Ergodic theorem on B-structures

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Abstract

In the paper the extended individual ergodic theorem for B-structures with a state is presented. The classical ergodic theorem is formulated for ergodic mapping on Ω, where (Ω, S, P) is a probability space and $\xi : \Omega \to R$ is an integrable random variable. In our case S is replaced by a B-structure B and integrable random variable is replaced by an integrable observable.
Keywords: B-structure, ergodic theorem.

1 Introduction

A B-structures were introduced by B. Riečan and K. Čunderlíková in [1]. By the B-structure we understand some bounded structure, where a partial ordering and a partial binary operation are defined. Many examples of this structure can be found in the fuzzy sets theory and in quantum structures. B-structures can be seen as a generalization of many algebraic structures. The aim of their introduction is a generalization of the probability theory for all these structures.

Definition 1 *A B-structure is a system $(B, \hat{\oplus}, \leq, 0_B, 1_B)$ such that*

(i) $\hat{\oplus}$ is a partial binary operation on B;
(ii) $\leq$ is a partial ordering on B;
(iii) 0_B is the smallest, 1_B is the largest element in $(B, \leq)$.

Definition 2 *A state on B is a mapping $m : B \to [0, 1]$ satisfying the following conditions:*
(I) $m(1_B) = 1$, $m(0_B) = 0$
(II) if $a = b\hat{\oplus}c$, then $m(a) = m(b) + m(c)$
(III) if $a_n \nearrow a$, then $m(a_n) \nearrow m(a)$.

We will show some examples of B-structures. We can find some in the theory of fuzzy ssts or there are some quantum structures satisfying all properties for B-structures.

Example 1 *The first example is the Łukasiewicz square $M = [0, 1]^2$. It is a poset with the ordering ($(x_1, y_1) \leq (x_2, y_2)$ iff $x_1 \leq x_2, y_1 \geq y_2$). The following partially binary operations are used*

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 \oplus x_2, y_1 \otimes y_2)$$
$$(x_1, y_1) \otimes (x_2, y_2) = (x_1 \otimes x_2, y_1 \oplus y_2),$$

where the operations $\oplus, \otimes$ are defined by the following equalities:

$$x_1 \oplus x_2 = (x_1 + x_2) \wedge 1, \; x_1 \otimes x_2 = (x_1 + x_2 - 1) \vee 0.$$

The smallest element is $(0,1)$ *and the largest element is* $(1,0)$ *here.*

Example 2 *The one of the most important examples of B-structure is the system* $B = (M, +, \leq, 0_M, 1_M)$ *depends on a pseudo-MV-algebra* $M = (M, \oplus, \odot, *, ', 0_M, 1_M)$ *by the following way:*

a partial binary operation $+$ *on* M *is defined by*

$$a + b \text{ is defined iff } a \leq b^* \text{ and then: } a + b := a \oplus b,$$

We can say that $B = (M, +, \leq, 0_M, 1_M)$ *is a B-structure coinciding with the pseudo-MV-algebra M and the states on B are corresponding with the states on pseudo-MV-algebra M defined in [4]. Similarly we can define a B-structure from MV-algebra, which is a commutative case of a pseudo-MV-algebra.*

2 Probability on B-structures

Let $(\Omega, \mathcal{S}, P)$ be a probability space. The random variable is a mapping from Ω to R such that:

$$\text{if } A \in \mathcal{B}(R) \text{ then } \xi^{-1}(A) \in \mathcal{S}.$$

We denote a family of all Borrel sets by the $\mathcal{B}(R)$.

If we define the mapping $x : \mathcal{B}(R) \to \mathcal{S}$ by the law: $x : A \to \xi^{-1}(A)$ then x is a σ-homomorphism. This mapping substituting random variable on B-structures will be called an observable.

Definition 3 *Let* $B = \left(B, \hat{\oplus}, \leq, 0_B, 1_B\right)$ *be a B-structure. An observable of* B *is a mapping* $x : \mathcal{B}(R) \to B$ *satisfying the following conditions:*

(i) $x(R) = 1_B$, $x(\emptyset) = 0_B$;

(ii) if $A, B \in \mathcal{B}(R)$ *and* $A \cap B = \emptyset$, *then* $x(A \cup B) = x(A) \hat{\oplus} x(B)$;

(iii) if $A_n \in \mathcal{B}(R)$, $A_n \nearrow A$, *then* $x(A_n) \nearrow x(A)$.

