16th Int. Conf. on IFSs, Sofia, 9–10 Sept. 2012
Notes on Intuitionistic Fuzzy Sets
Vol. 18, 2012, No. 3, 30–38

On the intuitionistic fuzzy sets with metric type relation between the membership and non-membership functions

Peter Vassilev

Bioinformatics and Mathematical Modelling Department
Institute of Biophysics and Biomedical Engineering
Bulgarian Academy of Sciences
105 Acad. G. Bonchev Str., Sofia 1113, Bulgaria
e-mail: peter.vassilev@gmail.com

Abstract: In the paper the so-called d_φ-intuitionistic fuzzy set (d_φ-*IFS*), over the non-empty universe E, are considered for the case when d_φ is $\mathcal{R}^2$-metric induced by an arbitrary fixed absolute normalized $\mathcal{R}^2$-norm φ. It is proved that there exists a bijective isomorphism between the class of all such sets and the class of all intuitionistic fuzzy sets over E.
Keywords: Intuitionistic fuzzy set, d-intuitionistic fuzzy set, d_φ-intuitionistic fuzzy set, Norm, Absolute norm, Normalized norm, Absolute normalized norm, Injection, Surjection, Bijection, Isomorphism
AMS Classification: 03E72.

1 Basic definitions

The following definition (in another form) is contained in [1]:

Definition 1. Let $A \subset E$ and $\mu_A : E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ are mappings such that for any $x \in E$ the inequality

$$\mu_A(x) + \nu_A(x) \leq 1$$

holds. The set

$$\tilde{A} = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$$

is called intuitionistic fuzzy set (or Atanassov set) over E.

The mappings μ_A and ν_A are called membership and non-membership function, respectively. The map $\pi_A : E \to [0,1]$, that for $x \in E$ is introduced by

$$\pi_A(x) \stackrel{\text{def}}{=} 1 - \mu_A(x) - \nu_A(x),$$

is called hesitancy function.

The class of all intuitionistic fuzzy sets over E is denoted by IFS(E).

Definition 2. An $\mathcal{R}^2$-norm φ is called normalized norm if the equality

$$\varphi((1,0)) = \varphi((0,1)) = 1$$

holds. The class of all normalized $\mathcal{R}^2$-norms is denoted by N_2.

Definition 3. An $\mathcal{R}^2$-norm φ is called absolute norm if for any $(\mu,\nu) \in \mathcal{R}^2$ the equality

$$\varphi((\mu,\nu)) = \varphi((|\mu|,|\nu|))$$

holds. The class of all absolute normalized $\mathcal{R}^2$-norms is denoted by AN_2.

Let $\varphi \in N_2$. Then φ induced $\mathcal{R}^2$-metric d_φ by the formula

$$d_\varphi((\mu_1,\nu_1),(\mu_2,\nu_2)) = \varphi((|\mu_1-\mu_2|,|\nu_1-\nu_2|)).$$

Let d is $\mathcal{R}^2$-metric. In [8], for the first time, the notion d-intuitionistic fuzzy set (d-*IFS*) over E was introduced. Below we give the following

Definition 4. Let $\varphi \in N_2$, $A \subset E$ and $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ are mappings such that for any $x \in E$ the inequality

$$\varphi((\mu_A(x),\nu_A(x))) \le 1$$

holds. The set

$$\tilde{A} = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$$

is called d_φ-intuitionistic fuzzy set (d_φ-*IFS*) over E. The mappings μ_A and ν_A are called membership and non-membership function, respectively. The map $\pi_A : E \to [0,1]$, that for $x \in E$ is introduced by

$$\pi_A(x) \stackrel{\text{def}}{=} 1 - \varphi((\mu_A(x),\nu_A(x))),$$

is called hesitancy function.

The class of all d_φ-intuitionistic fuzzy sets over E is denoted by d_φ-IFS(E).

