Notes on Intuitionistic Fuzzy Sets
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An investigation of group action on intuitionistic fuzzy primary and semiprimary ideals of Γ-ring

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Abstract: This article is a continuation of the author's earlier work [11]. Here we extend the study of group actions on intuitionistic fuzzy structures by focusing on intuitionistic fuzzy primary and semiprimary ideals of Γ-rings. We introduce and characterize intuitionistic fuzzy $\mathcal{G}$-primary and $\mathcal{G}$-semiprimary ideals, establishing their relationships with $\mathcal{G}$-invariant intuitionistic fuzzy ideals. Several structural properties and related results are investigated to deepen the understanding of these generalized ideals under group actions. Furthermore, we explore the behavior of intuitionistic fuzzy $\mathcal{G}$-primary and $\mathcal{G}$-semiprimary ideals under $\mathcal{G}$-homomorphisms, specifically analyzing their image and pre-image. These results contribute to the ongoing development of intuitionistic fuzzy algebraic structures in the context of Γ-rings.
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1 Introduction

Γ-rings, introduced by Nobusawa [6] as a generalization of rings, offer a flexible algebraic structure that encompasses a wider class of algebraic systems. Barnes [3] later refined Nobusawa's definition by relaxing certain constraints, thereby establishing a more versatile framework applicable to various mathematical contexts. Notable contributions to the development of Γ-ring theory include the structural investigations of Barnes [3] and Kyuno [5], Warsi's detailed analysis of the decomposition of primary ideals [17], and Paul's comprehensive study of different classes of Γ-ideals and their associated operator rings [10].

The concept of intuitionistic fuzzy sets, introduced by Atanassov [1] in 1983, extends Zadeh's fuzzy sets [18] by incorporating both membership and non-membership degrees, thereby providing a more nuanced framework for representing uncertainty. This approach has found successful applications in various algebraic structures, including rings, modules, and more recently, Γ-rings (see [4, 7–10, 12–16]). In a previous study by the author [11], the influence of group actions on intuitionistic fuzzy ideals of Γ-rings was examined, leading to structural insights into $\mathcal{G}$-invariant intuitionistic fuzzy ideals.

The present article builds upon this foundation by investigating more specialized types of intuitionistic fuzzy ideals, namely intuitionistic fuzzy primary and semiprimary ideals, in the context of group actions on Γ-rings. We introduce and characterize the notions of intuitionistic fuzzy $\mathcal{G}$-primary ideals and intuitionistic fuzzy $\mathcal{G}$-semiprimary ideals, highlighting their properties and interrelationships with $\mathcal{G}$-invariant intuitionistic fuzzy ideals. These generalizations not only extend classical ideal theory but also deepen our understanding of how symmetry, expressed via group actions, interacts with fuzzy and intuitionistic fuzzy structures.

Additionally, we examine the image and pre-image of these ideals under $\mathcal{G}$-homomorphisms, shedding light on how these structures transform under morphic mappings that respect the group action. The results presented in this paper aim to contribute to the theoretical development of intuitionistic fuzzy algebra in Γ-ring environments and stimulate further exploration in this direction.

2 Preliminaries

Let us recall some definitions and results, which are necessary for the development of the paper.

Definition 2.1. ([3, 6]) If $(\mathcal{M}, +)$ and $(\Gamma, +)$ are additive Abelian groups, then $\mathcal{M}$ is called a Γ-ring (in the sense of Barnes [3]) if there exist mapping $\mathcal{M} \times \Gamma \times \mathcal{M} \to \mathcal{M}$ [image of (x, α, y) is denoted by $x\alpha y$, $x, y \in \mathcal{M}, \gamma \in \Gamma$] satisfying the following conditions:

(1) $x\alpha y \in \mathcal{M}$.

(2) $(x+y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)y = x\alpha y + x\beta y$, $x\alpha(y+z) = x\alpha y + x\alpha z$.

(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$. for all $x, y, z \in \mathcal{M}$, and $\gamma \in \Gamma$.

The Γ-ring $\mathcal{M}$ is called commutative if $x\gamma y = y\gamma x, \forall x, y \in \mathcal{M}, \gamma \in \Gamma$. An element $1 \in \mathcal{M}$ is said to be the unity of $\mathcal{M}$ if for each $x \in \mathcal{M}$ there exists $\gamma \in \Gamma$ such that $x\gamma 1 = 1\gamma x = x$.

A subset $\mathcal{N}$ of a Γ-ring $\mathcal{M}$ is a left (right) ideal of $\mathcal{M}$ if $\mathcal{N}$ is an additive subgroup of $\mathcal{M}$ and $\mathcal{M}\Gamma\mathcal{N} = \{x\alpha y \mid x \in \mathcal{M}, \alpha \in \Gamma, y \in \mathcal{N}\}$, ("$\mathcal{N}\Gamma\mathcal{M} = \{x\alpha y \mid x \in \mathcal{N}, \alpha \in \Gamma, y \in \mathcal{M}\}$") is contained in $\mathcal{N}$. If $\mathcal{N}$ is both a left and a right ideal, then $\mathcal{N}$ is a two-sided ideal, or simply an ideal of $\mathcal{M}$. A Γ-ring $\mathcal{M}$ is said to be commutative if $a\gamma b = b\gamma a$ for all $a, b \in \mathcal{M}$ and $\gamma \in \Gamma$. A mapping $\sigma : \mathcal{M} \to \mathcal{M}'$ of Γ-rings is called a Γ-homomorphism [3] if $\sigma(x + y) = \sigma(x) + \sigma(y)$ and $\sigma(x\alpha y) = \sigma(x)\alpha\sigma(y)$ for all $x, y \in \mathcal{M}, \alpha \in \Gamma$.

Definition 2.2. ([17]) Let $\mathcal{M}$ be a Γ-ring. A proper ideal $\mathcal{L}$ of $\mathcal{M}$ is called prime if for all pair of ideals $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{M}$, $\mathcal{S}\Gamma\mathcal{T} \subseteq \mathcal{L}$ implies that $\mathcal{S} \subseteq \mathcal{L}$ or $\mathcal{T} \subseteq \mathcal{L}$.

Theorem 2.3. ([10]) *If $\mathcal{L}$ is an ideal of a Γ-ring $\mathcal{M}$, the following conditions are equivalent:*

(i) $\mathcal{L}$ is a prime ideal of $\mathcal{M}$;

(ii) If $a, b \in \mathcal{M}$ and $a\Gamma\mathcal{M}\Gamma b \subseteq \mathcal{L}$ then $a \in \mathcal{L}$ or $b \in \mathcal{L}$.

Definition 2.4. ([17]) Let $\mathcal{M}$ be a Γ-ring. Then the radical of an ideal $\mathcal{K}$ of $\mathcal{M}$ is denoted by $\sqrt{\mathcal{K}}$ and is defined as the set

$$\sqrt{\mathcal{K}} = \{x \in \mathcal{M} : (x\gamma)^{n-1}x \in \mathcal{K}, \text{ for some } n \in \mathbb{N} \text{ and for any } \gamma \in \Gamma\},$$

where $(x\gamma)^{n-1}x = x$ for $n = 1$.

