# On the operators and partial orderings of intuitionistic fuzzy sets 

Evgeniy Marinov<br>Dept. of Bioinformatics and Mathematical Modelling<br>IBPhBME - Bulgarian Academy of Sciences 105 Acad. Georgi Bonchev Str., 1113 Sofia, Bulgaria email: evgeniy.marinov@biomed.bas.bg


#### Abstract

Atanassov's extension of the notion of fuzzy set has proved to be an important field of real-life applications and theoretical research. In the paper [5], there was introduced a new ordering on IF-sets, the so called $\pi$-ordering, which turns out to be a key concept for this paper. In the last section we introduce some new operators on IF-sets and investigate their properties in respect of the two base partial orderings on the class of IF-sets. The standard operators are classified according to the $\pi$-ordering as well. The theoretical basis is provided trough the investigation of more general partial orderings on the vector space $\mathbb{R}^{2}$ and their properties are carried over the triangular representation of IF-sets.


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## 1 Intuitionistic fuzzy sets and quasi-orderings

A fuzzy set in $X$ (Zadeh [8]) is given by

$$
\begin{equation*}
A^{\prime}=\left\{<x, \mu_{A^{\prime}}(x)>\mid x \in X\right\} \tag{1}
\end{equation*}
$$

where $\mu_{A^{\prime}}(x) \in[0,1]$ is the membership function of the fuzzy set $A^{\prime}$. As opposed to the Zadeh's fuzzy set (abbreviated F-set or just FS), Atanassov extended its definition to an intuitionistic fuzzy set (IF-set) (Atanassov [1], Atanassov [2]) $A$, given by

$$
\begin{equation*}
A=\left\{<x, \mu_{A}(x), \nu_{A}(x)>\mid x \in X\right\} \tag{2}
\end{equation*}
$$

where: $\mu_{A}: X \rightarrow[0,1]$ and $\nu_{A}: X \rightarrow[0,1]$ such that

$$
\begin{equation*}
0 \leq \mu_{A}(x)+\nu_{A}(x) \leq 1 \tag{3}
\end{equation*}
$$

and $\mu_{A}(x), \nu_{A}(x) \in[0,1]$ denote a degree of membership and a degree of non-membership of $x \in A$, respectively. (Two approaches to the assigning memberships and non-memberships for

IFSs are proposed by Szmidt and Baldwin [7]). An additional concept for each IFS in $X$, that is an obvious result of (2) and (3), is called

$$
\begin{equation*}
\pi_{A}(x)=1-\mu_{A}(x)-\nu_{A}(x) \tag{4}
\end{equation*}
$$

a hesitation margin of $x \in A$. It expresses a lack of knowledge of whether $x$ belongs to $A$ or not (cf. Atanassov [1]). It is obvious that $0 \leq \pi_{A}(x) \leq 1$, for each $x \in X$. Hesitation margins turn out to be relevant for both - applications and the development of theory of IFSs. For instance, distances between IFSs are calculated in the literature in two ways, using two parameters only or all three parameters.

When talking about partial ordering on IF-sets, we will by default mean $(\operatorname{IFS}(X), \leq)$ where $\leq$ stands for the standard partial ordering on $\operatorname{IFS}(X)$. That is, for any two $A$ and $B \in I F S(X)$ : $A \leq B$ is satisfied if and only if $\mu_{A}(x) \leq \mu_{B}(x)$ and $\nu_{A}(x) \geq \nu_{B}(x)$ for any $x \in X$. But we will often use another important partial ordering, the so called $\pi$-ordering (cf. Marinov and Atanassov [5]).

Definition 1 For any two intuitionistic fuzzy sets $A$ and $B \in \operatorname{IFS}(X)$, we define the following binary relation:

$$
A \preceq_{\pi} B \text { iff }(\forall x \in X)\left(\mu_{A}(x) \leq \mu_{B}(x) \& \nu_{A}(x) \leq \nu_{B}(x)\right)
$$

This relation is actually a partial ordering, the so called $\pi$-ordering.
Let us now remind the modal quasi-orderings on IF-sets and some of their properties (cf. Atanassov [1] and Marinov and Atanassov [5]). "Necessity" and "possibility" operators (denoted $\square$ and $\diamond$ respectively) applied on an intuitionistic fuzzy set $A \in \operatorname{IFS}(X)$ have been defined as:

$$
\left.\begin{array}{lll}
\square A=\{<x, & \mu_{A}(x), & 1-\mu_{A}(x) \\
\diamond \mid x \in X\} \\
\diamond A=\{<x, & 1-\nu(x), & \nu_{A}(x)
\end{array}>\mid x \in X\right\}
$$