Theorem 1 *Let* $x : \mathcal{B}(R) \to B$ *be an observable and* $m : B \to [0,1]$ *be a state. Then the transformation* $m \circ x = m_x : \mathcal{B}(R) \to [0,1]$ *is a probability measure.*

Proof: Let $m : B \to [0,1]$ be a state on B and $x : \mathcal{B}(R) \to B$ be an observable. Then we will prove, that the map $m_x = m \circ x$ is probability, so this map is continuous, additive and the boundary conditions are satisfied. So

(i) $m_x(R) = m(x(R)) = m(1_B) = 1$, $m_x(\emptyset) = m(x(\emptyset)) = m(0_B) = 0$

(ii) let A, B be two arbitrary disjoint sets from $\mathcal{B}(R)$, then
$m_x(A \cup B) = m(x(A \cup B)) = m\left(x(A) \hat{\oplus} x(B)\right) = m(x(A)) + m(x(B)) = m_x(A) + m_x(B)$

(iii) let $A_n \in \mathcal{B}, n \in N$ that $A_n \nearrow A$, so $x(A_n) \nearrow x(A)$ and then:
$m_x(A_n) = m(x(A_n)) \nearrow m(x(A)) = m_x(A)$
and so m_x is σ-additive.

Definition 4 *Let* $x : \mathcal{B}(R) \to B$ *be an observable on a B-structure* B *with a state* m. *The mapping* x *is integrable if there exists the expected value of the observable defined by the equation:*

$$E(x) = \int_R t \, dm_x(t);$$

where $m_x : \mathcal{B}(R) \to [0,1]$ *is the transformation* $m_x = m \circ x$.

3 Ergodic theorem

Let us recall the classical definition of a dynamical system with an ergodic transformation. By the dynamical system we mean a system $(\Omega, \mathcal{S}, P, T)$, where $(\Omega, \mathcal{S}, P)$ is a probability space and $T : \Omega \to \Omega$ is a probability preserving mapping.

The mapping $T : \Omega \to \Omega$ is called ergodic, if the following statements are satisfied:
(i) if $A \in \mathcal{S}$, then $T^{-1}(A) \in \mathcal{S}$ and $P\left(T^{-1}(A)\right) = P(A)$,
(ii) if $A = T^{-1}(A)$ then $P(A) = 0$ or $P(A) = 1$.

The following proposition is very well known.

Proposition 1 *Let (Ω, S, P, T) be a dynamical system, $T : \Omega \to \Omega$ be an ergodic mapping and $\xi : \Omega \to R$ be an integrable random variable. Then P-almost everywhere there holds:*

$$\lim_{n\to\infty} \tfrac{1}{n} \sum_{i=0}^{n-1} \xi \circ T^i = E(\xi).$$

We can define the ergodicity of the mapping on B-structures similarly to the classical case.

Definition 5 *Let B be a B-structure with a state m. A mapping $\lambda : B \to B$ is called m-preserving map, if for all elements a from B there holds:*

$$m(\lambda(a)) = m(a).$$

Definition 6 *A mapping $\lambda : B \to B$ is called ergodic with respect to an observable x, if the following rules are satisfied:*
(i) λ is m-preserving map,
(ii) for all $n \in N$ there exists σ-homomorphism $h_n : \mathcal{B}(R^n) \to B$, such that the equalities holds

$$m\left(h_n\left(A_1 \times A_2 ... \times A_n\right)\right) = m\left(x(A_1) \wedge (\lambda \circ x)(A_2) \wedge ... \wedge \left(\lambda^{n-1} \circ x\right)(A_n)\right) =$$
$$m\left(x(A_1)\right) \cdot m\left((\lambda \circ x)(A_2)\right) \cdot ... \cdot m\left(\left(\lambda^{n-1} \circ x\right)(A_n)\right).$$

Kolmogorov's construction:
Let x be an observable on a B-structure B with a state m.
Let $\lambda : B \to B$ be an ergodic mapping with respect to the observable x.
Let $C = \left\{\pi_n^{-1}(B) ; B \in \mathcal{B}(R^n), n \in N\right\}$ be the set of all cylinders, where the function $\pi_n : R^N \to R^n$ defined by $\pi_n\left((u_i)_{n=1}^{\infty}\right) = (u_1, u_2, ..., u_n)$ is called the n-th coordinate projection of random vector.

