \in A$ implies that either $x_{(p,q)} \in A$ or $r_{(s,t)} \in (A : \chi_M)$.

Definition 4.3. ([25]) Suppose A and B be two IFSMs of an R-module M. Then

$$(A : B) = \bigcup\{r_{(\alpha,\beta)} : r \in R, \alpha, \beta \in (0,1] \text{ with } \alpha + \beta \leq 1, r_{(\alpha,\beta)}B \subseteq A\}$$

is an IFI in R. Further, if A is a non-constant IFPSM of M, then $(A : \chi_M)$ is an IFPI of R.

Theorem 4.4. ([26]) *(a) Let N be a prime submodule of an R-module M and $\alpha, \beta \in (0,1)$ such that $\alpha + \beta < 1$. If A is an IFS of M defined by*

$$\mu_A(x) = \begin{cases} 1, & \textit{if } x \in N \\ \alpha, & \textit{otherwise} \end{cases} \quad ; \quad \nu_A(x) = \begin{cases} 0, & \textit{if } x \in N \\ \beta, & \textit{otherwise} \end{cases}$$

for all $x \in M$. Then, A is an intuitionistic fuzzy prime submodule of M.

(b) Conversely, any intuitionistic fuzzy prime submodule can be obtained as in (a).

Definition 4.5. ([23]) An R-module M is called an intuitionistic fuzzy multiplication module if and only if for each intuitionistic fuzzy submodule A of M: there exists an intuitionistic fuzzy ideal C of R with $C(0_R) = (1,0)$ such that $A = C\chi_M$. One can easily show that if $A = C\chi_M$ then $A = (A : \chi_M)\chi_M$.

Theorem 4.6. ([23]) *Let M be an R-module. Then M is a multiplication module if and only if M is an intuitionistic fuzzy multiplication module.*

Now, will introduce the definition of intuitionistic fuzzy almost prime submodules.

Definition 4.7. A non-constant IFSM A of an R-module M is said to be an intuitionistic fuzzy almost prime submodule (IFAPSM) of M, if for $r \in R$, $x \in M$ with $\mu_A(rx) > \min_{y\in M}\{\mu_A(y)\}$, $\nu_A(rx) < \max_{y\in M}\{\nu_A(y)\}$ and $\mu_{(I_A \circ A)}(rx) = \min_{y\in M}\{\mu_A(y)\}$ and $\nu_{(I_A\circ A)}(rx) = \max_{y\in M}\{\nu_A(y)\}$, then $\mu_A(rx) = \mu_A(x), \nu_A(rx) = \nu_A(x)$ or $\mu_{I_A}(r) > \min_{y\in M}\{\mu_A(y)\}, \nu_{I_A}(r) < \max_{y\in M}\{\nu_A(y)\}$, where I_A is an IFS of R defined by

$$\mu_{I_A}(r) = \begin{cases} \min_{y\in M}\{\mu_A(ry)\}, & \text{if for all } x \in M, \mu_A(rx) > \min_{y\in M}\{\mu_A(y)\} \\ \min_{y\in M}\{\mu_A(y)\}, & \text{otherwise,} \end{cases}$$

$$\nu_{I_A}(r) = \begin{cases} \max_{y\in M}\{\nu_A(ry)\}, & \text{if for all } x \in M, \nu_A(rx) < \max_{y\in M}\{\nu_A(y)\} \\ \max_{y\in M}\{\nu_A(y)\}, & \text{otherwise.} \end{cases}$$

Next, it has been shown that every IFPSM is an IFAPSM but the converse needs not to be true.

Proposition 4.8. *If A is an IFPSM of an R-module M, then A is also an IFAPSM of M.*

Proof. Let A be an IFPSM of an R-module M, then by Theorem (4.3) there exists a prime submodule N of M such that

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in N \\ \alpha, & \text{otherwise} \end{cases} \quad ; \quad \nu_P(x) = \begin{cases} 0, & \text{if } x \in N \\ \beta, & \text{otherwise} \end{cases}$$

where $\alpha, \beta \in (0,1)$ such that $\alpha + \beta < 1$.