From the above definition it is evident that

$$
\begin{equation*}
\star: I F S(X) \longrightarrow F S(X) \tag{5}
\end{equation*}
$$

where $\star$ is the prefix operator $\star \in\{\square, \diamond\}$, operating on the class of intuitionistic fuzzy sets. On can easily check that the two modal operators are non-decreasing in respect of the standard ordering $(\operatorname{IFS}(X), \leq)$ and the set of fixed points of both of them is $F S(X)$ - the family of fuzzy sets on $X$ considered as a subset of $\operatorname{IFS}(X)$.

Quasi-ordered set is a set $Y$ with a binary relation $\preceq$ satisfying reflexivity and transitivity, where the anti-symmetric property may not be in general satisfied. For detailed information about quasi-ordered sets the reader can consult Birkhoff [4], Ch. II.1. For any $A, B \in \operatorname{IFS}(X)$ let us define the following relations:

$$
A \leq_{\square} B \text { iff } \mu_{A} \leq \mu_{B} \text { on } X
$$

and

$$
A \leq_{\diamond} B \text { iff } \nu_{A} \geq \mu_{B} \text { on } X
$$

corresponding the the modal operators. Obviously both relations $\leq_{\square}$ and $\leq_{\diamond}$ are reflexive and transitive, i.e. they are quasi-orderings on $\operatorname{IFS}(\mathrm{X})$. They have been defined as quasi $\square$-ordering and quasi $\diamond$-ordering respectively. Taking any $A$ and $B$ from $\operatorname{IFS}(X)$, let us write down the following obvious properties.


Figure 1. A linear operator $l \in \mathcal{L}\left(\mathbb{R}^{2}\right)$ and its kernel, defining the positive and respectively the negative half plane.

1. $\square A \leq A \leq \diamond A$
2. $A \leq B$ iff $A \leq_{\square} B$ and $A \leq_{\diamond} B$
3. $A \leq_{\pi} B$ iff $A \leq_{\square} B$ and $A \geq_{\diamond} B$

It is clear that the two orderings - the $\pi$-ordering and the standard one can be defined in terms of the two modal quasi-orderings $\leq_{\square}$ and $\leq_{\diamond}$.

The next section will give us the theoretical basis to classify some of the operators on IF-sets with respect of the orderings on IF-sets.

## 2 Partial orderings over $\mathbb{R}^{2}$ and IF-sets

Let us now define a family of partial orderings in $\mathbb{R}^{2}$ inspired by to the two orderings on IF-sets and explain their geometrical interpretations. We introduce first partial orderings in a more general way. Suppose that $l$ and $g$ are two non-zero and linear functionals over the two dimensional real vector space $\mathbb{R}^{2}$, i.e. $l, g \in \mathcal{L}\left(\mathbb{R}^{2}\right)$ such that $\operatorname{dim}(\operatorname{Ker}(l))=\operatorname{dim}(\operatorname{Ker}(g))=1 . \operatorname{Ker}(l)$ is the kernel of the operator $l$ and is defined as $\operatorname{Ker}(l)=l^{-1}(0,0)$. We also want $l$ and $g$ to be linearly independent which is equivalent to say that $\operatorname{Ker}(l) \neq \operatorname{Ker}(g)$ in our two dimensional vector space. Therefore, $\operatorname{Ker}(l)$ and $\operatorname{Ker}(g)$ represent two different lines passing through ( 0,0 ) (hyperplanes in general). From the linear algebra it is known that any $l \in \mathcal{L}(V) \backslash\{\overline{0}\}$ ( $V$ being any finite dimensional vector space) splits the vector space $V$ in two parts - the so called positive and negative half spaces in the following way

$$
l_{+}:=l^{-1}([0,+\infty)) \text { and } l_{-}:=l^{-1}((-\infty, 0]) .
$$

We also have that $\operatorname{Ker}(l)=l_{+} \cap l_{-}$and $V=l_{+} \cup l_{-}$which can be geometrically seen in Fig 1 .
Note that if for a fixed base of the vector space, the linear operator is represented as $l(x, y)=$ $a x+b y$ then the positive half plane is determined by the direction of the vector with coordinates $(a, b)$. This vector is in fact perpendicular to $\operatorname{Ker}(l)$. Let us give the following definition.


