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Cartesian products of intuitionistic fuzzy normal subgroups via level subsets

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Abstract: We study normality for specified finite min–max products of intuitionistic fuzzy subgroups (IFSGs). Using the known level-subset identity for min–max products, we separate preservation of normality from recovery of factor normality. Normality of all factors implies that their product is an intuitionistic fuzzy normal subgroup (IFNSG), with no uniformity requirement on identity grades. Uniform membership and non-membership grades at the factor identities suffice to recover normality of every factor from normality of the product. An example on $S_3 \times \mathbb{Z}_2$ gives two IFSG factors with a normal product and a nonnormal factor. At a threshold detecting nonnormality, the other factor has an empty level subset, so the defect is invisible in the product. This shows that uniformity cannot simply be omitted from the general recovery theorem. We also establish normalized projection formulas and illustrate the results with the matrix groups $\mathrm{GL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{R})$.

Keywords: Cartesian product, Intuitionistic fuzzy normal subgroup, Level subset, Direct product, Group theory.
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1 Introduction

Zadeh [14] introduced fuzzy sets. Atanassov [3, 4] developed intuitionistic fuzzy sets, in which membership and non-membership grades lie in $[0, 1]$ and have sum at most 1. Recording both quantities distinguishes non-membership from hesitation.

Rosenfeld [12] developed fuzzy group theory, and Biswas [8] studied intuitionistic fuzzy subgroups. Hur, Jang, and Kang [10] investigated intuitionistic fuzzy normal subgroups and intuitionistic fuzzy cosets. Here normality is expressed through the behavior of the membership and non-membership functions under conjugation.

Cartesian products of intuitionistic fuzzy sets and their properties were studied in [5, 9]. For IFSGs defined on a common group G, Abdurrahman [1, Theorem 3.6 and Corollary 3.7] proves that their min–max product is an IFSG on $G \times G$ and, more generally, on G^n for a finite family of factors.

For general IFSs, Deschrijver and Kerre [9, Section 2.7] studied cutting formulas for Cartesian products. In the IFSG literature, Sharma [13, Proposition 3.2] recorded the two-factor level-subset identity

$$C_{\alpha,\beta}(A \times B) = C_{\alpha,\beta}(A) \times C_{\alpha,\beta}(B).$$

He also proved that normality of the factors implies normality of their min–max product [13, Theorem 3.4]. His Remark 3.5 and Example 3.6 show that a product can be an intuitionistic fuzzy subgroup even when one factor fails the IFSG condition. Proposition 3.8 gives a sufficient condition for recovering one factor: identity-grade bounds ensure that restriction of the product to a coordinate subgroup reproduces that factor.

Nonempty admissible level subsets of an IFSG are ordinary subgroups.

The present paper studies recovery of factor normality when every factor is already an IFSG. Product normality constrains only those level subsets that are simultaneously nonempty in all factors. We record the finite-factor forms of the level-subset identity and normality-preservation result, then use level subsets to analyze recovery. Uniform identity grades make the coordinate restrictions reproduce every factor and hence suffice to recover factor normality from product normality. This uses the restriction mechanism underlying Sharma's Proposition 3.8. Example 3.6 on $S_3 \times \mathbb{Z}_2$ has two IFSG factors and a normal product, while the first factor is nonnormal. The level at which that factor fails normality is absent from the product because the other factor has an empty level at the same threshold. This shows that uniformity cannot simply be omitted from the general recovery theorem. The results do not classify all nonuniform families for which recovery holds. The normalized projection formulas and the matrix-group example provide supporting consequences and illustrations.

The paper is organized as follows. Section 2 recalls the background. Section 3 presents the level-subset identity, preservation of normality, recovery under uniform identity grades, a counterexample to recovery without uniformity, and normalized projections. Section 4 gives the matrix-group example.

2 Preliminaries

We recall the basic notions needed later, mainly to fix notation and to isolate those properties that interact well with level subsets and Cartesian products.

