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INTUITIONISTIC FUZZY SET ENERGIES

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Abstract: The concepts of energies of intuitionistic fuzzy sets that were for the first time paid attention to in [9] and [5] and thoroughly discussed in [8] are now being enriched by the so-called intuitionistic energy and take account of a characteristic description parameter of an intuitionistic fuzzy set, the so-called hesitancy margin (intuitionistic index).

Key word: intuitionistic fuzzy set, hesitancy margin, valuation, informational energy, energy of a fuzzy set.

1. The notion of energy and other key notions

Almost simultaneously with the start of L. A. Zadeh's investigations of fuzzy set theory [12] and its practical applications there appeared conceptions of measuring and comparing the "fuzziness" of a fuzzy set. Especially in the constructing of mathematical models of various systems, it was important to answer the question which of the two given fuzzy sets is fuzzier and what differences and what relations hold between a fuzzy set and the corresponding non-fuzzy set, and also what calculable advantages flow from the application of the apparatus of fuzzy set theory in the situation being examined. This problem was taken up in the 1970's by A. de Luca and S. Termini [7], starting the investigations of the so-called fuzzy set entropy. In their conceptions they were basing themselves on the notion of Shannon's entropy, taking pattern by a probabilistic entropy of random events. A competitive idea of measuring a fuzzy set was proposed by D. Dumitrescu's definition of the so–called informational energy of a fuzzy set [4], with that he based himself on O. Onicescu's paper [10] which also reaches probability theory with its roots, applying the notion of informational energy (this is the way of the constructing of information theory, alternative to Shannon's entropy).

Parallel to the process of creating tools for fuzzy set theory, there developed conceptions improving and generalizing the idea of a fuzzy set. One of the latest is K. Atanassov's intuitionistic fuzzy set theory [1], [2] from 1983, gaining more and more acceptance.

The present paper gives various conceptions of the measurement of intuitionistic fuzzy sets by means of differently defined energies and is a continuation of the conceptions presented originally in [4] and [8]. It may also be treated as a complement to paper [6] in which the concept of the entropy of an intuitionistic fuzzy set was presented in a similar manner.

Our considerations are confined here to the discrete case since we assume it to be a prevailing case in the majority of practical problems (although it may happen that the multitude of cases will be very big – after all, finite).

So, let $U = \{x_1, x_2, ..., x_N\}$ be a finite space of considerations.

DEFINITION 1.1. By a fuzzy set [12] in the space *U* we mean a set *A* characterized by the so-called membership function $\mu_A : U \to \langle 0, 1 \rangle$ and being the structure

$$A = \{(x, \mu_A(x)) : x \in U\}. \tag{1}$$

The family of fuzzy sets in *U* is denoted by *FS*(*U*).

The problem of measuring the indefiniteness or, in other words, „fuzziness" of a fuzzy set (i.e. entropy of such a set) is based on three natural axioms proposed on the empirical way by A. de Luca and S. Termini in [7]:

A1) the fuzziness is zero when the set is not fuzzy (i. e. $\mu(x) \in \{0,1\}$,

A2) the fuzziness is maximal when the set is the fuzziest (i.e. $\mu(x) \equiv \frac{1}{2}$),

A3) the fuzziness of a „fuzzier" set is greater than that of a „less fuzzy" set (where the relation of the „greater fuzziness" is defined by the conditions

$$(A << B) \Leftrightarrow (\mu_A(x) \le \mu_B(x) \le \frac{1}{2} \text{ and } \mu_A(x) \ge \mu_B(x) \ge \frac{1}{2})).$$

In paper [4] D. Dumitrescu proposed a conception of a fuzziness measure, competitive to [7], measuring the precision (definiteness, non-fuzziness) of a fuzzy set. He called this quantity an informational energy of a fuzzy set and required that it satisfy the postulates similar to A1, A2, A3 [4];

B1) the energy is the least when the set is the fuzziest (i.e. $\mu(x) \equiv \frac{1}{2}$),

B2) the energy is the greatest when the set is not fuzzy (i.e. $\mu(x) \in \{0,1\})$, B3) the energy of a „fuzzier” set is less than that of a „less fuzzy” set (where the relation of the” greater fuzziness” is defined in A3).

Considering the above conditions and their natural interpretation, the energy of a fuzzy set may be treated as a sharpness measure of a fuzzy set (contrary to the entropy which is treated as a fuzziness measure).

In [4] D. Dumitrescu proposed the following formula to be an informational energy of the set $A \in FS(U)$:

$$e(A) = p(\mu_A) + p(1-\mu_A) \tag{2}$$

where the function p is of the form

$$p(\mu) = \sum_{i=1}^{N} \mu^2(x_i) \tag{3}$$

(other formulae for the function p are also admitted).

Of course, formula (2) satisfies axioms B1, B2, B3, and the conception itself serves as a model for the construction of a sharpness measure of a fuzzy set, conditions B1, B2, B3 being treated as desired ones.