The Kolmogorov's construction $\mathbf{P}$ of the measures $m \circ h_i, i = 1, 2, ...$ on the space $\left(R^N, \sigma(C)\right)$ is defined by

$$\mathbf{P}\left\{(u_i)_1^{\infty} \in R^N ; u_1 \in A_1, ..., u_n \in A_n\right\} = m\left(h_n\left(A_1 \times A_2 ... \times A_n\right)\right)$$

for each $n \in N$ and every $A_1, ..., A_n \in \mathcal{B}(R)$.

It is easy to see that the function $\xi : R^N \to R$ given by $\xi\left((u_i)_1^{\infty}\right) = u_1$ is a random variable of $\left(R^N, \sigma(C), \mathbf{P}\right)$ and $P_\xi = m_x$.

In the classical theory with a probability space $(\Omega, \mathcal{S}, P)$, we say that the sequence of random variables ξ_n converges to 0 P-almost everywhere, if

$$P\left(\{\omega, \xi_n(\omega) \to 0\}\right) = 1.$$

This property can be rewritten by the following form:

$$(\forall \varepsilon < 0)\,(\forall \omega \in D)\,(\exists k \in N)\,(\forall n \geq k)\,(|\xi_n(\omega)| < \varepsilon)$$

If we denote by D the set $\{\xi_n(\omega) \to 0\}$, then the previous equality means:

$$(\forall l \in N)\,(\exists k \in N)\,(\forall n \geq k)\left(\xi_n^{-1}\left(-\tfrac{1}{l}, \tfrac{1}{l}\right)\right) \subset D.$$

Then we can define the analogous type of convergence for the sequence of random variables $(\xi_n)_{n=1}^{\infty}$ by this formula:

$$\lim_{l\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} P\left(\bigcap_{n=k}^{k+i} \xi_n^{-1}\left(-\tfrac{1}{l}, \tfrac{1}{l}\right)\right) = 1.$$

Similarly we define the convergence m-almost everywhere for a sequence of observables on B-structures.

Definition 7 *Let $(y_i)_{i=1}^{\infty}$ be the sequence of the observables on a B-structure B with a state m on B. We say, that the sequence converges m-almost everywhere to 0, if there holds:*

$$\lim_{l\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\left(\bigwedge_{n=1}^{k+i} y_n\left(-\tfrac{1}{l}, \tfrac{1}{l}\right)\right) = 1.$$

Proposition 2 *Let x be an observable on a B-structure B with a state m and $\lambda : B \to B$ be an ergodic mapping with respect to the observable x. A mapping $h_n : \mathcal{B}(R^n) \to B$ is the σ-homomorphism from the definition of the ergodic map. Let P be the probability measure generates by the Kolmogorov construction. For all natural numbers n g_n is a Borrel function from R^n to R. We define an observable $y_n = g_n\left(x, \lambda \circ x, ..., \lambda^{n-1} \circ x\right) : \mathcal{B}(R) \to B$ by the equality $y_n = h_n \circ g_n^{-1}$. If π_n is the projection R^N to R^n given by the equality: $\pi_n\left((u_i)_{i=1}^{\infty}\right) = (u_1, u_2, ..., u_n)$, then there holds: $P \circ \pi_n^{-1} \circ g_n^{-1} = m \circ h_n \circ g_n^{-1} = m \circ y_n$. If the sequence $(g_n \circ \pi_n)_{n=1}^{\infty}$ converges P-almost everywhere to 0, then the sequence $(y_i)_{i=1}^{\infty}$ converges m-almost everywhere to 0.*