Definition 2.5. ([3]) An ideal $\mathcal{K}$ of a commutative Γ-ring $\mathcal{M}$ is said to be primary if for any two ideals $\mathcal{I}$ and $\mathcal{J}$ of $\mathcal{M}$, $\mathcal{I}\Gamma\mathcal{J} \subseteq \mathcal{K}$ implies either $\mathcal{I} \subseteq \mathcal{K}$ or $\mathcal{J} \subseteq \sqrt{\mathcal{K}}$, where $\sqrt{\mathcal{K}}$ is the prime radical of $\mathcal{K}$.

We now review some intuitionistic fuzzy logic concepts. We refer the reader to follow [1] and [7] for complete details.

Definition 2.6. ([1, 2]) An intuitionistic fuzzy set A in $\mathcal{X}$ can be represented as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathcal{X}\}$, where the functions $\mu_A, \nu_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in \mathcal{X}$ to A respectively and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in \mathcal{X}$.

Remark 2.7. ([1,2]) When $\mu_A(x) + \nu_A(x) = 1, \forall x \in \mathcal{X}$, then A is called a fuzzy set.

If $A, B \in IFS(\mathcal{X})$, then $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x), \forall x \in \mathcal{X}$. Also, $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$. For any subset $\mathcal{Y}$ of $\mathcal{X}$, the intuitionistic fuzzy characteristic function $\chi_\mathcal{Y}$ is an intuitionistic fuzzy set of $\mathcal{X}$, defined by: $\chi_\mathcal{Y}(x) = (1, 0), \forall x \in \mathcal{Y}$ and $\chi_\mathcal{Y}(x) = (0, 1), \forall x \in \mathcal{X}\backslash\mathcal{Y}$.

Let $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. Then the crisp set

$$A_{(\alpha,\beta)} = \{x \in \mathcal{X} \mid \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta\}$$

is called the (α, β)-level cut subset of A. Also, the IFS $x_{(\alpha,\beta)}$ of $\mathcal{X}$ defined as $x_{(\alpha,\beta)}(y) = (\alpha, \beta)$, if $y = x$, otherwise $(0, 1)$ is called the intuitionistic fuzzy point (IFP) in $\mathcal{X}$ with support x. By $x_{(\alpha,\beta)} \in A$ we mean $\mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta$. Furthermore, if $\sigma : \mathcal{X} \to \mathcal{Y}$ is a mapping, and A and B are IFS of $\mathcal{X}$ and $\mathcal{Y}$, respectively, then the image $\sigma(A)$, is an IFS of $\mathcal{Y}$, defined as: $\mu_{\sigma(A)}(y) = \sup\{\mu_A(x) : \sigma(x) = y\}$, $\nu_{\sigma(A)}(y) = \inf\{\nu_A(x) : \sigma(x) = y\}$, for all $y \in \mathcal{Y}$. The inverse image $\sigma^{-1}(B)$, is an IFS of $\mathcal{X}$, defined as: $\mu_{\sigma^{-1}(B)}(x) = \mu_B(\sigma(x))$, $\nu_{\sigma^{-1}(B)}(x) = \nu_B(\sigma(x))$, for all $x \in \mathcal{X}$, i.e., $\sigma^{-1}(B)(x) = B(\sigma(x))$, for all $x \in \mathcal{X}$. Furthermore, the IFS A of $\mathcal{X}$ is said to be σ-invariant if for any $x, y \in \mathcal{X}$, whenever $\sigma(x) = \sigma(y)$ implies $A(x) = A(y)$.

Definition 2.8. ([8, 13]) Let A and B be two IFSs of a Γ-ring $\mathcal{M}$ and $\gamma \in \Gamma$. Then the product $A\Gamma B$ and the composition $A \circ B$ of A and B are defined by

$$(A\Gamma B)(x) = \begin{cases} \Big(\vee_{x=y\gamma z} \left(\mu_A(y) \wedge \mu_B(z)\right), \wedge_{x=y\gamma z}(\nu_A(y) \vee \nu_B(z)\Big), & \text{if } x = y\gamma z \\ (0,1), & \text{otherwise} \end{cases}$$

and

$$(A \circ B)(x) = \begin{cases} \Big(\vee_{x=\sum\limits_{i=1}^{n} y_i\gamma z_i} (\mu_A(y_i) \wedge \mu_B(z_i)), \wedge_{x=\sum\limits_{i=1}^{n} y_i\gamma z_i}(\nu_A(y_i) \vee \nu_B(z_i))\Big), & \text{if } x = \sum\limits_{i=1}^{n} y_i\gamma z_i \\ (0,1), & \text{otherwise} \end{cases}$$

Remark 2.9. ([8, 13]) If A and B are two IFSs of a Γ-ring $\mathcal{M}$, then $A\Gamma B \subseteq A \circ B \subseteq A \cap B$.

Definition 2.10. ([8, 13]) Let A be an IFS of a Γ-ring $\mathcal{M}$. Then A is called an intuitionistic fuzzy ideal (IFI) of $\mathcal{M}$ if for all $x, y \in \mathcal{M}, \gamma \in \Gamma$, the following assertions are satisfied:

(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$; (ii) $\mu_A(x\alpha y) \geq \mu_A(x) \vee \mu_A(y)$;

(iii) $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$; (iv) $\nu_A(x\alpha y) \leq \nu_A(x) \wedge \nu_A(y)$.

The set of all intuitionistic fuzzy ideals of Γ-ring $\mathcal{M}$ is denoted by $IFI(\mathcal{M})$. Note that if $A \in IFI(\mathcal{M})$, then $\mu_A(0_{\mathcal{M}}) \geq \mu_A(x)$ and $\nu_A(0_{\mathcal{M}}) \leq \nu_A(x), \forall x \in \mathcal{M}$ (See [4]).

Remark 2.11. ([7–9]) If A, B and C be IFIs of a Γ-ring $\mathcal{M}$, then $A\Gamma B$, $A \circ B$, $A \cap B$ are also IFI of $\mathcal{M}$. Further, $A\Gamma B \subseteq C$ if and only if $A \circ B \subseteq C$.

Definition 2.12. ([7, 14]) Let Q be a non-constant IFI of a Γ-ring $\mathcal{M}$. Then Q is said to be an intuitionistic fuzzy prime (primary) ideal of $\mathcal{M}$ if for any two IFIs A, B of $\mathcal{M}$ such that $A\Gamma B \subseteq Q$ implies that either $A \subseteq Q$ or $B \subseteq Q$ ($A \subseteq Q$ or $B \subseteq \sqrt{Q}$), where $\sqrt{Q}$ defined by

$$\mu_{\sqrt{Q}}(x) = \vee\{\mu_Q((x\gamma)^{n-1}x) : n \in \mathbb{N}\} \quad \text{and} \quad \nu_{\sqrt{Q}}(x) = \wedge\{\nu_Q((x\gamma)^{n-1}) : n \in \mathbb{N}\}$$

is called the intuitionistic fuzzy prime radical of Q, where $(x\gamma)^{n-1}x = x$, for $n = 1, \gamma \in \Gamma$.