Let $r \in R, x \in M$ with $\mu_A(rx) > \min_{y \in M}\{\mu_A(y)\} = \alpha$, $\nu_A(rx) < \max_{y \in M}\{\nu_A(y)\} = \beta$ and $\mu_{(I_A \circ A)}(rx) = \min_{y \in M}\{\mu_A(y)\} = \alpha$ and $\nu_{(I_A \circ A)}(rx) = \max_{y \in M}\{\nu_A(y)\} = \beta$, then $rx \in N$ and $\sup\{\mu_{I_A}(r_1) \wedge \mu_A(x_1) : r_1 \in R, x_1 \in M\} = \alpha$ and $\inf\{\nu_{I_A}(r_1) \vee \nu_A(x_1) : r_1 \in R, x_1 \in M\} = \beta$. So, either $\mu_{I_A}(r) = \alpha, \nu_{I_A}(r) = \beta$ or $\mu_A(x) = \alpha, \nu_A(x) = \beta$. The first case yields that there exists $y \in M$ such that $\mu_A(ry) = \alpha, \nu_A(ry) = \beta$, which further implies that $ry \notin N$, i.e., $r \notin (N : M)$, as N is a prime ideal, we get $x \in N$ and then $\mu_A(x) = \mu_A(rx) = 1, \nu_A(x) = \nu_A(rx) = 0$. Also, the second case yields that $r \in (N : M)$.

So, for every $y \in M$, $ry \in N$ and $\mu_A(ry) = 1$, $\nu_A(ry) = 0$, which implies that $\mu_{I_A}(r) = 1 > \alpha = \min_{y \in M}\{\mu_A(y)\}$, $\nu_{I_A}(r) = 0 < \beta = \max_{y \in M}\{\nu_A(y)\}$. Therefore, A is an IFAPSM of M. □

Example 4.9. Consider $R = \mathbb{Z}$, and $M = \mathbf{Z_9}$ is an R-module. Define the IFS A of M as

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in \langle \bar{0} \rangle \\ \alpha, & \text{if } x \in \langle \bar{3} \rangle - \langle \bar{0} \rangle \\ 0, & \text{otherwise} \end{cases}; \quad \nu_P(x) = \begin{cases} 0, & \text{if } x \in \langle \bar{0} \rangle \\ \beta, & \text{if } x \in \langle \bar{3} \rangle - \langle \bar{0} \rangle \\ 1, & \text{otherwise} \end{cases},$$

where $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta < 1$. It is easy to check that A is an IFSM of M. Now, before showing that A is an IFAPSM of M, we shall define IFS I_A of R by

$$\mu_{I_A}(r) = \begin{cases} \min_{y \in \mathbf{Z_9}}\{\mu_A(ry)\}, & \text{if } \forall x \in M, \mu_A(rx) > \min_{y \in \mathbf{Z_9}}\{\mu_A(y)\} \\ \min_{y \in \mathbf{Z_9}}\{\mu_A(y)\}, & \text{otherwise} \end{cases}$$

$$\nu_{I_A}(r) = \begin{cases} \max_{y \in \mathbf{Z_9}}\{\nu_A(ry)\}, & \text{if } \forall x \in M, \nu_A(rx) < \max_{y \in \mathbf{Z_9}}\{\nu_A(y)\} \\ \max_{y \in \mathbf{Z_9}}\{\nu_A(y)\}, & \text{otherwise} \end{cases},$$

and we shall consider the following three cases.

Case (i) When $r \notin \langle 3 \rangle$, then $r.\bar{1} \notin \langle \bar{3} \rangle$ and $\mu_A(r.\bar{1}) = 0 = \min_{y \in \mathbf{Z_9}}\{\mu_A(y)\}$, $\nu_A(r.\bar{1}) = 1 = \max_{y \in \mathbf{Z_9}}\{\nu_A(y)\}$. Hence $\mu_{I_A}(r) = 0, \nu_{I_A}(r) = 1$.