Figure 2. The linear operators $l, g \in \mathcal{L}\left(\mathbb{R}^{2}\right)(\operatorname{Ker}(l) \neq \operatorname{Ker}(g))$, translated in $A \in \mathbb{R}^{2}$. In this way, $l$ and $g$ split the plane in four parts. The two by two opposite parts define the elements which are $\leq_{\pi(l, g)}$ and $\leq_{(l, g)}$ resp. less or greater than $A$.

Definition 2 For the above chosen pair of linear operators (l,g) belonging to $\mathcal{L}\left(\mathbb{R}^{2}\right)$ we define the following binary relations on $\mathbb{R}^{2}$. Taking any two elements $A, B \in \mathbb{R}^{2}$, let us write

$$
\begin{equation*}
A \leq_{\pi(l, g)} B \text { iff } l(A) \leq l(B) \text { and } g(A) \leq g(B) \tag{6}
\end{equation*}
$$

and

$$
\begin{equation*}
A \leq{ }_{(l, g)} B \text { iff } l(A) \leq l(B) \text { and } g(A) \geq g(B) \tag{7}
\end{equation*}
$$

respectively.
We can state the following proposition.
Proposition 1 The above defined relations, under the assumptions imposed, are actually partial orderings.

The reflexivity and transitivity are easy to check and therefore the above relations are quasiorderings. For the anti-symmetric property, let us take any $A, B \in \mathbb{R}^{2}$ with $A \leq_{\pi(l, g)} B$ and $A \geq_{\pi(l, g)} B$. Thereby, $l(A)=l(B)$ and $g(A)=g(B)$ which provides that $l(\overrightarrow{A B})=0=g(\overrightarrow{A B})$. From the last expression we get that $\overrightarrow{A B} \in \operatorname{Ker}(l) \cap \operatorname{Ker}(g)=\{\overrightarrow{0}\}$ and therefore $\overrightarrow{A B}=\overrightarrow{0}$ which means that $A=B$. The anti-symmetric property is proved and hence $\leq_{\pi(l, g)}$ is a partial ordering in $\mathbb{R}^{2}$. In the same way we conclude that $\leq_{(l, g)}$ is a partial ordering as well.

Let us write down an explicit expression for which pairs of elements from the vector space $\mathbb{R}^{2}$ are not $\leq_{\pi(l, g)}$ comparable and respectively not $\leq_{(l, g)}$ comparable.

Remark 1 In the above assumptions about $l$ and $g$ and the defined partial orderings $\leq_{\pi(l, g)}$ we have that any $A$ and $B \in \mathbb{R}^{2}$ are

- $\leq_{\pi(l, g)}$-incomparable iff $(l(B)-l(A))(g(B)-g(A))<0$
- $\leq(l, g)$-incomparable iff $(l(B)-l(A))(g(B)-g(A))>0$

On Fig. 2 we can see how such two linear operators look like. We have $l, g \in \mathcal{L}\left(\mathbb{R}^{2}\right)(\operatorname{Ker}(l) \neq$ $\operatorname{Ker}(g))$ with the coordinate system translated in $A$ with its positive/negative half planes. That way $l$ and $g$ split the plane in four parts. The two by two opposite parts define the elements which are $\leq_{\pi(l, g)}$ and $\leq_{(l, g)}$ resp. less/greater than $A$. On the picture we have $B, C, D$ and $E$ - three points from $\mathbb{R}^{2}$ which lie on different parts split by $l$ and $g$. For those points on the picture the following relations hold

- $D \leq_{\pi(l, g)} A$ and $A \leq_{\pi(l, g)} B$
- $C \leq_{(l, g)} A$ and $A \leq_{(l, g)} E$
and let us also remark that
- $B$ and $D$ are $\leq_{(l, g)}$ incomparable with respect of $A$
- $C$ and $E$ are $\leq_{\pi(l, g)}$ incomparable with respect of $A$

From the last remarks and the geometric representations we leave for the reader the easy proof of the following two $\leq_{\pi(l, g)}$ and $\leq_{(l, g)}$ classifying propositions.