Theorem 2.13. ([7, 14]) *Let $\mathcal{M}$ be a commutative Γ-ring and Q be an IFI of $\mathcal{M}$. Then for any two IFPs $x_{(p,q)}, y_{(t,s)} \in IFP(\mathcal{M})$ the following are equivalent:*

(i) Q is an intuitionistic fuzzy prime (primary) ideal of $\mathcal{M}$

(ii) $x_{(p,q)}\Gamma y_{(t,s)} \subseteq Q$ implies $x_{(p,q)} \subseteq Q$ or $y_{(t,s)} \subseteq Q$ ($x_{(p,q)} \subseteq Q$ or $y_{(t,s)} \subseteq \sqrt{Q}$).

Theorem 2.14. ([7, 14]) *If Q is an intuitionistic fuzzy prime (primary) ideal of a Γ-ring $\mathcal{M}$, then the following conditions hold:*

(i) $Q(0_{\mathcal{M}}) = (1, 0)$,

(ii) Q_* is a prime (primary) ideal of $\mathcal{M}$,

(iii) $\mathrm{Img}(Q) = \{(1, 0), (t, s)\}$, where $t, s \in [0, 1)$ such that $t + s \leq 1$.

Definition 2.15. ([16]) A non-constant intuitionistic fuzzy ideal P of a Γ-ring $\mathcal{M}$ is called intuitionistic fuzzy semiprime ideal if for any IFIs A of $\mathcal{M}$, $A\Gamma A \subseteq P$ implies $A \subseteq P$.

Proposition 2.16. ([16]) *Let P be a non-constant intuitionistic fuzzy ideal of a Γ-ring $\mathcal{M}$, then the following conditions are equivalent:*

(i) P is an intuitionistic fuzzy semiprime ideal of $\mathcal{M}$

(ii) For any $a \in \mathcal{M}$, $\inf_{m\in\mathcal{M}\gamma_1,\gamma_2\in\Gamma}\{\mu_P(a\gamma_1 m\gamma_2 a)\} = \mu_P(a)$ and $\sup_{m\in\mathcal{M},\gamma_1,\gamma_2\in\Gamma}\{\nu_P(a\gamma_1 m\gamma_2 a)\} = \nu_P(a)$.

Remark 2.17.

(i) From the definition it is clear that if A is an intuitionistic fuzzy semiprimary ideal of $\mathcal{M}$, then $\sqrt{A}$ is an intuitionistic fuzzy prime ideal of $\mathcal{M}$.

(ii) If A is an intuitionistic fuzzy primary ideal of $\mathcal{M}$, then A is also an intuitionistic fuzzy semiprimary ideal of $\mathcal{M}$. But converse of it is not true. See the following example

Example 2.18. Consider $\mathcal{M} = \mathbb{Z}_8 = \{0,1,2,3,4,5,6,7\}, \Gamma = \mathbb{Z}_2 = \{0,1\}$. Then $\mathcal{M}$ is a Γ-ring. Define the IFS A of $\mathcal{M}$ by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0.6, & \text{if } x = 2,4,6 \\ 0, & \text{if } x = 1,3,5,7 \end{cases} ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.3, & \text{if } x = 2,4,6 \\ 0.5, & \text{if } x = 1,3,5,7 \end{cases} .$$

Now, it is easy to verify that A is an IFI of $\mathcal{M}$ such that $\sqrt{A} = \chi_I$, where $I = \{0,2,4,6\}$ is a prime ideal of $\mathcal{M}$ and by Theorem (2.14) $\sqrt{A}$ is an intuitionistic fuzzy prime ideal of $\mathcal{M}$. Then by definition A is an intuitionistic fuzzy semiprimary ideal of $\mathcal{M}$. Moreover, by Theorem (2.14) A is not an intuitionistic fuzzy primary ideal of $\mathcal{M}$, for $|\mathrm{Img}(A)| = 3$.

Theorem 2.19. ([14]) *Let f be a homomorphism of a Γ-ring $\mathcal{M}$ onto a Γ-ring $\mathcal{M}'$. If B is an intuitionistic fuzzy primary ideal of $\mathcal{M}'$, then $f^{-1}(B)$ is an intuitionistic fuzzy primary ideal of $\mathcal{M}$.*

Theorem 2.20. ([14]) *Let f be a homomorphism of a Γ-ring $\mathcal{M}$ onto a Γ-ring $\mathcal{M}'$. If A is an f-invariant intuitionistic fuzzy primary ideal of $\mathcal{M}$, then $f(A)$ is an intuitionistic fuzzy primary ideal of $\mathcal{M}'$.*

Theorem 2.21. ([14]) *Let f be a homomorphism of a Γ-ring $\mathcal{M}$ onto a Γ-ring $\mathcal{M}'$. If A is an IFI of $\mathcal{M}$ such that A is constant on* $\operatorname{Ker} f$, *then* $\sqrt{f(A)} = f(\sqrt{A})$.

3 Group action on intuitionistic fuzzy ideal of a Γ-ring

Definition 3.1. ([11]) Let $\mathcal{G}$ be a group and $\mathcal{S}$ be a non-empty set. Then the mapping $\phi : \mathcal{G}\times\mathcal{S}\to\mathcal{S}$, with $\phi(g,x)$ written as $g * x$, is an action of $\mathcal{G}$ on $\mathcal{S}$ if and only if, for all $g,h \in \mathcal{G}$ and $x \in \mathcal{S}$, the following conditions hold:

1. $g * (h * x) = (gh) * x$,
2. $e * x = x$, where e is the identity element of the group $\mathcal{G}$.

We assume that $\mathcal{M}$ is a Γ-ring and $\mathcal{G}$ is a finite group such that $\mathcal{G}$ acts on a subset $\mathcal{S}$ of $\mathcal{M}$. (For example: $\forall g \in \mathcal{G}, x \in \mathcal{S}, x^g = gxg^{-1} \in \mathcal{S}$, where x^g define the action of the element g on the element x of $\mathcal{S}$.) Here in this section, we define the group action of $\mathcal{G}$ on an IFS A of a Γ-ring $\mathcal{M}$.

Definition 3.2. ([11]) The group action of $\mathcal{G}$ on an IFS A of a Γ-ring $\mathcal{M}$ is denoted by A^g and is defined as $A^g = \{\langle x, \mu_{A^g}(x), \nu_{A^g}(x)\rangle : x \in \mathcal{M}\}$, where $\mu_{A^g}(x) = \mu_A(x^g)$ and $\nu_{A^g}(x) = \nu_A(x^g)$ for any $x \in \mathcal{M}, g \in \mathcal{G}$.

From the definition of group action on IFS, following results can be seen in [11]

Proposition 3.3. ([11]) *Let A be an IFI of a Γ-ring $\mathcal{M}$, and let $\mathcal{G}$ be a finite group which acts on A. Then A^g is also an IFI of $\mathcal{M}$.*

Remark 3.4. The converse of Proposition (3.3) need not be true (see Remark 3.7 of [11]).

Definition 3.5. ([11]) Let A be an IFI of a Γ-ring $\mathcal{M}$, and let $\mathcal{G}$ be a finite group which acts on A. Then A is called an intuitionistic fuzzy $\mathcal{G}$-ideal of $\mathcal{M}$ if A^g is an IFI of $\mathcal{M}$ for all $g \in \mathcal{G}$.