Definition 2.1. [3, 4, 6] *Let X be a nonempty set. An intuitionistic fuzzy set (IFS) A in X is defined as*

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\},$$

where $\mu_A, \nu_A : X \to [0, 1]$ satisfy

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \quad \textit{for all } x \in X.$$

Here, $\mu_A(x)$ represents the degree of membership of x in A, while $\nu_A(x)$ represents the degree of non-membership of x in A. The quantity

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$$

is called the hesitation degree (or intuitionistic fuzzy index), which measures the uncertainty associated with the membership of x in A.

Definition 2.2. [8] *Let G be a group. An IFS $A = (\mu_A, \nu_A)$ in G is called an* intuitionistic fuzzy subgroup *(IFSG) of G if, for all $x, y \in G$,*

$$\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}, \quad \nu_A(xy) \leq \max\{\nu_A(x), \nu_A(y)\},$$

$$\mu_A(x^{-1}) \geq \mu_A(x), \quad \nu_A(x^{-1}) \leq \nu_A(x).$$

The inverse inequalities are equalities, since applying them to x^{-1} reverses them. Applying the product inequalities to $xx^{-1} = e$ and using the inverse equalities gives $\mu_A(e) \geq \mu_A(x)$ and $\nu_A(e) \leq \nu_A(x)$ for every $x \in G$. Component-wise, μ_A is a fuzzy subgroup and ν_A an anti-fuzzy subgroup [7], subject to the IFS compatibility condition.

Definition 2.3. [10] *An IFSG $A = (\mu_A, \nu_A)$ of a group G is called an* intuitionistic fuzzy normal subgroup *(IFNSG) if, for all $x, g \in G$,*

$$\mu_A(xgx^{-1}) \geq \mu_A(g), \quad \nu_A(xgx^{-1}) \leq \nu_A(g).$$

This condition can also be interpreted component-wise: the function μ_A satisfies the condition of a fuzzy normal subgroup, while the function ν_A satisfies the condition of an anti-fuzzy normal subgroup. This reflects a dual stability under conjugation in the intuitionistic fuzzy setting.

Definition 2.4. [3, Section III] *Let $A = (\mu_A, \nu_A)$ be an IFS in a group G. For $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$, the* (α, β)-level subset *of A is*

$$C_{\alpha,\beta}(A) = \{\, x \in G \mid \mu_A(x) \geq \alpha,\ \nu_A(x) \leq \beta \,\}.$$

Remark 2.5. Atanassov [3, Section III] used $A_{\alpha,\beta}$ for this two-threshold level construction. We write $C_{\alpha,\beta}(A)$ and use the admissibility convention $\alpha + \beta \leq 1$, as in Sharma [13, Definition 2.3]. Ahn, Hur, and Jang [2] studied level subgroups of intuitionistic fuzzy subgroups.

For an IFSG A of a group G, every nonempty admissible $C_{\alpha,\beta}(A)$ is a subgroup of G; cf. Sharma [13, Theorem 2.4]. We use the standard normality criterion: A is an IFNSG if and only if every nonempty admissible $C_{\alpha,\beta}(A)$ is normal in G; cf. [13, Proposition 2.8]. Empty levels impose no condition in this criterion.

3 Main results

We recall the finite-product level identity, then examine recovery of factor normality and its obstruction.

3.1 Cartesian products and level subsets

We use the min–max Cartesian product of IFSs, described as the type 4 product in the IFS literature [6,9]. Its use for intuitionistic fuzzy subgroups appears in [1,13]. Fix an integer $n \geq 1$. Let G_i be groups and let $A_i = (\mu_{A_i}, \nu_{A_i})$ be IFSs on G_i, for $i = 1, \ldots, n$.

Definition 3.1. [6,9] *The* min–max Cartesian product *of* $\{A_i\}_{i=1}^n$ *is the IFS*

$$\mathcal{A} := \prod_{i=1}^{n} A_i \quad \textit{on} \quad \mathcal{G} := \prod_{i=1}^{n} G_i,$$

defined by

$$\mu_{\mathcal{A}}(x_1, \ldots, x_n) = \min_{1 \leq i \leq n} \mu_{A_i}(x_i), \qquad \nu_{\mathcal{A}}(x_1, \ldots, x_n) = \max_{1 \leq i \leq n} \nu_{A_i}(x_i), \tag{1}$$

for all $(x_1, \ldots, x_n) \in \mathcal{G}$.