Definition 5. By Ψ_2 is denoted the class of all convex functions $\psi \in C[0,1]$ that for $t \in [0,1]$ satisfy the condition

$$\max(1-t,t) \le \psi(t) \le 1.$$

2 Introduction

The class AN_2 is well studied but yet still being investigated. For example, one may see: [3–7, 9]. Here we must note that for the $\mathcal{R}^2$-norm

$$\varphi((\mu,\nu)) \stackrel{\text{def}}{=} \sup_{t\in[0,1]} |\mu - t\nu|$$

we have $\varphi \in N_2$ but $\varphi \notin AN_2$, since $\varphi((1,-1)) = 2 \neq 1 = \varphi((|1|,|-1|))$.

The fundamental result for the class AN_2 is given by Bonsall and Duncan [2, p. 37, Lemma 3]. Below we give in the following form:

Theorem 1. *There exists a bijection between AN_2 and Ψ_2. Moreover, for any $\psi \in \Psi_2$ there exist a unique $\varphi \in AN_2$ such that*

$$(\forall t \in [0,1])(\psi(t) = \varphi((1-t,t))) \tag{1}$$

and for any $\varphi \in AN_2$ there exists a unique $\psi \in \Psi_2$, such that for $(\mu,\nu) \in \mathcal{R}^2$ we have

$$\varphi((\mu,\nu)) = \begin{cases} (|\mu|+|\nu|)\psi\left(\frac{|\nu|}{|\mu|+|\nu|}\right), \ \textit{if}\ (\mu,\nu) \neq (0,0) \\ 0,\ \textit{if}\ (\mu,\nu) = (0,0). \end{cases} \tag{2}$$

3 Main result

The following is the main result of the paper, showing the connection between d_φ-IFS(E) and IFS(E).

Theorem 2. *Let $\varphi \in AN_2$. Then there exists a bijective isomorphism between d_φ-*IFS(E) *and* IFS(E).

Proof. Let $\varphi \in AN_2$ be a fixed norm, $\psi \in \Psi_2$ be given by (1) and let T_φ be the mapping which juxtaposes to the set

$$A \stackrel{\text{def}}{=} \{\langle x, \mu(x), \nu(x)\rangle | x \in E\} \in d_\varphi\text{-IFS}(E)$$

the set B, that is given by

$$B \stackrel{\text{def}}{=} \{\langle x, \mu^*(x), \nu^*(x)\rangle | x \in E\},$$

where:

$$\mu^*(x) = \begin{cases} \mu(x)\psi\left(\frac{\nu(x)}{\mu(x)+\nu(x)}\right), \ \text{if } \mu(x)+\nu(x) > 0 \\ 0,\ \text{if } \mu(x)+\nu(x) = 0 \end{cases} ; \tag{3}$$

$$\nu^*(x) = \begin{cases} \nu(x)\psi\left(\frac{\nu(x)}{\mu(x)+\nu(x)}\right), \ \text{if } \mu(x)+\nu(x) > 0 \\ 0,\ \text{if } \mu(x)+\nu(x) = 0. \end{cases} \tag{4}$$

We will show that T_φ is a bijective isomorphism between d_φ-IFS(E) and IFS(E).

First we must establish that $B \in$ IFS(E).

The condition $A \in d_\varphi$-IFS(E) implies:

$$(\forall x \in E)(\varphi((\mu(x), \nu(x))) \leq 1). \tag{5}$$

From (2), (3) and (4) it follows

$$\mu^*(x) + \nu^*(x) = \begin{cases} (\mu(x) + \nu(x))\psi\left(\frac{\nu(x)}{\mu(x)+\nu(x)}\right), & \text{if } \mu(x) + \nu(x) > 0 \\ 0, \text{ if } \mu(x) + \nu(x) = 0 \end{cases} =$$

$$\varphi((\mu(x), \nu(x))) \leq 1.$$

Hence $B \in$ IFS(E).

Second, we will prove that T_φ is injection.

Let us assume the opposite. Then there would exist mappings $\mu_i : E \to [0,1]$, $\nu_i : E \to [0,1]$, $i = 1, 2$, such that:

$$(\mu_1, \nu_1) \neq (\mu_2, \nu_2); \tag{6}$$

$$(\mu_1^*, \nu_1^*) = (\mu_2^*, \nu_2^*) \tag{7}$$

Obviously, (6) means that the following condition holds:

(i_1) There exists $x_0 \in E$, such that at least one of the equalities:

$$\mu_1(x_0) = \mu_2(x_0);\ \nu_1(x_0) = \nu_2(x_0)$$

is violated.