Proposition 3.6. *If $\sqrt{A}$ is an intuitionistic fuzzy radical of an IFI A of a Γ-ring $\mathcal{M}$ and $\mathcal{G}$ is a finite group which acts on A, then $\sqrt{A^g} = (\sqrt{A})^g$, for all $g \in \mathcal{G}$.*

Proof. Let $x \in \mathcal{M}, g \in \mathcal{G}$ be any element. Then

$$\begin{aligned}
\mu_{\sqrt{A^g}}(x) &= \inf\{\mu_{A^g}((x\gamma)^{n-1}x) : n \in \mathbb{N}\} \\
&= \inf\{\mu_A[(x\gamma)(x\gamma)\cdots(x\gamma)x]^g : n \in \mathbb{N}\} \\
&= \inf\{\mu_A[(x^g\gamma)(x^g\gamma)\cdots(x^g\gamma)x^g] : n \in \mathbb{N}\} \\
&= \inf\{\mu_A((x^g\gamma)^{n-1}x^g) : n \in \mathbb{N}\} \\
&= \mu_{\sqrt{A}}(x^g) \\
&= \mu_{(\sqrt{A})^g}(x).
\end{aligned}$$

Similarly, we can show that $\nu_{\sqrt{A^g}}(x) = \nu_{(\sqrt{A})^g}(x)$. Hence $\sqrt{A^g} = (\sqrt{A})^g$, for all $g \in \mathcal{G}$. □

Proposition 3.7. *If P is an intuitionistic fuzzy primary ideal of a Γ-ring $\mathcal{M}$, then P^g is also an intuitionistic fuzzy primary ideal of $\mathcal{M}$, where $g \in \mathcal{G}$ be any element.*

Proof. Let A, B be IFIs of Γ-ring $\mathcal{M}$ such that $A\Gamma B \subseteq P^g$, where $g \in \mathcal{G}$ be any element.
Now, we claim that $A^{g^{-1}}\Gamma B^{g^{-1}} \subseteq P$. It is sufficient to show that $\mu_{A^{g^{-1}}\Gamma B^{g^{-1}}}(x) \leq \mu_P(x)$ and $\nu_{A^{g^{-1}}\Gamma B^{g^{-1}}}(x) \geq \nu_P(x), \forall x \in \mathcal{M}$.

$$\begin{aligned}
\mu_{A^{g^{-1}}\Gamma B^{g^{-1}}}(x) &= \sup_{x=a\gamma b}\{\min(\mu_{A^{g^{-1}}}(a), \mu_{B^{g^{-1}}}(b))\} \\
&= \sup_{x^{g^{-1}}=a^{g^{-1}}\gamma b^{g^{-1}}}\{\min(\mu_A(a^{g^{-1}}), \mu_B(b^{g^{-1}}))\} \\
&= \mu_{A\Gamma B}(x^{g^{-1}}) \\
&\leq \mu_{P^g}(x^{g^{-1}}) \qquad [\because A\Gamma B \subseteq P^g] \\
&= \mu_P(x).
\end{aligned}$$

Thus $\mu_{A^{g^{-1}}\Gamma B^{g^{-1}}}(x) \leq \mu_P(x)$. Similarly we can show that $\nu_{A^{g^{-1}}\Gamma B^{g^{-1}}}(x) \geq \nu_P(x)$. Hence $A^{g^{-1}}\Gamma B^{g^{-1}} \subseteq P$, which implies that either $A^{g^{-1}} \subseteq P$ or $B^{g^{-1}} \subseteq \sqrt{P}$. If $A^{g^{-1}} \subseteq P$, then $\mu_A(x) = \mu_A((x^g)^{g^{-1}}) = \mu_{A^{g^{-1}}}(x^g) \leq \mu_P(x^g) = \mu_{P^g}(x)$. Similarly, we have $\nu_A(x) \geq \nu_{P^g}(x)$. Thus $A \subseteq P^g$. In a same way we can achieve that, if $B^{g^{-1}} \subseteq \sqrt{P}$, then $B \subseteq (\sqrt{P})^g = \sqrt{P^g}$. Hence P^g is an intuitionistic fuzzy primary ideal of $\mathcal{M}$. $\square$

Similarly, we can prove the following proposition.

Proposition 3.8. *If P is an intuitionistic fuzzy semiprimary ideal of a Γ-ring $\mathcal{M}$, then P^g is also an intuitionistic fuzzy semiprimary ideal of $\mathcal{M}$, where $g \in \mathcal{G}$ be any element.*

4 Intuitionistic fuzzy $\mathcal{G}$-primary and $\mathcal{G}$-semiprimary ideal

Following the definition of $\mathcal{G}$-invariant ideal of a Γ-ring $\mathcal{M}$, we define $\mathcal{G}$-invariant intuitionistic fuzzy ideal and $\mathcal{G}$-invariant intuitionistic fuzzy primary and semiprimary ideal of $\mathcal{M}$.

Definition 4.1. ([11]) Let A be an IFS of a Γ-ring $\mathcal{M}$, and let $\mathcal{G}$ be a group which acts on $\mathcal{M}$. Then A is said to be $\mathcal{G}$-invariant IFS of $\mathcal{M}$ if and only if

$$\mu_{A^g}(x) = \mu_A(x^g) \geq \mu_A(x), \nu_{A^g}(x) = \nu_A(x^g) \leq \nu_A(x), \forall x \in \mathcal{M}, \forall g \in \mathcal{G}.$$

Proposition 4.2. ([11]) *Let A be an IFS of a Γ-ring $\mathcal{M}$, and let $\mathcal{G}$ be a finite group which acts on $\mathcal{M}$. Then A is $\mathcal{G}$-invariant IFS of $\mathcal{M}$ if and only if $A^g = A$, for all $g \in \mathcal{G}$.*

Theorem 4.3. ([11]) *Let A be an IFS of a Γ-ring $\mathcal{M}$, and let $\mathcal{G}$ be a finite group which acts on $\mathcal{M}$. Let $A^{\mathcal{G}} = \bigcap_{g\in\mathcal{G}} A^g$. Then $A^{\mathcal{G}} = (\mu_{A^{\mathcal{G}}}, \nu_{A^{\mathcal{G}}})$, where $\mu_{A^{\mathcal{G}}}(x) = \min\{\mu_A(x^g) : g \in \mathcal{G}\}$ and $\nu_{A^{\mathcal{G}}}(x) = \max\{\nu_A(x^g) : g \in \mathcal{G}\}, \forall x \in \mathcal{M}$. Moreover, $A^{\mathcal{G}}$ is the largest $\mathcal{G}$-invariant IFS of $\mathcal{M}$ contained in A.*

Proposition 4.4. ([11]) *Let A be an IFI of a Γ-ring $\mathcal{M}$, and let $\mathcal{G}$ be a finite group which acts on $\mathcal{M}$. Then $A^{\mathcal{G}}$ is the largest $\mathcal{G}$-invariant IFI of $\mathcal{M}$ contained in A.*

Proposition 4.5. ([11]) *An IFI A of a Γ-ring $\mathcal{M}$ is $\mathcal{G}$-invariant IFI of $\mathcal{M}$ if and only if $A^{\mathcal{G}} = A$.*

Proposition 4.6. *If A is a $\mathcal{G}$-invariant IFI of a Γ-ring $\mathcal{M}$ and $\mathcal{G}$ be a finite group which acts on A, then $(\sqrt{A})^{\mathcal{G}} = \sqrt{A^{\mathcal{G}}}$.*