For each $x \in \mathcal{G}$, choose j attaining $\max_i \nu_{A_i}(x_i)$. Then $\mu_{\mathcal{A}}(x) + \nu_{\mathcal{A}}(x) \leq \mu_{A_j}(x_j) + \nu_{A_j}(x_j) \leq 1$, so the product is indeed an IFS.

The following identity is the finite-factor form of the two-factor identity recorded by Sharma [13, Proposition 3.2]; related cutting formulas for general IFSs appear earlier in Deschrijver and Kerre [9, Section 2.7].

Lemma 3.2. *Let* $\mathcal{A} = \prod_{i=1}^{n} A_i$ *be the min–max Cartesian product given in Definition 3.1. Then, for every* $\alpha, \beta \in [0,1]$ *with* $\alpha + \beta \leq 1$*, we have*

$$C_{\alpha,\beta}(\mathcal{A}) = \prod_{i=1}^{n} C_{\alpha,\beta}(A_i).$$

Proof. For $x = (x_1, \ldots, x_n) \in \mathcal{G}$,

$$\begin{aligned} x \in C_{\alpha,\beta}(\mathcal{A}) &\iff \min_i \mu_{A_i}(x_i) \geq \alpha \text{ and } \max_i \nu_{A_i}(x_i) \leq \beta \\ &\iff \mu_{A_i}(x_i) \geq \alpha \text{ and } \nu_{A_i}(x_i) \leq \beta \quad \text{for every } i \\ &\iff x \in \prod_{i=1}^{n} C_{\alpha,\beta}(A_i). \end{aligned}$$

This proves the identity, including cases with an empty factor level. □

3.2 Normality of Cartesian products

Common identity grades make each nonempty factor level visible in the product, providing a sufficient hypothesis for recovery of factor normality.

Theorem 3.3. *Let $A_i = (\mu_{A_i}, \nu_{A_i})$ be intuitionistic fuzzy subgroups of G_i for $i = 1, \ldots, n$, and let*

$$\mathcal{A} = \prod_{i=1}^{n} A_i$$

be their min–max Cartesian product on

$$\mathcal{G} = \prod_{i=1}^{n} G_i.$$

Assume that there exist $s, t \in [0, 1]$ with $s + t \leq 1$ such that

$$\mu_{A_i}(e_i) = s \quad \textit{and} \quad \nu_{A_i}(e_i) = t \qquad \textit{for all } i = 1, \ldots, n,$$

where e_i denotes the identity element of G_i. Then $\mathcal{A}$ is an IFNSG of $\mathcal{G}$ if and only if each A_i is an IFNSG of G_i.

Proof. The factor hypotheses imply that $\mathcal{A}$ is an IFSG, by finite iteration of Sharma [13, Theorem 3.3].

Preservation. Suppose each A_i is an IFNSG. By Lemma 3.2, every nonempty admissible level of $\mathcal{A}$ is a product of nonempty normal subgroups $C_{\alpha,\beta}(A_i) \trianglelefteq G_i$. It is therefore normal in $\mathcal{G}$, so $\mathcal{A}$ is an IFNSG. This argument does not use uniform identity grades.

Recovery. Suppose $\mathcal{A}$ is an IFNSG and fix k. For an admissible pair with $H_k = C_{\alpha,\beta}(A_k) \neq \varnothing$, the identity bounds for an IFSG give $\alpha \leq s$ and $t \leq \beta$. Hence $e_j \in H_j := C_{\alpha,\beta}(A_j)$ for every j, and

$$K = C_{\alpha,\beta}(\mathcal{A}) = \prod_{j=1}^{n} H_j$$

is nonempty and normal in $\mathcal{G}$. For $g \in G_k$ and $h \in H_k$, conjugating $(e_1, \ldots, h, \ldots, e_n) \in K$ by $(e_1, \ldots, g, \ldots, e_n)$ shows that $ghg^{-1} \in H_k$. Thus $H_k \trianglelefteq G_k$. The nonempty-level criterion gives that A_k is an IFNSG, and k was arbitrary. □

3.3 Preservation and recovery of normality

The preservation argument above is the finite-factor form of Sharma [13, Theorem 3.4] and requires no uniformity.