On the other hand, (7) means that for any $x \in E$ it is fulfilled:

$$\mu_1^*(x) = \mu_2^*(x);\ \nu_1^*(x) = \nu_2^*(x).$$

In particular:

$$\mu_1^*(x_0) = \mu_2^*(x_0);\ \nu_1^*(x_0) = \nu_2^*(x_0). \tag{8}$$

For x_0 we have

(i_2) At least one of the equalities:

$$\mu_1(x_0) + \nu_1(x_0) = 0\ ;\ \mu_2(x_0) + \nu_2(x_0) = 0,$$

is violated.

The assumption that (i_2) is not true, yields:

$$\mu_1(x_0) = \nu_1(x_0) = \mu_2(x_0) = \nu_2(x_0) = 0,$$

which contradicts to (i_1).

Therefore, because of (i_2), there are only three possible cases:

(I) $\mu_1(x_0) + \nu_1(x_0) > 0$ & $\mu_2(x_0) + \nu_2(x_0) = 0$;

(II) $\mu_1(x_0)+\nu_1(x_0)=0$ & $\mu_2(x_0)+\nu_2(x_0)>0$;

(III) $\mu_1(x_0)+\nu_1(x_0)>0$ & $\mu_2(x_0)+\nu_2(x_0)>0$.

Let (I) hold. Then

$$\mu_2(x_0)=\nu_2(x_0)=0.$$

From (3) and (4) with: $\mu=\mu_2$; $\mu^*=\mu_2^*$; $\nu=\nu_2$; $\nu^*=\nu_2^*$; $x=x_0$, it follows:

$$\mu_2^*(x_0)=\nu_2^*(x_0)=0.$$

The above equalities and (8) yield:

$$\mu_1^*(x_0)=\mu_2^*(x_0)=0;\ \nu_1^*(x_0)=\nu_2^*(x_0). \tag{9}$$

Definition 5 provides

(i_3) $(\forall t\in[0,1])(\psi(t)>0)$.

Putting $\mu=\mu_1$; $\mu^*=\mu_1^*$; $\nu=\nu_1$; $\nu^*=\nu_1^*$; $x=x_0$ in (3) and (4), from (9) and (i_3) we obtain

$$\mu_1(x_0)=\nu_1(x_0)=0.$$

But the last contradicts to (I).

In the same manner the case (II) leads us to contradiction.

Let (III) hold. We put:

$$\psi\left(\frac{\nu_1(x_0)}{\mu_1(x_0)+\nu_1(x_0)}\right)=z;\ \psi\left(\frac{\nu_2(x_0)}{\mu_2(x_0)+\nu_2(x_0)}\right)=-w. \tag{10}$$

From (3), for: $\mu=\mu_1$; $\nu=\nu_1$; $\mu^*=\mu_1^*$; $x=x_0$, we obtain

$$\mu_1^*(x_0)=\mu_1(x_0)z \tag{11}$$

and for: $\mu=\mu_2$; $\nu=\nu_2$; $\mu^*=\mu_2^*$; $x=x_0$, we obtain

$$\mu_2^*(x_0)=-\mu_2(x_0)w. \tag{12}$$

From (4), for: $\mu=\mu_1$; $\nu=\nu_1$; $\nu^*=\nu_1^*$; $x=x_0$, we obtain

$$\nu_1^*(x_0)=\nu_1(x_0)z \tag{13}$$

and for: $\mu=\mu_2$; $\nu=\nu_2$; $\nu^*=\nu_2^*$; $x=x_0$, we obtain

$$\nu_2^*(x_0)=-\nu_2(x_0)w. \tag{14}$$

Then, because of (8) we get the following linear homogeneous system with unknowns z and w :

$$\begin{cases}\mu_1(x_0)z+\mu_2(x_0)w=0\\ \nu_1(x_0)z+\nu_2(x_0)w=0.\end{cases} \tag{15}$$

Now (i_3) and (10) imply $z \neq 0$ and $w \neq 0$, i.e. the linear homogeneous system has a non-trivial soslution. Then, because of the well known result of the linear algebra, the determinant:

$$\Delta = \begin{vmatrix} \mu_1(x_0) & \mu_2(x_0) \\ \nu_1(x_0) & \nu_2(x_0) \end{vmatrix}$$

equals to 0.