Case (ii) If $r \in \langle 3 \rangle - \langle 9 \rangle$, then $r = 3k$, where $k \in \mathbb{Z} - 3\mathbb{Z}$ so $rx = 3kx \in \langle \bar{3} \rangle$, for every $x \in \mathbf{Z_9}$. It follows that $\min_{y \in \mathbf{Z_9}}\{\mu_A(y)\} = \alpha$, $\max_{y \in \mathbf{Z_9}}\{\nu_A(y)\} = \beta$ and $\mu_{I_A}(r) = \alpha, \nu_{I_A}(r) = \beta$.

Case (iii) If $r \in \langle 9 \rangle$, then $r = 9k$ with $k \in \mathbb{Z}$ also $rx = 9kx \in \langle \bar{0} \rangle$, for $x \in \mathbf{Z_9}$, it follows that $\min_{y \in \mathbf{Z_9}}\{\mu_A(y)\} = 1, \max_{y \in \mathbf{Z_9}}\{\nu_A(y)\} = 0$ and $\mu_{I_A}(r) = 0, \nu_{I_A}(r) = 1$, where I_A is defined as

$$\mu_{I_A}(r) = \begin{cases} 1, & \text{if } x \in \langle 9 \rangle \\ \alpha, & \text{if } x \in \langle 3 \rangle - \langle 9 \rangle \\ 0, & \text{otherwise} \end{cases}; \quad \nu_{I_A}(r) = \begin{cases} 0, & \text{if } x \in \langle 9 \rangle \\ \beta, & \text{if } x \in \langle 3 \rangle - \langle 9 \rangle \\ 1, & \text{otherwise}. \end{cases}$$

Moreover, it is required to show that $(\langle \bar{3} \rangle : \mathbf{Z_9}) = \langle 3 \rangle$ and $\langle \bar{3} \rangle$ is a prime submodule in $\mathbf{Z_9}$. So, let $r \in (\langle \bar{3} \rangle : \mathbf{Z_9})$, then $r.\bar{1} \in \langle 3 \rangle$ also $r(1 + 3p) = 3q; p, q \in \mathbb{Z}$. It follows that $r = 3k$ with $k \in \mathbb{Z}$ and $r \in \langle 3 \rangle$. Therefore, $(\langle \bar{3} \rangle : \mathbf{Z_9}) \subseteq \langle 3 \rangle$. Now, let $r \in \langle 3 \rangle$ and $x \in \mathbf{Z_9}$, then $rx = 3kx \in \langle \bar{3} \rangle$ and $r \in (\langle \bar{3} \rangle : \mathbf{Z_9}) \subseteq \langle 3 \rangle$. Now, let $r \in \mathbb{Z}, x \in \mathbf{Z_9}$ with $rx \in \langle \bar{3} \rangle$ and $x \notin \langle \bar{3} \rangle$, then $x = p + 3q$ with $3 \nmid p$ and $p, q \in \mathbb{Z}$ and $rx = 3k; k \in \mathbb{Z}$, also $rx = rp + 3rq = 3k$ implies that $rp = 3k_1$ and $3|rp$. So, $3|r$, then $r \in \langle 3 \rangle = (\langle \bar{3} \rangle : \mathbf{Z_9})$ and $\langle \bar{3} \rangle$ is a prime submodule of $\mathbf{Z_9}$.

Now, we shall show that A is an IFAPSM in $\mathbf{Z_9}$.

Let $r \in \mathbb{Z}, x \in \mathbf{Z_9}$ with $\mu_A(rx) > 0, \nu_A(rx) < 1$ and $\mu_{(I_A \circ A)}(rx) = 0, \nu_{(I_A \circ A)}(rx) = 1$, then $rx \in \langle \bar{3} \rangle$. Therefore, there will be two cases.