Proposition 3.4. *Let $A_i = (\mu_{A_i}, \nu_{A_i})$ be intuitionistic fuzzy subgroups of G_i for $i = 1, \ldots, n$, and let $\mathcal{A} = \prod_{i=1}^{n} A_i$ be their min–max Cartesian product on $\mathcal{G} = \prod_{i=1}^{n} G_i$. If each A_i is an IFNSG of G_i, then $\mathcal{A}$ is an IFNSG of $\mathcal{G}$.*

Proof. The preservation argument in the proof of Theorem 3.3 uses no uniformity assumption on identity grades and applies here. □

Remark 3.5. *Normality of a min–max product controls the factor levels only at thresholds where all factor levels are nonempty. Uniform identity grades ensure that this occurs whenever any one factor level is nonempty.*

Example 3.6. We construct two groups G_1, G_2 with IFSGs A_1, A_2 having non-uniform identity values, such that $A_1 \times A_2$ is an IFNSG of $G_1 \times G_2$ yet A_1 is not an IFNSG of G_1.

Let $G_1 = S_3 = \{e, r, r^2, s, sr, sr^2\}$, where $r = (123)$ and $s = (12)$, so that $r^3 = s^2 = e$ and $sr = r^{-1}s$. Let $G_2 = \mathbb{Z}_2 = \{0, 1\}$ under addition modulo 2.

Define $A_1 = (\mu_1, \nu_1)$ on S_3 by

$$\mu_1(x) = \begin{cases} 0.8 & \text{if } x = e, \\ 0.65 & \text{if } x = s, \\ 0.45 & \text{if } x \in \{r, r^2, sr, sr^2\}, \end{cases} \qquad \nu_1(x) = \begin{cases} 0.1 & \text{if } x = e, \\ 0.15 & \text{if } x = s, \\ 0.3 & \text{if } x \in \{r, r^2, sr, sr^2\}. \end{cases}$$

The nonempty membership superlevel sets and non-membership sublevel sets of A_1 belong to the subgroup chain $\{e\} \leq \{e, s\} \leq S_3$. Closure of these subgroups under products and inverses gives the IFSG inequalities. The identity grades are $\mu_1(e) = 0.8$ and $\nu_1(e) = 0.1$.

Define $A_2 = (\mu_2, \nu_2)$ on $\mathbb{Z}_2$ by

$$\mu_2(0) = 0.4, \quad \mu_2(1) = 0.3, \qquad \nu_2(0) = 0.3, \quad \nu_2(1) = 0.5.$$

Then A_2 is an IFSG of $\mathbb{Z}_2$, with identity values $\mu_2(0) = 0.4$ and $\nu_2(0) = 0.3$. Since $\mathbb{Z}_2$ is Abelian, A_2 is also an IFNSG of $\mathbb{Z}_2$. Note that the identity values are non-uniform: $\mu_1(e) = 0.8 \neq 0.4 = \mu_2(0)$.

Step 1: A_1 is not an IFNSG of S_3. Consider the admissible pair $(\alpha, \beta) = (0.55,\ 0.2)$ with $\alpha + \beta = 0.75 \leq 1$. Then

$$C_{0.55,\,0.2}(A_1) = \{x \in S_3 \mid \mu_1(x) \geq 0.55,\ \nu_1(x) \leq 0.2\} = \{e, s\}.$$

The set $\{e, s\} = \{e, (12)\}$ is a subgroup of S_3, but it is not normal: indeed, $r\, s\, r^{-1} = r(12)r^{-1} = (23) = sr \notin \{e, s\}$. Since $C_{0.55,\,0.2}(A_1)$ is not a normal subgroup of S_3, we conclude that A_1 is not an IFNSG of S_3.