This means that the vector-columns of Δ are linearly dependent. Then, due to (III), these vectors are different from the zero-vector. Hence, there exists a real number $k \neq 0$, such that:

$$\mu_2(x_0) = k\mu_1(x_0);\ \nu_2(x_0) = k\nu_1(x_0).$$

The last two equalities imply:

$$\psi\left(\frac{\nu_2(x_0)}{\mu_2(x_0) + \nu_2(x_0)}\right) = \psi\left(\frac{k\nu_1(x_0)}{k\mu_1(x_0) + k\nu_1(x_0)}\right) = \psi\left(\frac{\nu_1(x_0)}{\mu_1(x_0) + \nu_1(x_0)}\right).$$

The above equalities and (10) yield $-w = z$. Hence, from (11)-(14), we obtain:

$$\mu_1^*(x_0) = z\mu_1(x_0);\ \mu_2^*(x_0) = z\mu_2(x_0);\ \nu_1^*(x_0) = z\nu_1(x_0);\ \nu_2^*(x_0) = z\nu_2(x_0).$$

The last equalities and (8) yield:

$$z\mu_1(x_0) = z\mu_2(x_0);\ z\nu_1(x_0) = z\nu_2(x_0).$$

Hence:

$$\mu_1(x_0) = \mu_2(x_0);\ \nu_1(x_0) = \nu_2(x_0),$$

since $z \neq 0$. But the last contradicts to (i_1), and therefore to (6).

Thus, we proved that T_φ is injection.

Third, we will prove that T_φ is surjection.

Let $B \stackrel{\text{def}}{=} \{\langle x, \mu^*(x), \nu^*(x)\rangle | x \in E\} \in \text{IFS}(E)$. For any $x \in E$ we put:

$$\mu(x) = \begin{cases} \frac{\mu^*(x)}{\psi\left(\frac{\nu^*(x)}{\mu^*(x)+\nu^*(x)}\right)},\ \text{if } \mu^*(x) + \nu^*(x) > 0 \\ 0,\ \text{if } \mu^*(x) + \nu^*(x) = 0; \end{cases} \tag{16}$$

$$\nu(x) = \begin{cases} \frac{\nu^*(x)}{\psi\left(\frac{\nu^*(x)}{\mu^*(x)+\nu^*(x)}\right)},\ \text{if } \mu^*(x) + \nu^*(x) > 0 \\ 0,\ \text{if } \mu^*(x) + \nu^*(x) = 0. \end{cases} \tag{17}$$

We will show that:

$$\mu : E \to [0,1]; \nu : E \to [0,1]. \tag{18}$$

Let $x \in E$ is such that $\mu^*(x) + \nu^*(x) = 0$. Then (16) and (17) imply: $\mu(x) = 0$ and $\nu(x) = 0$, i.e. $\mu(x), \nu(x) \in [0,1]$.

Let $x \in E$ is such that $\mu^*(x) + \nu^*(x) > 0$. Then (16) and (17) yield:

$$\mu(x) = \frac{\mu^*(x)}{\psi\left(\frac{\nu^*(x)}{\mu^*(x)+\nu^*(x)}\right)};\ \nu(x) = \frac{\nu^*(x)}{\psi\left(\frac{\nu^*(x)}{\mu^*(x)+\nu^*(x)}\right)}\ . \tag{19}$$

We put

$$t = \frac{\nu^*(x)}{\mu^*(x) + \nu^*(x)}.$$

Since $\psi \in \Psi_2$, then Definition 5 implies:

$$\psi\left(\frac{\nu^*(x)}{\mu^*(x) + \nu^*(x)}\right) = \psi(t) \geq \max(t, 1-t) = \max\left(\frac{\nu^*(x)}{\mu^*(x) + \nu^*(x)}, \frac{\mu^*(x)}{\mu^*(x) + \nu^*(x)}\right).$$