Proof. Let $x \in \mathcal{M}, g \in \mathcal{G}$ be any element. Then

$$\begin{aligned} \mu_{(\sqrt{A})^{\mathcal{G}}}(x) &= \min\{\mu_{\sqrt{A}}(x^g) : g \in \mathcal{G}\} \\ &= \min\{\sup\{\mu_A((x^g\gamma)^{m-1}x^g) : m \in \mathbb{N}\} : g \in \mathcal{G}\} \\ &= \min\{\sup\{\mu_A(((x\gamma)^{m-1}x)^g) : m \in \mathbb{N}\} : g \in \mathcal{G}\} \\ &= \sup\{\min\{\mu_A(((x\gamma)^{m-1}x)^g) : g \in \mathcal{G}\} : m \in \mathbb{N}\} \\ &= \sup\{\mu_{A^{\mathcal{G}}}((x\gamma)^{m-1}x) : m \in \mathbb{N}\} \\ &= \mu_{\sqrt{A^{\mathcal{G}}}}(x). \end{aligned}$$

Similarly, we can show that $\nu_{(\sqrt{A})^{\mathcal{G}}}(x) = \nu_{\sqrt{A^{\mathcal{G}}}}(x), \forall x \in \mathcal{M}$.
This implies that $(\sqrt{A})^{\mathcal{G}} = \sqrt{A^{\mathcal{G}}}$. $\square$

Theorem 4.7. ([11]) *If A and B are $\mathcal{G}$-invariant IFIs of a Γ-ring $\mathcal{M}$, then $A + B$ and $A\Gamma B$ are also $\mathcal{G}$-invariant IFIs of $\mathcal{M}$.*

Definition 4.8. Let P be a non-constant IFI of a Γ-ring $\mathcal{M}$ and $\mathcal{G}$ be a finite group which acts on P. Then P is said to be an intuitionistic fuzzy $\mathcal{G}$-primary ideal of $\mathcal{M}$ if P is $\mathcal{G}$-invariant intuitionistic fuzzy primary ideal of $\mathcal{M}$.

Definition 4.9. Let P be a non-constant IFI of a Γ-ring $\mathcal{M}$ and $\mathcal{G}$ be a finite group which acts on P. Then P is said to be an intuitionistic fuzzy $\mathcal{G}$-semiprimary ideal of $\mathcal{M}$ if P is $\mathcal{G}$-invariant intuitionistic fuzzy semiprimary ideal of $\mathcal{M}$.

Proposition 4.10. *Let P be an intuitionistic fuzzy $\mathcal{G}$-primary ideal of $\mathcal{M}$. Then $P_{(s,t)}$ is a $\mathcal{G}$-primary ideal of $\mathcal{M}$, where $s \in [\mu_P(1), \mu_P(0)]$ and $t \in [\nu_P(0), \nu_P(1)]$ such that $s + t \leq 1$.*

Proof. It is easy to show that $P_{(s,t)}$ is an ideal of $\mathcal{M}$. We show that $P_{(s,t)}$ is $\mathcal{G}$-invariant. Let $x \in P_{(s,t)}, g \in \mathcal{G}$ be any element. Since P is $\mathcal{G}$-invariant intuitionistic fuzzy primary ideal of $\mathcal{M}$, so $\mu_P(x^g) = \mu_P(x) \geq s$ and $\nu_P(x^g) = \nu_P(x) \leq t$, for any $g \in \mathcal{G}$ implies that $x^g \in P_{(s,t)}, \forall g \in \mathcal{G}$. Hence $P_{(s,t)}$ is $\mathcal{G}$-invariant.

Next we show that $P_{(s,t)}$ is primary ideal of $\mathcal{M}$. Let I and J be two $\mathcal{G}$-invariant ideals of $\mathcal{M}$ such that $I\Gamma J \subseteq P_{(s,t)}$. Define two IFSs $A = \chi_I$ and $B = \chi_J$. It is easy to check that A and B are $\mathcal{G}$-invariant IFIs of $\mathcal{M}$ (as I and J are $\mathcal{G}$-invariant ideals). We claim that $A\Gamma B \subseteq P$. Let $x \in \mathcal{M}$ be any element. If $A\Gamma B(x) = (0, 1)$, there is nothing to prove. If $A\Gamma B(x) \neq (0, 1)$, then

$$\mu_{A\Gamma B}(x) = \sup_{x=y\gamma z} (\mu_A(y) \wedge \mu_B(z)) = \sup_{x=y\gamma z} (\chi_I(y) \wedge \chi_J(z)) \neq 0,$$
$$\nu_{A\Gamma B}(x) = \inf_{x=y\gamma z} (\nu_A(y) \vee \nu_B(z)) = \inf_{x=y\gamma z} (\chi_I(y) \vee \chi_J(z)) \neq 1.$$

This implies that there exist $y \in I, z \in J, \gamma \in \Gamma$ such that $x = y\gamma z$. Moreover, $A\Gamma B(x) = (s, t)$. Thus $x = y\gamma z \in I\Gamma J \subseteq P_{(s,t)}$. So $\mu_P(x) \geq s, \nu_P(x) \leq t$. Hence $A\Gamma B \subseteq P$. Since P is an intuitionistic fuzzy $\mathcal{G}$-primary ideal of $\mathcal{M}$, either $A \subseteq P$ or $B \subseteq \sqrt{P}$. Suppose that, $A \subseteq P$, then $I \subseteq P_{(s,t)}$. For, if $I \nsubseteq P_{(s,t)}$, then there exists an element $a \in \mathcal{M}$ such that $a \in I$, but $a \notin P_{(s,t)}$. This implies that $\mu_A(a) = \mu_{\chi_I}(a) = s$ and $\nu_A(a) = \nu_{\chi_I}(a) = t$, but $\mu_P(a) < s$ and $\nu_P(a) > t$. Thus $\mu_A(a) = s > \mu_P(a)$ and $\nu_A(a) = t < \nu_P(a)$. Hence $A \nsubseteq P$, a contradiction. Similarly, if $B \subseteq \sqrt{P}$, then $J \subseteq \sqrt{P_{(s,t)}}$. Hence $P_{(s,t)}$ is $\mathcal{G}$-primary ideal of $\mathcal{M}$. □

Similarly, we can prove the following proposition.