Step 2: $A_1 \times A_2$ is an IFNSG of $S_3 \times \mathbb{Z}_2$. Both factors are IFSGs, so their min–max product is an IFSG. For every $x \in S_3$ and $z \in \mathbb{Z}_2$,

$$\mu_1(x) \geq 0.45 > \mu_2(z), \qquad \nu_1(x) \leq 0.3 \leq \nu_2(z).$$

Consequently,

$$\mu_{A_1 \times A_2}(x, z) = \mu_2(z), \qquad \nu_{A_1 \times A_2}(x, z) = \nu_2(z).$$

Thus $C_{\alpha,\beta}(A_1 \times A_2) = S_3 \times C_{\alpha,\beta}(A_2)$. It is empty exactly when $\alpha > 0.4$ or $\beta < 0.3$. Otherwise it is $S_3 \times \{0\}$ or $S_3 \times \mathbb{Z}_2$, both normal in $S_3 \times \mathbb{Z}_2$. Empty levels impose no condition, so the nonempty-level criterion proves product normality.

The product grades depend only on A_2 and conceal the nonnormal level $C_{0.55,\,0.2}(A_1) = \{e, s\}$, since $C_{0.55,\,0.2}(A_2) = \varnothing$.

In Sharma's Example 3.6 [13], one factor fails the IFSG condition. In Example 3.6, both factors are IFSGs, while the first is nonnormal. The example therefore isolates the obstruction to recovering normality within the class of IFSG factors.

Remark 3.7. *Proposition 3.4 gives preservation of normality without uniformity. Theorem 3.3 gives recovery under common identity grades, while Example 3.6 shows that recovery can fail without them. Thus uniformity cannot simply be omitted from the general recovery theorem; these results do not classify all nonuniform families allowing recovery.*

3.4 Projection principle

In this subsection, A_i are IFSs on G_i and $\mathcal{A} = \prod_i A_i$. For an IFS, $\pi_k(\mathcal{A})$ denotes its extension-principle image; for an ordinary subset S, $\pi_k(S)$ denotes its set image. The normalization below concerns only the other factor identities.

Remark 3.8. *Normalizing all factor identities to* $(1, 0)$ *implies the uniformity hypothesis of Theorem 3.3. For a fixed projection onto* G_k*, only the factors with* $j \neq k$ *are normalized; the identity grades of* A_k *need not be* $(1, 0)$.

Write $\iota_k(y) = (e_1, \ldots, e_{k-1}, y, e_{k+1}, \ldots, e_n)$ for the coordinate embedding.

Lemma 3.9 (Projection formula)**.** *Let* $\pi_k : \mathcal{G} \to G_k$ *be the canonical projection onto the* k*-th coordinate. Assume that for every* $j \neq k$*,*

$$\mu_{A_j}(e_j) = 1 \quad \textit{and} \quad \nu_{A_j}(e_j) = 0,$$

where e_j *denotes the identity element of* G_j*. Then*

$$\pi_k(\mathcal{A}) = A_k.$$

Proof. For $y \in G_k$, the extension principle [11, p. 40, Equations (30)–(31)] gives

$$\mu_{\pi_k(\mathcal{A})}(y) = \sup_{x_k = y} \mu_{\mathcal{A}}(x), \qquad \nu_{\pi_k(\mathcal{A})}(y) = \inf_{x_k = y} \nu_{\mathcal{A}}(x).$$

For every such x, the product definition gives $\mu_{\mathcal{A}}(x) \leq \mu_{A_k}(y)$ and $\nu_{\mathcal{A}}(x) \geq \nu_{A_k}(y)$. At $x = \iota_k(y)$, auxiliary normalization makes both inequalities equalities. The supremum and infimum therefore equal the corresponding grades of A_k. □

The same identity witness also recovers the target level under ordinary set projection.

Theorem 3.10. *Let* $\mathcal{A} = \prod_{i=1}^n A_i$ *be the min–max Cartesian product on*

$$\mathcal{G} = \prod_{i=1}^{n} G_i,$$

and let $\pi_k : \mathcal{G} \to G_k$ *be the canonical projection onto the* k*-th coordinate. Assume that for every* $j \neq k$*,*

$$\mu_{A_j}(e_j) = 1 \quad \textit{and} \quad \nu_{A_j}(e_j) = 0.$$

Then, for every admissible pair (α, β) *with* $\alpha + \beta \leq 1$*,*

$$\pi_k\big(C_{\alpha,\beta}(\mathcal{A})\big) = C_{\alpha,\beta}(A_k).$$

Proof. For each $j \neq k$, normalization gives $e_j \in C_{\alpha,\beta}(A_j)$. Lemma 3.2 implies that the k-th coordinate of every product-level element belongs to $C_{\alpha,\beta}(A_k)$. Conversely, if $y \in C_{\alpha,\beta}(A_k)$, then $\iota_k(y) \in C_{\alpha,\beta}(\mathcal{A})$. These two observations prove the equality, also when the target level is empty. □

Normality of the target factor follows as well.