The last and (19) imply that (18) will be proved if the following inequalities hold:

$$\mu^*(x) \leq \max\left(\frac{\nu^*(x)}{\mu^*(x) + \nu^*(x)}, \frac{\mu^*(x)}{\mu^*(x) + \nu^*(x)}\right)$$

$$\nu^*(x) \leq \max\left(\frac{\nu^*(x)}{\mu^*(x) + \nu^*(x)}, \frac{\mu^*(x)}{\mu^*(x) + \nu^*(x)}\right).$$

But these inequalities follow from the inequality

$$\mu^*(x) + \nu^*(x) \leq 1, \tag{20}$$

which is true, since $B \in \mathrm{IFS}(E)$.

We will prove that $\mu(x)$ and $\nu(x)$, given by (16) and (17), satisfy (5).

According to (2) we have

$$\varphi(\mu(x), \nu(x)) = \begin{cases} (\mu(x) + \nu(x))\psi\left(\frac{\nu(x)}{\mu(x)+\nu(x)}\right), \text{ if } \mu(x) + \nu(x) > 0 \\ 0, \text{ if } \mu(x) + \nu(x) = 0. \end{cases} \tag{21}$$

Equalities (16) , (17) and (i_3) imply that

$$\mu(x) + \nu(x) = 0 \text{ if and only if } \mu^*(x) + \nu^*(x) = 0.$$

From the last it follows that (21) may be rewritten as

$$\varphi(\mu(x), \nu(x)) = \begin{cases} (\mu(x) + \nu(x))\psi\left(\frac{\nu(x)}{\mu(x)+\nu(x)}\right), \text{ if } \mu^*(x) + \nu^*(x) > 0 \\ 0, \text{ if } \mu^*(x) + \nu^*(x) = 0. \end{cases} \tag{22}$$

Let $x \in E$ is such that $\mu^*(x) + \nu^*(x) = 0$. Then $\mu(x) + \nu(x) = 0$. Hence: $\mu(x) = 0;\ \nu(x) = 0$ and $\varphi(\mu(x), \nu(x)) = 0$, i.e. (5) holds.

Let $x \in E$ is such that $\mu^*(x) + \nu^*(x) > 0$. Then (16), (17) and (22) yield

$$\varphi(\mu(x), \nu(x)) = \frac{\mu^*(x) + \nu^*(x)}{\psi\left(\frac{\nu^*(x)}{\mu^*(x)+\nu^*(x)}\right)}\psi\left(\frac{\nu(x)}{\mu(x) + \nu(x)}\right). \tag{23}$$

Equalities (16) and (17) imply

$$\psi\left(\frac{\nu(x)}{\mu(x) + \nu(x)}\right) = \psi\left(\frac{\dfrac{\nu^*(x)}{\psi\left(\dfrac{\nu^*(x)}{\mu^*(x) + \nu^*(x)}\right)}}{\dfrac{\mu^*(x)}{\psi\left(\dfrac{\nu^*(x)}{\mu^*(x) + \nu^*(x)}\right)} + \dfrac{\nu^*(x)}{\psi\left(\dfrac{\nu^*(x)}{\mu^*(x) + \nu^*(x)}\right)}}\right).$$

Hence (because of (i_3))

$$\psi\left(\frac{\nu(x)}{\mu(x)+\nu(x)}\right)=\psi\left(\frac{\nu^*(x)}{\mu^*(x)+\nu^*(x)}\right). \tag{24}$$

Equalities (23) and (24) yield

$$\varphi(\mu(x),\nu(x))=\mu^*(x)+\nu^*(x).$$

The last equality and (20) immediately prove (5).

Let $A \stackrel{\text{def}}{=} \{\langle x,\mu(x),\nu(x)\rangle | x\in E\}$. From the proved (5) and (18) it follows: $A\in d_\varphi$-IFS(E).

Equalities (3), (4) and (24) immediately yield

$$T_\varphi(A)=B.$$

Hence: T_φ is surjection. Therefore, T_φ is bijection.