Proposition 4.11. *Let P be an intuitionistic fuzzy $\mathcal{G}$-semiprimary ideal of $\mathcal{M}$. Then $P_{(s,t)}$ is a $\mathcal{G}$-semiprimary ideal of $\mathcal{M}$, where $s \in [\mu_P(1), \mu_P(0)]$ and $t \in [\nu_P(0), \nu_P(1)]$ such that $s + t \leq 1$.*

Proposition 4.12. *If P is an intuitionistic fuzzy $\mathcal{G}$-semiprimary ideal of $\mathcal{M}$, then $P = (\sqrt{P})^{\mathcal{G}}$.*

Proof. Let P is an intuitionistic fuzzy $\mathcal{G}$-semiprimary ideal of $\mathcal{M}$. Therefore, P is $\mathcal{G}$-invariant intuitionistic fuzzy semiprimary ideal of $\mathcal{M}$ and so, $P^{\mathcal{G}} = P$.

$$\begin{aligned}
\mu_{(\sqrt{P})^{\mathcal{G}}}(x) &= \min\{\mu_{\sqrt{P}}(x^g) : g \in \mathcal{G}\} \\
&= \min\{\sup\{\mu_P((x^g\gamma)^{m-1}x^g) : m \in \mathbb{N}\} : g \in \mathcal{G}\} \\
&= \min\{\sup\{\mu_P(((x\gamma)^{m-1}x)^g) : m \in \mathbb{N}\} : g \in \mathcal{G}\} \\
&= \sup\{\min\{\mu_P((((x\gamma)^{m-1}x)^g) : g \in \mathcal{G}\} : m \in \mathbb{N}\} \\
&= \sup\{\mu_P((x\gamma)^{m-1}x) : m \in \mathbb{N}\} \qquad [\because \mu_P(t^g) = \mu_P(t), \forall t \in \mathcal{M}, g \in \mathcal{G}] \\
&= \mu_{\sqrt{P}}(x) \\
&\leq \mu_P(x).
\end{aligned}$$

Similarly, we can show that $\nu_{(\sqrt{P})^{\mathcal{G}}}(x) \geq \nu_P(x)$. Thus $(\sqrt{P})^{\mathcal{G}} \subseteq P$.

For the other inclusion, we have $\mu_P(x) = \mu_{P^{\mathcal{G}}}(x) = \min\{\mu_P(x^g) : g \in \mathcal{G}\}$. As A is an IFI of $\mathcal{M}$, we have $\mu_A(x) \leq \mu_A((x\gamma)^{m-1}x), \forall m \in \mathbb{N}, x \in \mathcal{M}, \gamma \in \Gamma$ and so

$$\begin{aligned}
\min\{\mu_P(x^g) : g \in G\} &\leq \min\{\mu_P((x^g\gamma)^{m-1}x^g) : g \in \mathcal{G}, m \in \mathbb{N}\} \\
&\leq \min\{\sup\{\mu_P((x^g\gamma)^{m-1}x^g) : m \in \mathbb{N}\} : g \in \mathcal{G}\} \\
&= \min\{\mu_{\sqrt{P}}(x^g) : g \in \mathcal{G}\} \\
&= \mu_{(\sqrt{P})^{\mathcal{G}}}(x).
\end{aligned}$$

Thus $\mu_P(x) \leq \mu_{(\sqrt{P})^{\mathcal{G}}}(x)$. Similarly, we can show that $\nu_P(x) \geq \nu_{(\sqrt{P})^{\mathcal{G}}}(x)$.
Thus we have $P \subseteq (\sqrt{P})^{\mathcal{G}}$. Hence the result proved. □

From the above discussion on the results on intuitionistic fuzzy primary (semiprimary) ideals that are $\mathcal{G}$-invariant also. We can also define intuitionistic fuzzy $\mathcal{G}$-primary ($\mathcal{G}$-semiprimary) ideals in the following ways, too.

Definition 4.13. A non-constant $\mathcal{G}$-invariant IFI P of a Γ-ring $\mathcal{M}$ is said to be $\mathcal{G}$-primary IFI if for any two $\mathcal{G}$-invariant IFIs A and B of $\mathcal{M}$ such that $A\Gamma B \subseteq P$ implies either $A \subseteq P$ or $B \subseteq \sqrt{P}$.

Definition 4.14. A non-constant $\mathcal{G}$-invariant IFI P of a Γ-ring $\mathcal{M}$ is said to be $\mathcal{G}$-semiprimary IFI if for any two $\mathcal{G}$-invariant IFIs A and B of $\mathcal{M}$ such that $A\Gamma B \subseteq P$ implies either $A \subseteq \sqrt{P}$ or $B \subseteq \sqrt{P}$.

Proposition 4.15. *Let P be an intuitionistic fuzzy $\mathcal{G}$-invariant ideal of $\mathcal{M}$. Then the following statements are equivalent:*

1. *P is an intuitionistic fuzzy $\mathcal{G}$-primary ideal of $\mathcal{M}$;*
2. *For any $x_{(p,q)}, y_{(s,t)} \in IFP(\mathcal{M})$, where $x, y \in \mathcal{M}$ are $\mathcal{G}$-invariant points such that $x_{(p,q)}\Gamma y_{(s,t)} \subseteq P$, this implies that either $x_{(p,q)} \subseteq P$ or $y_{(s,t)} \subseteq \sqrt{P}$.*

Proof. $\underline{(1) \Rightarrow (2)}$ Suppose that P is an intuitionistic fuzzy $\mathcal{G}$-primary ideal of $\mathcal{M}$.
Let $x_{(p,q)}, y_{(s,t)} \in IFP(\mathcal{M})$, where $x, y \in \mathcal{M}$ are $\mathcal{G}$-invariant points such that $x_{(p,q)}\Gamma y_{(s,t)} \subseteq P$. Then $x_{(p,q)}\Gamma y_{(s,t)} = (x\Gamma y)_{(p\wedge s, q\vee t)}$, where $\mu_P(x\gamma y) \geq p \wedge s$ and $\nu_P(x\gamma y) \leq q \vee t$. Let us define IFSs A, B of $\mathcal{M}$ by $A = \chi_{\langle x\rangle}$ and $B = \chi_{\langle y\rangle}$. Clearly, A and B are $\mathcal{G}$-invariant IFIs of $\mathcal{M}$. Now $\mu_{A\Gamma B}(z) = \sup_{z=u\gamma v}(\mu_A(u) \wedge \mu_B(v)) = p \wedge s$ and $\nu_{A\Gamma B}(z) = \inf_{z=u\gamma v}(\nu_A(u) \vee \nu_B(v)) = q \vee t$, where $u \in \langle x\rangle$ and $v \in \langle y\rangle$. Thus $\mu_{A\Gamma B}(z) = p \wedge s \leq \mu_P(z)$ and $\nu_{A\Gamma B}(z) = q \vee t \geq \nu_P(z)$, when $z = u\gamma v$, where $u \in \langle x\rangle$ and $v \in \langle y\rangle$. Otherwise $A\Gamma B(z) = (0, 1)$, i.e., $A\Gamma B \subseteq P$. As P is an intuitionistic fuzzy $\mathcal{G}$-primary ideal, so either $A \subseteq P$ or $B \subseteq \sqrt{P}$. Then $x_{(p,q)} \subseteq A \subseteq P$ or $y_{(s,t)} \subseteq \sqrt{B} \subseteq \sqrt{P}$.