Proposition 3.11. *Assume the normalization hypotheses of Theorem 3.10. If $\mathcal{A}$ is an IFNSG of $\mathcal{G}$, then A_k is an IFNSG of G_k. In particular, by Lemma 3.9, the projection $\pi_k(\mathcal{A})$ is an IFNSG of G_k.*

Proof. Under auxiliary normalization,

$$\mu_{A_k} = \mu_{\mathcal{A}} \circ \iota_k, \qquad \nu_{A_k} = \nu_{\mathcal{A}} \circ \iota_k.$$

Since ι_k is a homomorphism, the IFSG axioms of $\mathcal{A}$ transfer to A_k. If $H = C_{\alpha,\beta}(A_k)$ is nonempty, Theorem 3.10 gives $H = \pi_k(K)$ for $K = C_{\alpha,\beta}(\mathcal{A})$, so K is nonempty. Product normality gives $K \trianglelefteq \mathcal{G}$. Its image under the surjective homomorphism π_k is normal in G_k. The nonempty-level criterion therefore makes A_k an IFNSG. Finally, Lemma 3.9 gives $\pi_k(\mathcal{A}) = A_k$. □

4 Illustrative non-Abelian example

Let $G = \mathrm{GL}_2(\mathbb{R})$ and $H = \mathrm{SL}_2(\mathbb{R}) = \ker(\det : G \to \mathbb{R}^\times)$. Thus $H \trianglelefteq G$. Define

$$(\mu_A(X), \nu_A(X)) = \begin{cases} (1,0), & X \in H, \\ (0.6, 0.3), & X \notin H. \end{cases}$$

The IFS constraint holds. If $X, Y \in H$, then $XY \in H$; otherwise the required membership minimum and non-membership maximum are 0.6 and 0.3, which bound all grades of A. Inversion preserves membership in H, and $\det(YXY^{-1}) = \det(X)$ gives conjugation invariance of both grades. Hence A is an IFNSG of G.

For every admissible pair,

$$C_{\alpha,\beta}(A) = \begin{cases} G, & \alpha \leq 0.6 \text{ and } \beta \geq 0.3, \\ H, & \text{otherwise.} \end{cases}$$

Proposition 3.4 makes $A \times A$ an IFNSG of $G \times G$. By Lemma 3.2, its levels are $G \times G$ and $H \times H$ at the corresponding thresholds. Since $(\mu_A(I), \nu_A(I)) = (1, 0)$, Lemma 3.9 also gives $\pi_1(A \times A) = \pi_2(A \times A) = A$.

5 Conclusion

For a specified finite min–max product of IFSGs, Theorem 3.3 gives, under common identity grades, an equivalence between product normality and normality of all the given factors. The product representation is part of its hypotheses; a factorization criterion for an arbitrary IFNSG

on a direct-product group is not established here. We record the finite-factor form of the known level-subset identity for min–max products [9, 13] and the preservation-of-normality result [13]. Normality is preserved without uniformity, while common identity grades suffice for recovery of factor normality. The example on $S_3 \times \mathbb{Z}_2$ shows that recovery can fail without uniformity because a nonnormal factor level can be absent from the product. Uniformity therefore cannot simply be omitted from the general recovery theorem. The results do not classify all nonuniform families for which recovery still holds. The normalized projection formulas and matrix-group example provide supporting consequences and illustrations.

A separate question for further study is to identify conditions under which an IFNSG on a direct-product group admits a min–max representation by IFSGs on the factor groups. By Lemma 3.2, such a representation requires a coordinatewise Cartesian product structure at every admissible level. Building on existing intuitionistic fuzzy coset theory [10], one may also ask how a precisely specified intuitionistic fuzzy quotient construction interacts with the min–max product and which additional hypotheses ensure compatibility.

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