Proposition 2 For any two different points $A$ and $B \in \mathbb{R}^{2}$ such that $l(A) \neq l(B)$ and $g(A) \neq$ $g(B)$ we have that exactly one of the following relations hold

1. $A \int_{\pi(l, g)} B$ iff $l(A)<l(B)$ and $g(A)<g(B)$
2. $B \leq \pi(l, g)$ iff $l(A)>l(B)$ and $g(A)>g(B)$
3. $A \lesseqgtr(l, g)$ iff $l(A)<l(B)$ and $g(A)>g(B)$
4. $B \underset{f(l, g)}{ } A$ iff $l(A)>l(B)$ and $g(A)<g(B)$

Proposition 3 For any two different points $A$ and $B \in \mathbb{R}^{2}$ we have that they are both $\leq_{\pi(l, g)}$ and $\leq_{(l, g)}$ comparable only in the following cases

- $A \int_{\pi(l, g)} B$ and $A \int_{(l, g)} B$ iff $l(A)<l(B)$ and $g(A)=g(B)$
- $A \not \gtrless_{(l, g)} B$ and $A \not{ }_{(l, g)} B$ iff $l(A)>l(B)$ and $g(A)=g(B)$
- $A \lessgtr_{\sim(l, g)} B$ and $A \not \chi_{(l, g)} B$ iff $l(A)=l(B)$ and $g(A)<g(B)$
- $A \not \gtrless_{\pi(l, g)} B$ and $A \lesseqgtr(l, g)$ iff $l(A)=l(B)$ and $g(A)>g(B)$

We have now the theoretical basis and all the properties needed to classify some of the operators over IF-sets with respect of the orderings over IF-sets defined in the first section.

## 3 Operators classified by the orderings on IF-sets

Let us recall the definitions of some of the operators on IF-sets (cf. Atanassov [2] and [1]) and introduce then some new operators. In what follows $A$ will be an arbitrarily chosen IF-set from over $X$ and $f_{A}(x)$ means the point of $\mathbb{R}$ such that $p r_{1}\left(f_{A}(x)\right)=\mu_{A}(x)$ and $p r_{2}\left(f_{A}(x)\right)=\nu_{A}(x)$.

Take $\alpha$ and $\beta \in[0,1]$ and let us write the definition of the following operators

$$
H_{\alpha, \beta}, J_{\alpha, \beta}: \operatorname{IFS}(X) \rightarrow \operatorname{IFS}(X)
$$

such that

- $\mu_{H_{\alpha, \beta}(A)}=\alpha \mu_{A}$ and $\nu_{H_{\alpha, \beta}(A)}=\nu_{A}+\beta \pi_{A}$
- $\mu_{J_{\alpha, \beta}(A)}=\mu_{A}+\alpha \pi_{A}$ and $\nu_{J_{\alpha, \beta}(A)}=\beta \nu_{A}$


Figure 3. The marked areas in the triangles $\triangle_{I}$ are the ranges of the corresponding operators $H_{\alpha, \beta}$ and $J_{\alpha, \beta}$.

The geometrical representation is given on Fig. 3. It is obvious that the operators $H_{\alpha, \beta}$ and $J_{\alpha, \beta}$ are non-decreasing with respect of $\leq$, i.e. if $A \leq B$ then $\star_{\alpha, \beta}(A) \leq \star_{\alpha, \beta}(B)$ (with $\star \in$ $\{H, J\}$ ).

Let $I$ stand for the closed interval $[0,1] . I \times I$ (and any other subset of $\mathbb{R}^{2}$ ) derives the partial orderings from $\mathbb{R}^{2}$. In particular we note that for $l(x, y)=x$ and $g(x, y)=y$ the standard $(\leq)$ and the $\pi\left(\leq_{\pi}\right)$ partial orderings on the closed triangle $\triangle_{I}=\{(x, y) \mid x \in I \& y \in I \& x+y \leq 1\}$ coincide with $\leq_{(l, g)}$ and resp. $\leq_{\pi(l, g)}$. The last two partial orderings have been defined and investigated in detail in the previous section.