Theorem 2 is proved. □

Remark 1. From the proof of Theorem 2 it is seen that T_φ is injection also for the case: $\varphi\in N\setminus AN_2$. But in this case it is not guaranteed that T_φ is surjection. The last means that for $\varphi\in N\setminus AN_2$ it is not certain (in the general case) that T_φ is bijection.

From the proof of Theorem 2 we obtain the following

Corollary 1. *The mappings T_φ and T_φ^{-1} admit the representations:*

$$T_\varphi\langle\mu(x),\nu(x)\rangle=\begin{cases}\langle\frac{\mu(x)\varphi((\mu(x),\nu(x)))}{\mu(x)+\nu(x)},\frac{\nu(x)\varphi((\mu(x),\nu(x)))}{\mu(x)+\nu(x)}\rangle, & \textit{if } \mu(x)+\nu(x)\neq 0\\ \langle 0,0\rangle, \textit{ if } \mu(x)=\nu(x)=0,\end{cases}$$

where μ and ν are the membership and non-membership functions of an element from the class d_φ-IFS*(E);*

$$T_\varphi^{-1}\langle\mu(x),\nu(x)\rangle=\begin{cases}\langle\mu(x)\frac{\mu(x)+\nu(x)}{\varphi((\mu(x),\nu(x)))},\nu(x)\frac{\mu(x)+\nu(x)}{\varphi((\mu(x),\nu(x)))}\rangle, & \textit{if } \mu(x)+\nu(x)>0\\ \langle 0,0\rangle, \textit{ if } \mu(x)=\nu(x)=0,\end{cases}$$

where μ and ν are the membership and non-membership functions of an element from the class IFS*(E).*

Another Corollary from Theorem 2 is:

Theorem 3. *Let $\varphi,\varphi^*\in AN_2$. Then the mapping $T_{\varphi,\varphi^*}: d_\varphi$-IFS$(E)\to d_{\varphi^*}$-IFS$(E)$, which is given by*

$$T_{\varphi,\varphi^*}\stackrel{\text{def}}{=}T_{\varphi^*}^{-1}T_\varphi,$$

is a bijective isomorphism between d_φ-IFS(E) and d_{φ^*}-IFS(E).

Acknowledgements

The author is grateful for the support provided by the National Science Fund of Bulgaria under grants DID-02-29/2009 and DMU-03-38/2011.

References

[1] Atanassov, K. *Intuitionistic Fuzzy Sets.* Springer Physica-Verlag, Heidelberg, 1999.

[2] Bonsall, F., J. Duncan. Numerical Ranges II, *London Math. Soc., Lecture Notes Ser.,* Vol. 10, 1973.

[3] Kato, M., T. Tamura. Weakly nearly uniform smoothness and worth property of Ψ-direct sums of Banach spaces. *Comment. Math. Prace Mat.,* Vol. 46, 2006, 113—129

[4] Mitani K., K. Saito, T. Suzuki. Smoothness of absolute norms on $\mathbb{C}^n$, *J. Convex Anal.* Vol. 10, 2003, 89-–107.

[5] Saito, K.-S., M. Kato, Y. Takahashi. Absolute norms on $\mathbb{C}^n$, *J. Math. Anal. Appl.,* Vol. 252, 2000, 879-–905.

[6] Suzuki, T., A. Yamano, M. Kato. The James Constant of 2-Dimensional Lorentz Sequence Spaces. *Bull. Kyushu Inst. Tech. Pure Appl. Math.* Vol. 53, 2006, 15–24

[7] Takahashi, Y., M. Kato, K.-S. Saito. Strict convexity of absolute norms on $\mathbb{C}^2$ and direct sums of Banach spaces, *J. Inequal. Appl.* Vol. 7, 2002, 179–186.

[8] Vassilev, P. A Metric Approach To Fuzzy Sets And Intuitionistic Fuzzy Sets. *Proc. of. First Int. Workshop on IFSs, GNs, KE,* Sept. 2006, 31–38.

[9] Zuo, Z. The Ptolemy constant of absolute normalized norms on $\mathbb{R}^2$. *Journal of Inequalities and Applications,* 2012, 2012:107