Case (i) When $\mu_{I_A}(r) = 0, \nu_{I_A}(r) = 1$ implies that $r \notin \langle 3 \rangle$ and $x \in \langle \bar{3} \rangle$. If $rx \in \langle \bar{0} \rangle$, then $x = 9k$ with $k \in \mathbb{Z}$ also $rx = (3r)(3k)$ implies that there exists $r_1 = 3r \in (\langle \bar{3} \rangle : \mathbf{Z_9})$, $x_1 = 3k$; $\mu_{I_A}(r_1) \wedge \mu_A(x_1) > 0, \nu_{I_A}(r_1) \vee \nu_A(x_1) < 1$. It follows that $\sup\{\mu_{I_A}(r_1) \wedge \mu_A(x_1) : r_1 \in R, x_1 \in M\} \neq 0$ and $\inf\{\nu_{I_A}(r_1) \vee \nu_A(x_1) : r_1 \in R, x_1 \in M\} \neq 1$ which is not true by the choice of r and x, then $x \in \langle \bar{3} \rangle - \langle \bar{0} \rangle$ and $rx \in \langle \bar{3} \rangle - \langle \bar{0} \rangle$, therefore, $\mu_A(rx) = \mu_A(x) = \alpha, \nu_A(rx) = \nu_A(x) = \beta$.

Case (ii) When $\mu_A(x) = 0, \nu_A(x) = 1$, implies that $x \notin \langle \bar{3} \rangle$ and $r \in (\langle \bar{3} \rangle : \mathbf{Z_9}) = \langle 3 \rangle$. It follows that $\mu_{I_A}(r) \geq \alpha > 0, \nu_{I_A}(r) \leq \beta < 1$. Hence A is an IFAPSM in $\mathbf{Z_9}$.

Proposition 4.10. *Let R be a PID and M be a finitely generated faithful multiplication R-module, and N be a proper submodule of M. Then the following are equivalent:*

1. *N is an almost prime submodule of M.*

2. *Let*

$$\mu_A(x) = \begin{cases} 1, & \textit{if } x \in (N : M)N \\ \alpha, & \textit{if } x \in N - (N : M)N \\ 0, & \textit{otherwise} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \textit{if } x \in (N : M)N \\ \beta, & \textit{if } x \in N - (N : M)N \\ 1, & \textit{otherwise} \end{cases}$$

   where $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta < 1$. Then A is an IFAPSM of M.

3. *I_A is an IFAPI of R.*

Proof. $(1) \Rightarrow (2)$ Let $r \in R, x \in M$,

$$\mu_A(rx) > \min_{y \in M}\{\mu_A(y)\} = 0, \quad \nu_A(rx) < \max_{y \in M}\{\nu_A(y)\} = 1$$

and $\mu_{(I_A \circ A)}(rx) = 0$ and $\nu_{(I_A \circ A)}(rx) = 1$. Assume that $rx \in (N : M)N$, then $\mu_{I_A}(r) > 0$, $\nu_{I_A}(r) < 1$. So $\mu_{(I_A \circ A)}(rx) = 0$ and $\nu_{(I_A \circ A)}(rx) = 1$ which is not true by the choice of r and x, then $rx \in N - (N : M)N$ also either $\mu_{I_A}(r) = 0, \nu_{I_A}(r) = 1$, which implies $r \notin (N : M), x \in N - (N : M)N$ and $\mu_A(rx) = \mu_A(x) = \alpha, \nu_A(rx) = \nu_A(x) = \beta$ or $\mu_A(x) = 0, \nu_A(x) = 1$ implies $x \notin N$ and $r \in (N : M)$. Therefore, $\mu_{I_A}(r) > 0, \nu_{I_A}(r) < 1$. Hence A is an IFAPSM of M.