Proposition 4 Let us consider the mappings $H$ and $J$ with domain $I \times I$ in the following way

$$
H, J: I \times I \rightarrow I F S(X)^{I F S(X)} .
$$

These two mappings are non-decreasing with respect to the standard partial ordering $\leq$ on $I \times I$ and $\operatorname{IFS}(X)^{I F S(X)}$. That is, for $\star \in\{H, J\}$ we have that

$$
\left(\forall(\alpha, \beta),\left(\alpha_{1}, \beta_{1}\right) \in I \times I\right)\left((\alpha, \beta) \leq\left(\alpha_{1}, \beta_{1}\right) \Rightarrow \star_{(\alpha, \beta)} \leq \star_{(\alpha, \beta)}\right)
$$

Let us explain more precisely the meaning of the relation $\leq$ for two mappings $\mathcal{U}$ and $\mathcal{V} \in$ $\operatorname{IFS}(X)^{I F S(X)}$. We assume the following conventions (definitions).

Definition 3 For the above $\mathcal{U}$ and $\mathcal{V}$ let us write

- $\mathcal{U} \leq \mathcal{V}$ iff $(\forall A \in \operatorname{IFS}(X))(\mathcal{U}(A) \leq \mathcal{V}(A))$
- $\mathcal{U} \leq_{\pi} \mathcal{V}$ iff $(\forall A \in \operatorname{IFS}(X))\left(\mathcal{U}(A) \leq_{\pi} \mathcal{V}(A)\right)$.

One can easily check that the above defined binary relations are partial orderings on $\operatorname{IFS}(X)^{\operatorname{IFS}(X)}$ carried over from the corresponding orderings on $\operatorname{IFS}(X)$.

As it has been shown (cf. Atanassov [2]) for some of the operators the pair-indexes ( $\alpha, \beta$ ) can be replaced by IF-sets. In this flow of thoughts we are going to define a few more operators. Let us consider the picture on Fig. 4 with $f_{A}(x)$ chosen for an arbitrary point $x$ from the underlying set $X$. For an IF-set $B \in \operatorname{IFS}(X)$ we can define $H_{B} \in \operatorname{IFS}(X)^{I F S(X)}$ taking $\alpha=\mu_{B}$ and $\beta=\nu_{B}$ as it has been introduced by Atanassov in [2]. That is, $\mu_{H_{B}(A)}=\left(1-\nu_{B}\right) \mu_{A}$ and $\nu_{H_{B}(A)}=\nu_{A}+\left(1-\mu_{B}\right) \pi_{A}$. But because of the restriction $\mu_{B}(x)+\nu_{B}(x) \leq 1$ for any point $x \in X$ we have that the range of $H_{B}$ is the closed triangle area of $\triangle S T R$, where the points $S, R$ and $T$ may vary with $x \in X$.


Figure 4. A more detailed picture of the range of some of the operators on IF-sets.

Definition $4\left(H^{\prime}{ }_{\alpha, \beta}\right.$ and $\left.J^{\prime}{ }_{\alpha, \beta}\right)$ For the pair-indexes $(\alpha, \beta) \in I \times I$ let us define $H^{\prime}{ }_{\alpha, \beta}$ and $J^{\prime}{ }_{\alpha, \beta} \in$ $\operatorname{IFS}(X)^{\operatorname{IFS}(X)}$ in the following way

- $\mu_{H^{\prime}{ }_{\alpha, \beta}}(A)=(1-\beta) \mu_{A}$ and $\nu_{H^{\prime}{ }_{\alpha, \beta}}(A)=\nu_{A}+(1-\alpha) \pi_{A}$
- $\mu_{J^{\prime}{ }_{\alpha, \beta}(A)}=\mu_{A}+(1-\beta) \pi_{A}$ and $\nu_{J^{\prime}{ }_{\alpha, \beta}(A)}=(1-\alpha) \nu_{A}$

Therefore, we have that $H^{\prime}{ }_{\alpha, \beta}=H_{1-\beta, 1-\alpha}$ and $J^{\prime}{ }_{\alpha, \beta}=J_{1-\beta, 1-\alpha}$ which provides that the two operators $J^{\prime}$ and $H^{\prime}$ are non-increasing in respect of the ordering $\leq$ on the indexes from $I \times I$. On the other hand for any pairs $(\alpha, \beta)$ the operators $H^{\prime}{ }_{\alpha, \beta}$ and $J^{\prime}{ }_{\alpha, \beta}$ with domains $\operatorname{IFS}(X)$ are non-decreasing.