$\underline{(2) \Rightarrow (1)}$ Let A and B be two $\mathcal{G}$-invariant IFIs of $\mathcal{M}$ such that $A\Gamma B \subseteq P$. Suppose $A \nsubseteq P$. Then there exists a $\mathcal{G}$-invariant element $x \in \mathcal{M}$ such that $\mu_A(x) > \mu_P(x)$ and $\nu_A(x) < \nu_P(x)$. Let $\mu_A(x) = p, \nu_A(x) = q$. Let $y \in \mathcal{M}$ be $\mathcal{G}$-invariant element of $\mathcal{M}$ such that $\mu_A(x) = r, \nu_A(x) = s$. If $z = x\gamma y$, then $x_{(p,q)}\Gamma y_{(s,t)} = (x\gamma y)_{(p\wedge s, q\vee t)}$. Hence

$$\mu_P(z) = \mu_P(x\gamma y) \geq \mu_{A\Gamma B}(x\gamma y) \geq [\mu_A(x) \wedge \mu_B(y)] = p \wedge s = \mu_{(x\gamma y)_{(p\wedge s, q\vee t)}}(z).$$

Similarly, we have $\nu_P(z) \leq \nu_{(x\gamma y)_{(p\wedge s, q\vee t)}}(z)$. Hence $x_{(p,q)}\Gamma y_{(s,t)} \subseteq P$, then by (1), we get either $x_{(p,q)} \subseteq P$ or $y_{(s,t)} \subseteq \sqrt{P}$, i.e., either $\mu_P(x) \geq p$, $\nu_P(x) \leq q$ or $\mu_{\sqrt{P}}(x) \geq s$, $\nu_{\sqrt{P}}(x) \leq t$. Since $\mu_P(x) \ngeq p, \nu_P(x) \nleq q$ and $\mu_B(y) = s \leq \mu_{\sqrt{P}}(y)$, $\nu_B(y) = t \geq \nu_{\sqrt{P}}(y)$, so, $B \subseteq \sqrt{P}$. Hence P is an intuitionistic fuzzy $\mathcal{G}$-primary ideal of $\mathcal{M}$. □

Similarly, we can prove the following proposition.

Proposition 4.16. *Let P be an intuitionistic fuzzy $\mathcal{G}$-invariant ideal of $\mathcal{M}$. Then the following are equivalent:*

1. *P is an intuitionistic fuzzy $\mathcal{G}$-semiprimary ideal of $\mathcal{M}$;*
2. *For any $x_{(p,q)}, y_{(s,t)} \in IFP(\mathcal{M})$, where $x, y \in \mathcal{M}$ are $\mathcal{G}$-invariant points such that $x_{(p,q)}\Gamma y_{(s,t)} \subseteq P$ implies that either $x_{(p,q)} \subseteq \sqrt{P}$ or $y_{(s,t)} \subseteq \sqrt{P}$.*

Proposition 4.17. *If P is an intuitionistic fuzzy primary ideal of a Γ-ring $\mathcal{M}$, then $P^{\mathcal{G}}$ is $\mathcal{G}$-primary IFI ideal of $\mathcal{M}$. Conversely, if Q is an intuitionistic fuzzy $\mathcal{G}$-primary ideal of $\mathcal{M}$, then there exists an intuitionistic fuzzy primary ideal P of $\mathcal{M}$ such that $P^{\mathcal{G}} = Q$.*

Proof. Let P be an intuitionistic fuzzy primary ideal of the Γ-ring $\mathcal{M}$ and let A and B be two $\mathcal{G}$-invariant IFIs of $\mathcal{M}$ such that $A\Gamma B \subseteq P^{\mathcal{G}}$. Then $A\Gamma B \subseteq P$ (since $P^{\mathcal{G}} \subseteq P$ always). So, either $A \subseteq P$ or $B \subseteq \sqrt{P}$. But $P^{\mathcal{G}}$ is the largest $\mathcal{G}$-invariant IFI of $\mathcal{M}$ contained in P. So, either $A \subseteq P^{\mathcal{G}}$ or $B \subseteq (\sqrt{P})^{\mathcal{G}} = \sqrt{P^{\mathcal{G}}}$. Hence $P^{\mathcal{G}}$ is $\mathcal{G}$-primary IFI ideal of $\mathcal{M}$.

For the converse part, suppose that Q is an IF $\mathcal{G}$-primary ideal of $\mathcal{M}$. Therefore, $Q^{\mathcal{G}} = Q$. Let $\mathcal{S} = \{P | P$ is an IFI of $\mathcal{M}$ with $P^{\mathcal{G}} \subseteq Q\}$. By Zorn's lemma, there exists an intuitionistic fuzzy maximal ideal P such that $P^{\mathcal{G}} \subseteq Q$. Let A and B be two IFIs of $\mathcal{M}$ such that $A\Gamma B \subseteq P$. Then $(A\Gamma B)^{\mathcal{G}} \subseteq P^{\mathcal{G}} \subseteq Q$. Since $A^{\mathcal{G}}$ and $B^{\mathcal{G}}$ are the largest IFIs of $\mathcal{M}$ contained in A and B, respectively, we claim that $A^{\mathcal{G}}\Gamma B^{\mathcal{G}} \subseteq A\Gamma B$ is a $\mathcal{G}$-invariant.

$$\begin{aligned}
\mu_{A^{\mathcal{G}}\Gamma B^{\mathcal{G}}}(x^g) &= \sup_{x^g = u\gamma v} \min\{\mu_{A^{\mathcal{G}}}(u), \mu_{B^{\mathcal{G}}}(v)\} \\
&= \sup_{x = u^{g^{-1}}\gamma v^{g^{-1}}} \min\{\mu_{A^{\mathcal{G}}}(u^{g^{-1}}), \mu_{B^{\mathcal{G}}}(v^{g^{-1}})\} \\
&= \mu_{A^{\mathcal{G}}\Gamma B^{\mathcal{G}}}(x).
\end{aligned}$$

Similarly, we can show that $\nu_{A^{\mathcal{G}}\Gamma B^{\mathcal{G}}}(x^g) = \nu_{A^{\mathcal{G}}\Gamma B^{\mathcal{G}}}(x)$. Hence $A^{\mathcal{G}}\Gamma B^{\mathcal{G}} \subseteq (A\Gamma B)^{\mathcal{G}} \subseteq Q$. Since Q is an IF $\mathcal{G}$-primary ideal of $\mathcal{M}$, then we have either $A^{\mathcal{G}} \subseteq Q$ or $B^{\mathcal{G}} \subseteq \sqrt{Q}$. By maximality of P either $A \subseteq P$ or $B \subseteq \sqrt{P}$. This implies that P is an IF primary ideal of $\mathcal{M}$. As $Q^{\mathcal{G}} = Q$, we have $Q \in \mathcal{S}$. But maximality of P gives that $Q \subseteq P$. Since P and $Q^{\mathcal{G}}$ are $\mathcal{G}$ invariant and $P^{\mathcal{G}}$ is the largest in P, we get $Q \subseteq P^{\mathcal{G}}$. Hence $P^{\mathcal{G}} = Q$. □

Similarly, we can prove the following proposition.

Proposition 4.18. *If P is an intuitionistic fuzzy semiprimary ideal of a Γ-ring $\mathcal{M}$, then $P^{\mathcal{G}}$ is $\mathcal{G}$-semiprimary IFI ideal of $\mathcal{M}$. Conversely, if Q is an intuitionistic fuzzy $\mathcal{G}$-semiprimary ideal of $\mathcal{M}$, then there exists an intuitionistic fuzzy semiprimary ideal P of $\mathcal{M}$ such that $P^{\mathcal{G}} = Q$.*

5 $\mathcal{G}$-Homomorphism of intuitionistic fuzzy $\mathcal{G}$-ideals

In this section, we study the image and preimage of intuitionistic fuzzy $\mathcal{G}$-primary and $\mathcal{G}$-semiprimary ideals under the Γ-ring homomorphism.