$(2) \Rightarrow (1)$ Let $r \in R, x \in M$ with $rx \in N - (N : M)N$, then $\mu_A(rx) = \alpha, \nu_A(rx) = \beta$ and $\mu_{I_A}(r) = 0, \nu_{I_A}(r) = 1$ also either $\mu_A(rx) = \mu_A(x) = \alpha, \nu_A(rx) = \nu_A(x) = \beta$ implies $x \in N$ or $\mu_{I_A}(r) > 0, \nu_{I_A}(r) < 1$. implies $\mu_A(ry) > 0, \nu_A(ry)) < 1, \forall y \in M$ and $r \in (N : M)$. Therefore, N is an almost prime submodule of M.

$(2) \Leftrightarrow (3)$ For every $r \in R$

$$\mu_{I_A}(r) = \begin{cases} \min_{y \in M}\{\mu_A(ry)\}, & \text{if for all } x \in M, \mu_A(rx) > \min_{y \in M}\{\mu_A(y)\} \\ \min_{y \in M}\{\mu_A(y)\}, & \text{otherwise,} \end{cases}$$

$$\nu_{I_A}(r) = \begin{cases} \max_{y\in M}\{\nu_A(ry)\}, & \text{if for all } x \in M, \nu_A(rx) < \max_{y\in M}\{\nu_A(y)\} \\ \max_{y\in M}\{\nu_A(y)\}, & \text{otherwise.} \end{cases}$$

Now, if $r \in (N : M)$, then there exists $y \in M$ with $ry \notin N$, $\mu_A(ry) = 0, \nu_A(ry)) = 1$ and $\mu_{I_A}(r) = 0, \nu_{I_A}(r) = 1$. If $r \in (N : M) - (N : M)^2$ (since $N \subset M$ implies that there exists $y \in M - N; ry \in N - (N : M)N$), then $\mu_A(ry) = \alpha, \nu_A(ry)) = \beta$ and $\mu_A(rx) > \alpha, \nu_A(rx)) < \beta$, for all $x \in M$, which implies that $\mu_{I_A}(r) = \alpha, \nu_{I_A}(r) = 1\beta$. If $r \in (N : M)^2$, then for every $x \in M$, $rx = r_1(r_2x) \in (N : M)N$ and then $r_1(r_2x) \in (N : M)N$, which implies that $\mu_A(rx) = 1, \nu_A(rx)) = 0$, for all $x \in M$. It follows that $\mu_{I_A}(r) = 1, \nu_{I_A}(r) = 0$.

$$\mu_{I_A}(r) = \begin{cases} 1, & \text{if } r \in (N : M)^2 \\ \alpha, & \text{if } r \in (N : M) - (N : M)^2 \\ 0, & \text{otherwise} \end{cases}, \quad \nu_{I_A}(r) = \begin{cases} 0, & \text{if } r \in (N : M)N \\ \beta, & \text{if } r \in N - (N : M)N \\ 1, & \text{otherwise} \end{cases},$$

where $r \in R$. Now, N is an almost prime submodule in M if and only if $(N : M)$ is an almost prime ideal in R (see Theorem (3.5) of [10]) if and only if I_A is IFAPI in R (by Proposition (3.7)) which completes the proof. □

5 Conclusion

In this paper we have introduced and studied the notions of intuitionistic fuzzy almost prime ideals of a commutative ring with unity and intuitionistic fuzzy almost prime submodules of a unitary module. Examples are used to show that the two notions are generalizations of intuitionistic fuzzy prime ideals and of intuitionistic fuzzy prime submodules, respectively. Relationships were established between the intuitionistic fuzzy prime ideals (submodules) and the intuitionistic fuzzy prime ideals (submodules). Many other related concepts have been defined and discussed. Further study of these concepts will open a new door to investigate new concepts such as intuitionistic fuzzy almost semiprime ideals and intuitionistic fuzzy almost semiprime submodules.

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