Proof: The equality $P \circ \pi_n^{-1} \circ g_n^{-1} = m \circ h_n \circ g_n^{-1} = m \circ y_n$ follows from the Kolmogorov's construction by the following way:

$$P\left(\pi_n^{-1}(A_1 \times A_2 \times ... \times A_n)\right) = P\left(A_1 \times A_2 \times ... \times A_n \times R \times R....\right) =$$

$$= m\left(x(A_1) \wedge (\lambda \circ x)(A_2) \wedge ... \wedge \left(\lambda^{n-1} \circ x\right)(A_n)\right) =$$
$$= m\left(h_n(A_1 \times A_2 ... \times A_n)\right).$$

It follows $P \circ \pi_n^{-1} = m \circ h_n$, hence also $P \circ \pi_n^{-1} \circ g_n^{-1} = m \circ h_n \circ g_n^{-1} = m \circ y_n$.

Next we show the second property. So let the sequence $(g_n \circ \pi_n)_{n=1}^{\infty}$ converges P-almost everywhere to 0 that is:

$$\lim_{l\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} P\left(\bigcap_{n=k}^{k+i} (g_n \circ \pi_n)^{-1}\left(-\tfrac{1}{l}, \tfrac{1}{l}\right)\right) = 1.$$

Then the following holds:

$$P\left(\bigcap_{n=k}^{k+i}\left(g_n\circ\pi_n\right)^{-1}\left(-\tfrac{1}{l},\tfrac{1}{l}\right)\right)=P\left(\bigcap_{n=k}^{k+i}\left(\pi_n^{-1}\circ g_n^{-1}\right)\left(-\tfrac{1}{l},\tfrac{1}{l}\right)\right)=$$
$$=P\left(\bigcap_{n=k}^{k+i}\pi_n^{-1}\left(g_n^{-1}\left(-\tfrac{1}{l},\tfrac{1}{l}\right)\right)\right)=P\left(\pi_{k+i}^{-1}\left(\bigcap_{n=k}^{k+i}g_n^{-1}\left(-\tfrac{1}{l},\tfrac{1}{l}\right)\right)\right)=$$
$$=\left(P\circ\pi_{k+i}^{-1}\right)\left(\bigcap_{n=k}^{k+i}g_n^{-1}\left(-\tfrac{1}{l},\tfrac{1}{l}\right)\right)=\left(m\circ h_{k+i}\right)\left(\bigcap_{n=k}^{k+i}g_n^{-1}\left(-\tfrac{1}{l},\tfrac{1}{l}\right)\right)=$$
$$=m\left(h_{k+i}\left(\bigcap_{n=k}^{k+i}g_n^{-1}\left(-\tfrac{1}{l},\tfrac{1}{l}\right)\right)\right)\leq m\left(\bigwedge_{n=k}^{k+i}h_{k+i}\circ g_n^{-1}\left(-\tfrac{1}{l},\tfrac{1}{l}\right)\right)=$$
$$=m\left(\bigwedge_{n=k}^{k+i}y_n\left(-\tfrac{1}{l},\tfrac{1}{l}\right)\right).$$

Then it holds:

$$P\left(\bigcap_{n=k}^{k+i}\left(g_n\circ\pi_n\right)^{-1}\left(-\tfrac{1}{l},\tfrac{1}{l}\right)\right)\leq m\left(\bigwedge_{n=k}^{k+i}y_n\left(-\tfrac{1}{l},\tfrac{1}{l}\right)\right)$$

and since:

$$\lim_{l\to\infty}\lim_{k\to\infty}\lim_{i\to\infty}P\left(\bigcap_{n=k}^{k+i}\left(g_n\circ\pi_n\right)^{-1}\left(-\tfrac{1}{l},\tfrac{1}{l}\right)\right)=1,$$

the following is satisfied:

$$\lim_{l\to\infty}\lim_{k\to\infty}\lim_{i\to\infty}m\left(\bigwedge_{n=k}^{k+i}y_n\left(-\tfrac{1}{l},\tfrac{1}{l}\right)\right)=1.$$

Now we recall the classical ergodic theorem for ergodic mapping and then can be introduce the ergodic theorem for the ergodic mapping on B-structures.