Let us remark that $H^{\prime}{ }_{B}$ for $B \in \operatorname{IFS}(X)$ defined by analogy of $H_{B}$ would range on the closed triangular area $\triangle P R T$. Obviously for any points $S_{1} \in \triangle S T R$ and $S_{2} \in \triangle P R T$ such that there exists $x_{0} \in X$ with $S_{1}=(1-\beta) \mu_{A}\left(x_{0}\right)$ and $S_{1}=\nu_{A}\left(x_{0}\right)+(1-\alpha) \pi_{A}\left(x_{0}\right)$ we have that $S_{1} \leq_{\pi} S_{2}$ because $\alpha \leq 1-\beta$ and $\beta \leq 1-\alpha$. We have just replaced above $\mu_{B}\left(x_{0}\right)$ by $\alpha$ and $\nu_{B}\left(x_{0}\right)$ by $\beta$ for simplicity. Therefore, in the above introduced notations the following expressions hold

1. $H_{B}(A) \leq A$ and $H^{\prime}{ }_{B}(A) \leq A$
2. $H_{B}(A) \leq_{\pi} H^{\prime}{ }_{B}(A)$
and respectively an analogous result for $J_{B}$ and $J^{\prime}{ }_{B}$
3. $A \leq J_{B}(A)$ and $A \leq J^{\prime}{ }_{B}(A)$
4. $J_{B}(A) \leq_{\pi} J^{\prime}{ }_{B}(A)$

For any IF-set $B \in \operatorname{IFS}(X)$ and $x_{0} \in X$ the image of $x_{0}$ through $J_{B}(A)$ is a point $f_{J_{B}(A)}\left(x_{0}\right)$ belonging to $\triangle K L T$ whereas the image of $x_{0}$ trough $J^{\prime}{ }_{B}(A)$ is a point $f_{J^{\prime} B_{B}(A)}\left(x_{0}\right)$ belonging to $\triangle L N T$. As it has been shown for $H$ and $H^{\prime}$ a similar reasoning provides that

$$
f_{J_{B}(A)}\left(x_{0}\right) \leq_{\pi} f_{J^{\prime}{ }_{B}(A)}\left(x_{0}\right) .
$$

Let us define another two operators over IF-sets.
Definition 5 ( $\dot{H}_{\alpha, \beta}$ and $\ddot{H}_{\alpha, \beta}$ ) For any $\alpha, \beta \in I$ and $A \in \operatorname{IFS}(X)$ let us define $\dot{H}_{\alpha, \beta}$ and $\ddot{H}_{\alpha, \beta} \in$ $\operatorname{IFS}(X)$ in the following way

1. $\mu_{\dot{H}_{\alpha, \beta}(A)}=(1-\alpha) \mu_{A}$ and $\nu_{\dot{H}_{\alpha, \beta}(A)}=\nu_{A}+\beta \pi_{A}$
2. $\mu_{\ddot{H}_{\alpha, \beta}(A)}=\beta \mu_{A}$ and $\nu_{\ddot{H}_{\alpha, \beta}(A)}=\nu_{A}+(1-\alpha) \pi_{A}$

As done before we can extend the above definitions to be indexed by IF-sets instead of pairs $(\alpha, \beta)$. For any IF-set $B \in \operatorname{IFS}(X)$ and $x_{0} \in X$ the image of $x_{0}$ through $\dot{H}_{B}(A)$ is a point $f_{\dot{H}_{B}(A)}\left(x_{0}\right)$ belonging to $\triangle S T P$ whereas the image of $x_{0}$ trough $\ddot{H}_{B}(A)$ is a point $f_{\ddot{H}_{B}(A)}\left(x_{0}\right)$ belonging to $\triangle S P R$. And moreover we have that

$$
f_{\ddot{H}_{B}(A)}\left(x_{0}\right) \leq f_{\dot{H}_{B}(A)}\left(x_{0}\right) .
$$

The reader may note that

$$
\ddot{H}_{\alpha, \beta}=\dot{H}_{1-\beta, 1-\alpha}
$$

Remark $2\left(\dot{J}_{\alpha, \beta}\right.$ and $\ddot{J}_{\alpha, \beta}$ ) As an easy exercise the reader may try to define the two operators $\dot{J}_{\alpha, \beta}$ and $\ddot{J}_{\alpha, \beta}$ by analogy of the last definition and state the corresponding relation $f_{\ddot{J}_{B}(A)}\left(x_{0}\right) \leq$ $f_{j_{B}(A)}\left(x_{0}\right)$.