In 1983 K. Atanassov generalized the notion of L. Zadeh’s fuzzy set by introducing the notion of an intuitionistic fuzzy set [2]:

DEFINITION 1.2. By an intuitionistic fuzzy set in U we mean a structure A of the form

$$A = \{(x : \mu_A(x), v_A(x)) : x \in U\} \tag{4}$$

where $\mu_A, v_A : U \to \langle 0,1 \rangle$ are, respectively, membership and non-membership functions of the set A and satisfy

$$0 \le \mu_A(x) + v_A(x) \le 1 . \tag{5}$$

With that, the difference

$$\pi_A(x) = 1 - \mu_A(x) - v_A(x) \tag{6}$$

is called a hesitancy margin of the membership of the object x in the set A (also – an intuitionistic index of the set A), and the family of intuitionistic fuzzy sets in U is denoted by the symbol $IFS(U)$.

Then a fuzzy set in the sense of Zadeh is written down as

$$A = \{(x; \mu_A(x), 1-\mu_A(x)) : x \in U\} ,$$

and a set in the ordinary sense – as

$$A = \{(x : \chi_A(x), 1-\chi_A(x)) : x \in U\}$$

where χ_A is the characteristic function of the set

($\chi_A(x) = 1$ when $x \in A$ and $\chi_A(x) = 0$ when $x \notin A$).

With that, the parameter $\pi(x) \in \langle 0,1 \rangle$ is a discriminant which accentuates the intuitionistic character of the intuitionistic fuzzy set and introduces an element of some elasticity in the description of reality with the use of Atanassov's sets. Of course, for fuzzy sets and – all the more – ordinary ones, we have $\pi(x) \equiv 0$.

2. Definitions and properties

In paper [8] the conception of D. Dumitrescu's informational energy of a fuzzy set was for the first time extended to the family of intuitionistic fuzzy sets. And so:

DEFINITIONS 2.1. By an arithmetical energy *E*(*A*) of a set $A \in IFS(U)$ we mean the number

$$E(A) = \frac{e(\mu_A) + e(v_A)}{2} = \frac{e(\mu_A) + e(1 - \mu_A - \pi_A)}{2} \tag{7}$$

where *e* stands for the energy calculated by formula (2) for a fuzzy set in the sense of Zadeh described by means of the membership functions μ_A and v_A, respectively.

In [8] it was demonstrated that *E*(*A*) satisfies the system of axioms B1, B2, B3 required as a model in [4]. Besides, the following theorems hold:

THEOREM 1. For any $A \in IFS(U)$,

$$E(A) = E(A') \tag{8}$$

(the proof is self-evident).

THEOREM 2. For $A, B \in IFS(U)$,

$$E(A \cup B) + E(A \cap B) = E(A) + E(B), \tag{9}$$

that is, *E* is a valuation.

Proof. $E(A \cup B) + E(A \cap B) = \frac{1}{2}\left[e(\mu_{A\cup B}) + e(v_{A\cup B}) + e(\mu_{A\cap B}) + e(v_{A\cap B})\right] =$

$$= \frac{1}{2}\left[e(\mu_A \vee \mu_B) + e(v_A \wedge v_B) + e(\mu_A \wedge \mu_B) + e(v_A \vee v_B)\right] =$$

$$= \frac{1}{2}\left[e(\mu_A) + e(v_A) + e(\mu_B) + (v_B))\right] = E(A) + E(B).$$

In the proof we have made use of the fact that *e* is a valuation and of the definitions of the union and the intersection in the family of sets *IFS*(*U*):

$$A \cup B = \{(x; \mu_{A\cup B}(x), v_{A\cup B}(x)) : x \in U\}$$

where

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x) = \mu_A(x) \vee \mu_B(x),$$

$v_{A\cup B}(x) = \min\{v_A(x), v_B(x) = v_A(x) \wedge v_B(x)$,

$A \cap B = \{(x;\ \mu_{A\cap B}(x), v_{A\cap B}(x)) : x \in U\}$

where

$\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x)$ and $v_{A\cap B}(x) = v_A(x) \vee\ v_B(x)$.

q.e.d.

It is worth stressing that the valuation property is a necessary condition for a functional to play the role of a set measure.

In paper [8], some other approach to the notion of an intuitionistic fuzzy set energy was presented:

DEFINITION 2.2. By a logical energy $\widetilde{E}(A)$ of a set $A \in IFS(U)$ we mean the number

$$\widetilde{E}(A) = p(\mu_A) + p(v_A) = p(\mu_A) + p(1 - \mu_A - \pi_A) \tag{10}$$

where p is the function defined earlier by formula (2).

Thus formula (10) can be written down equivalently as

$$\widetilde{E}(A) = \sum_{i=1}^{N} \left[\mu_A^2(x_i) + v_A^2(x_i)\right]. \tag{11}$$

With such a definition, however, one cannot prove the fulfillment of condition B1, whereas B2 and B3 hold, and also (analogously to (8) and (9))

$$\widetilde{E}(A) = \widetilde{E}(A'), \tag{12}$$

$$\widetilde{E}(A \cup B) + \widetilde{E}(A \cap B) = \widetilde{E}(A) + \widetilde{E}(B). \tag{13}$$

The proof of formula (12) needs no comment, while in the proof of (13) one makes use of the fact that p is a valuation and of the operations $\cup$ and $\cap$ on sets from $IFS(U)$.