Let $(\Omega,\mathcal{S},P)$ be a probability space and the mapping $T:\Omega\to\Omega$ be an ergodic transformation. Let $x:\Omega\to R$ be an integrable random variable with the mean value $E\left(x\right)$, then

$$\tfrac{1}{n}\sum_{i=0}^{n-1}x\circ T^i\to E\left(x\right)\text{ P-almost everywhere.}$$

Theorem 2 (Ergodic theorem) *Let B be a B-structure with a state m which satisfies: $\forall a,b\in B:\exists a\wedge b$. Let x be an integrable observable on B. Let the map $\lambda:B\to B$ be an ergodic mapping according to the observable x for which holds: $\lambda\left(a\wedge b\right)=\lambda\left(a\right)\wedge\lambda\left(b\right)$. Then the sequence $\left(y_n\right)_{n=1}^{\infty}$ defined by a formula:*

$y_n=\frac{1}{n}\sum\limits_{i=0}^{n-1}\lambda^i\circ x-E\left(x\right)$

converges m-almost everywhere to 0.

Proof: For a proof of this theorem we use the previous propositions. So we have the observable x with an ergodic mapping λ. The sequence y_n is equal to $g_n\left(x,\lambda\circ x,...,\lambda^{n-1}\circ x\right)$, where g_n is the Borel function defined by the following formula $g_n\left(u_1,u_2,...,u_n\right)=\frac{1}{n}\sum\limits_{i=1}^{n}u_i-E\left(x\right)$. Then we have the observable y_n from previous proposition.

Let $h_n:\mathcal{B}\left(R^n\right)\to B$ be the mapping from Definition 6 given by the equality:

$$m\left(h_n\left(A_1\times A_2\times...\times A_n\right)\right)=m\left(x\left(A_1\right)\wedge\left(\lambda\circ x\right)\left(A_2\right)\wedge...\wedge\left(\lambda^{n-1}\circ x\right)\left(A_n\right)\right).$$

Let $\left(R^N, \sigma(C)\right)$ be the probability space, on which we define a mapping $T : R^n \to R^n$ by this way:

$$T\left((u_i)_{i=1}^{\infty}\right) = (v_i)_{i=1}^{\infty}, v_i = u_{i+1}.$$

We shall prove that T is an ergodic transformation.

At first T is a measure preserving transformation.

$$\begin{gathered} P\left(T^{-1}\left(\pi_n^{-1}\left(A_1 \times A_2 \times ... \times A_n\right)\right)\right) = \\ = P\left(T^{-1}\left(A_1 \times A_2 \times ... \times A_n \times R \times R...\right)\right) = \\ = P\left(R \times A_1 \times A_2 \times ... \times A_n \times R \times R...\right) = \\ = m\left(x(R) \wedge (\lambda \circ x)(A_1) \wedge ... \wedge (\lambda^n \circ x)(A_n)\right) = \\ = m\left(\lambda\left(x(A_1) \wedge (\lambda \circ x)(A_2) \wedge ... \wedge \left(\lambda^{n-1} \circ x\right)(A_n)\right)\right) = \\ = m\left(x(A_1) \wedge (\lambda \circ x)(A_2) \wedge ... \wedge \left(\lambda^{n-1} \circ x\right)(A_n)\right) = \\ = P\left(\pi_n^{-1}\left(A_1 \times A_2 \times ... \times A_n\right)\right). \end{gathered}$$

So the equality $P \circ T^{-1} = P$ holds.

By the definition of ergodicity we obtain that:

$\left(R^N, \sigma(C), P, T\right)$ is the Bernoulli scheme, hence T is ergodic transformation.

Then we define an integrable random variable:

$\xi : R^N \to R, \xi\left((u_i)_{i=1}^{\infty}\right) = u_1$ and it holds $E(\xi) = E(x)$.

By results from proposition 1 and the equality:

$$g_n\left(\xi, \xi \circ T, ..., \xi \circ T^{n-1}\right) = \frac{1}{n} \sum_{i=0}^{n-1} \xi \circ T^i - E(\xi)$$

we have that P-almost everywhere the following holds:

$$\lim_{n \to \infty} \sum_{i=0}^{n-1} \xi \circ T^i = E(\xi).$$

By using the proposition 2 we can write:

$$\lim_{n \to \infty} \sum_{i=0}^{n-1} \lambda^i \circ x = E(x).$$

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