Let us now introduce the last two operators on IF-sets and state some properties of them.
Definition $6\left(\tilde{H}_{\alpha, \beta}\right.$ and $\tilde{J}_{\alpha, \beta}$ ) For any $\alpha, \beta \in I$ and $A \in \operatorname{IFS}(X)$ let us define $\tilde{H}_{\alpha, \beta}$ and $\tilde{J}_{\alpha, \beta} \in$ $\operatorname{IFS}(X)$ in the following way

1. $\mu_{\tilde{H}_{\alpha, \beta}(A)}=\alpha \mu_{A}$ and $\nu_{\tilde{H}_{\alpha, \beta}(A)}=1-\mu_{A}+\beta \mu_{A}$
2. $\mu_{\tilde{J}_{\alpha, \beta}(A)}=1-\nu_{A}+\alpha \nu_{A}$ and $\nu_{\tilde{J}_{\alpha, \beta}(A)}=\beta \nu_{A}$

A direct consequence from the definition is that the values of image of $\tilde{H}_{\alpha, \beta}$ lay in the triangular area $\triangle R P Q$ which is the lowest part beside all the figures on which the triangular area $\triangle_{I}$ has been split. On the other hand $\tilde{J}_{\alpha, \beta}$ lay in the triangular area $\triangle L M N$ - the greatest part beside the figures. From the above reasoning about all the introduced operators over IF-sets for any $A \in \operatorname{IFS}(X)$ the following expression hold

1. $\tilde{H}_{\alpha, \beta} \leq H_{\alpha, \beta}^{*} \leq H_{\alpha, \beta} \leq G_{\alpha, \beta}$
2. $\tilde{J}_{\alpha, \beta} \leq J_{\alpha, \beta}^{*} \leq J_{\alpha, \beta} \leq G_{\alpha, \beta}$
where $\alpha$ and $\beta \in I$. Following Atanassov [2] let us give the definition of the operators used in the above expressions and give their range as depicted on Fig. 4. That is

- $\mu_{G_{\alpha, \beta}(A)}=\alpha \mu_{A}$ and $\nu_{G_{\alpha, \beta}(A)}=\beta \nu_{A}$ (OKTS)
- $\mu_{H_{\alpha, \beta}^{*}(A)}=\alpha \mu_{A}$ and $\nu_{H_{\alpha, \beta}^{*}(A)}=\nu_{A}+\beta\left(1-\alpha \mu_{A}-\nu_{A}\right)$ (STPQ)
- $\mu_{J_{\alpha, \beta}^{*}(A)}=\mu_{A}+\alpha\left(1-\beta \nu_{A}-\mu_{A}\right)$ and $\nu_{H_{\alpha, \beta}^{*}(A)}=\beta \nu_{A}$ (KMNT)
- $\mu_{F_{\alpha, \beta}(A)}=\mu_{A}+\alpha \pi$ and $\nu_{F_{\alpha, \beta}(A)}=\nu_{A}+\beta \pi$ ( $\triangle T N P$ )

From the last picture and analytically on may easily check that for any $\alpha, \beta \in I$ and any $x_{0} \in X$ we have that

$$
f_{G_{\alpha, \beta}}\left(x_{0}\right) \leq_{\pi} f_{F_{\alpha, \beta}}\left(x_{0}\right) .
$$

Let us write down the last proposition similar to the Proposition 4, but for the operators $G$ and $F$.

Proposition 5 Let us consider the mappings $G$ and $F$ with domain $I \times I$ in the following way

$$
G, F: I \times I \rightarrow I F S(X)^{I F S(X)} .
$$

These two mappings are non-decreasing with respect to the $\pi$ partial ordering on $I \times I$ and $\operatorname{IFS}(X)^{I F S(X)}$. That is, for $\star \in\{G, F\}$ we have that

$$
\left(\forall(\alpha, \beta),\left(\alpha_{1}, \beta_{1}\right) \in I \times I\right)\left((\alpha, \beta) \leq_{\pi}\left(\alpha_{1}, \beta_{1}\right) \Rightarrow \star_{(\alpha, \beta)} \leq_{\pi} \star_{(\alpha, \beta)}\right) .
$$

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