Both the above conceptions treat the energy of an intuitionistic fuzzy set in a very stiff way – as a real number. As is known, such a „sharp" approach in the case of a probability of fuzzy events, proposed by L. Zadeh – for instance – in [13], raised certain objections and, consequently, led to a softer and more elastic approach, which found its reflection in R. Yager's paper [11]. The probability of a fuzzy event is there a fuzzy number from the interval $\langle 0,1 \rangle$. Following this track, we now offer some other approach to the notion of an energy of an intuitionistic fuzzy set, basing ourselves on the suggestions from paper [6]:

DEFINITION 2.3. For a set $A \in IFS(U)$, we define its intuitionistic fuzzy energy $E(A)$ as a number from the closed interval

$\langle E_{\min}(A), E_{\max}(A) \rangle$ (14 where

$$E_{\min}(A) = p(\mu_A) + p(1-\mu_A) = \sum_{i=1}^{N}\left[\mu_A^2(x_i) + (1-\mu_A(x_i))^2\right] \tag{15}$$

and

$$E_{\max}(A) = E_{\min}(A) + p(\pi_A) + p(1-\pi_A)\,. \tag{16}$$

Then $E_{\min}(A)$ may be treated as the least guaranteed energy of the set $A \in IFS(U)$, whereas $E_{\max}(A)$ represents the greatest possible energy of this set.

3. Numerical example and concluding remarks

Let $U = \{x_1, x_2, x_3, x_4\}$ be, similarly as in [6], the set of four examinations to be taken by students during the nearest examination session and let

$A = \{x_i;\, \mu_A(x_i), \nu_A(x_i), \pi_A(x_i)) :\; x_i \in U, i = 1,2,3,4\}$

be an intuitionistic fuzzy set of „difficult" exams:

$A = \{(x_1; 0.8, 0.1, 0.1), (x_2; 0.7, 0.1, 0.2), (x_3; 0.5, 0.4, 0.1), (x_4; 0.4, 0.4, 0.2)\}$.

Then the arithmetical energy $E(A)$ (according to (7)), the logical energy $\widetilde{E}(A)$ (according to (11) and the intuitionistic fuzzy energy $E(A)$ (according to (14)-(16)) are, respectively, equal to:

	x_1	x_2	x_3	x_4	Σ
$\mu_A(x_i)$	0.8	0.7	0.5	0.4	
$\nu_A(x_i)$	0.1	0.1	0.4	0.4	
$\pi_A(x_i)$	0.1	0.2	0.1	0.2	
$\mu_A^2(x_i)$	0.64	0.49	0.25	0.16	1.54
$(1-\mu_A(x_i))^2$	0.04	0.09	0.25	0.36	0.74
$\nu_A^2(x_i)$	0.01	0.01	0.16	0.16	0.34
$(1-\nu_A(x_i))^2$	0.81	0.81	0.36	0.36	2.34
$\pi_A^2(x_i)$	0.01	0.04	0.01	0.04	0.1
$(1-\pi_A(x_i))^2$	0.81	0.64	0.81	0.64	2.9

$E(A) = 2.48$,

$\widetilde{E}(A) = 1.88$

$E_{\min}(A) = 2.28$ and $E_{\max}(A) = 0.1 + 2.9 = 3$,

$\overline{E}(A) \in \langle 2.28; 3 \rangle$.

In paper [3] K. Atanassov proposed an algorithm for replacing the intuitionistic fuzzy set by the respective nearest fuzzy set in the sense of Zadeh. According to this procedure and the formula

$$A_{S_{0.5}} = \{(x; \mu_A(x) + 0.5\pi_A(A)) : x \in U\} \tag{17}$$

we get the set

$$A_{S_{0.5}} = \{(x_1; 0.85), (x_2; 0.8), (x_3; 0.55), (x_4; 0.5)\}$$

as a fuzzy equivalent $A_{S_{0.5}}$ of the set A of „difficult” exams and then, in virtue of formula (2), we have

$$e(A_{S_{0.5}}) \approx 2.43.$$

The result obtained differs from any of the results obtained by means of the mathematical apparatus designed for intuitionistic fuzzy sets.

It should be emphasized that each of the conceptions discussed here resolves itself, in the case of reducing fuzzy intuitionism to the fuzziness in the sense of Zadeh, into the case of the energy proposed originally by D. Dumitrescu in [4], Their practical use depends on the context and the situation, as well as on the computational possibilities. The conception of a logical energy seems to be the simplest in calculations, but it does not satisfy axiom B_1; the arithmetical conception fully satisfies the hitherto existing theory, but it is a bit more arduous in practice; the conception of an intuitionistic fuzzy energy, although quite elastic, is still little justified mathematically and still little investigated formally. Therefore it seems purposeful to take up further studies in this field.

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