Definition 5.1. ([11]) A Γ-ring homomorphism $\phi : \mathcal{M} \to \mathcal{M}'$ from a Γ-ring $\mathcal{M}$ to a Γ-ring $\mathcal{M}'$ with unity is called $\mathcal{G}$-homomorphism, if for all $g \in \mathcal{G}, x \in \mathcal{M}, \phi(g * x) = g * \phi(x)$, where group $\mathcal{G}$ acts on both the Γ-rings.

Lemma 5.2. ([11]) *Let $\mathcal{M}$ and $\mathcal{M}'$ be Γ-rings and $\mathcal{G}$ be a finite group which acts on $\mathcal{M}$ and $\mathcal{M}'$. Let $f : \mathcal{M} \to \mathcal{M}'$ is a function defined by $f(x^g) = (f(x))^g, \forall x \in \mathcal{M}, g \in \mathcal{G}$. Then f is a Γ-ring homomorphism. Moreover, f is also a $\mathcal{G}$-homomorphism.*

Lemma 5.3. ([11]) *Let $\mathcal{M}$ and $\mathcal{M}'$ be Γ-rings and $\mathcal{G}$ be a finite group which acts on $\mathcal{M}$ and $\mathcal{M}'$. Let $f : \mathcal{M} \to \mathcal{M}'$ be a $\mathcal{G}$-homomorphism and A, B are IFSs of $\mathcal{M}$ and $\mathcal{M}'$ respectively. Then*

1. $f^{-1}(B^g) = (f^{-1}(B))^g, \forall g \in \mathcal{G}$;
2. $f(A^g) = (f(A))^g, \forall g \in \mathcal{G}$.

Theorem 5.4. ([11]) *Let $\mathcal{M}$ and $\mathcal{M}'$ be Γ-rings and $\mathcal{G}$ be a finite group which acts on $\mathcal{M}$ and $\mathcal{M}'$. Let $f : \mathcal{M} \to \mathcal{M}'$ be a $\mathcal{G}$-homomorphism. If B is an intuitionistic fuzzy $\mathcal{G}$-ideal of $\mathcal{M}'$, then $f^{-1}(B)$ is an intuitionistic fuzzy $\mathcal{G}$-ideal of $\mathcal{M}$.*

Theorem 5.5. ([11]) *Let $\mathcal{M}$ and $\mathcal{M}'$ be Γ-rings and $\mathcal{G}$ be a finite group which acts on $\mathcal{M}$ and $\mathcal{M}'$. Let $f : \mathcal{M} \to \mathcal{M}'$ be a $\mathcal{G}$-epimorphism. If A is an intuitionistic fuzzy $\mathcal{G}$-ideal of $\mathcal{M}$ which is constant on* $\operatorname{Ker} f$ *of $\mathcal{M}$, then $f(A)$ is an intuitionistic fuzzy $\mathcal{G}$-ideal of $\mathcal{M}'$.*

Theorem 5.6. *Let $\mathcal{M}$ and $\mathcal{M}'$ be Γ-rings and $\mathcal{G}$ be a finite group which acts on $\mathcal{M}$ and $\mathcal{M}'$. Let $f : \mathcal{M} \to \mathcal{M}'$ be a $\mathcal{G}$-homomorphism. If P is an intuitionistic fuzzy $\mathcal{G}$-primary ideal of $\mathcal{M}'$, then $f^{-1}(P)$ is an intuitionistic fuzzy $\mathcal{G}$-primary ideal of $\mathcal{M}$.*

Proof. Since P is an intuitionistic fuzzy $\mathcal{G}$-primary ideal of $\mathcal{M}'$, so by Theorem (2.19) $f^{-1}(P)$ is also an intuitionistic fuzzy primary ideal of $\mathcal{M}$. So it remains to show that $f^{-1}(P)$ is $\mathcal{G}$-invariant. For this consider $x \in \mathcal{M}, g \in \mathcal{G}$ be any elements. Then we have

$$\mu_{f^{-1}(P)}(x^g) = \mu_P(f(x^g)) = \mu_P((f(x))^g) = \mu_P((f(x))) = \mu_{f^{-1}(P)}(x).$$

Similarly, we can show that $\nu_{f^{-1}(P)}(x^g) = \nu_{f^{-1}(P)}(x)$. Thus $f^{-1}(P)$ is $\mathcal{G}$-invariant. Hence $f^{-1}(P)$ is an intuitionistic fuzzy $\mathcal{G}$-primary ideal of $\mathcal{M}$. $\square$

Theorem 5.7. *Let $\mathcal{M}$ and $\mathcal{M}^{'}$ be Γ-rings and $\mathcal{G}$ be a finite group which acts on $\mathcal{M}$ and $\mathcal{M}^{'}$. Let $f : \mathcal{M} \to \mathcal{M}^{'}$ be a $\mathcal{G}$-epimorphism. If P is an intuitionistic fuzzy $\mathcal{G}$-primary ideal which is constant on* $\operatorname{Ker} f$ *of $\mathcal{M}$, then $f(P)$ is an intuitionistic fuzzy $\mathcal{G}$-primary ideal of $\mathcal{M}^{'}$.*

Proof. Since P is an intuitionistic fuzzy $\mathcal{G}$-primary ideal of $\mathcal{M}$ which is constant on $\operatorname{Ker} f$ of $\mathcal{M}$, so by Theorem (2.20) $f(P)$ is also an intuitionistic fuzzy primary ideal of $\mathcal{M}^{'}$. So it remains to show that $f(P)$ is $\mathcal{G}$-invariant. For this consider $y \in \mathcal{M}^{'}, g \in \mathcal{G}$ be any element. As f is an epimorphism so, there exists $x \in \mathcal{M}$ such that $f(x) = y$. Then we have

$$\begin{aligned}\mu_{f(P)}(y^g) &= \mu_{(f(P))^g}(y) = \mu_{f(P^g)}(y) = \mu_{P^g}(f^{-1}(y)) = \mu_{P^g}(x) \\ &= \mu_P(x^g) = \mu_P(x) = \mu_P(f^{-1}(y)) = \mu_{f(P)}(y).\end{aligned}$$

Similarly, we can show that $\nu_{f(P)}(y^g) = \nu_{f(P)}(y)$. Thus $f(P)$ is $\mathcal{G}$-invariant. Hence $f(P)$ is an intuitionistic fuzzy $\mathcal{G}$-primary ideal of $\mathcal{M}^{'}$. $\square$

Similarly, we can prove the following theorems.

Theorem 5.8. *Let $\mathcal{M}$ and $\mathcal{M}^{'}$ be Γ-rings and $\mathcal{G}$ be a finite group which acts on $\mathcal{M}$ and $\mathcal{M}^{'}$. Let $f : \mathcal{M} \to \mathcal{M}^{'}$ be a $\mathcal{G}$-homomorphism. If P is an intuitionistic fuzzy $\mathcal{G}$-semiprimary ideal of $\mathcal{M}^{'}$, then $f^{-1}(P)$ is an intuitionistic fuzzy $\mathcal{G}$-semiprimary ideal of $\mathcal{M}$.*

Theorem 5.9. *Let $\mathcal{M}$ and $\mathcal{M}^{'}$ be Γ-rings and $\mathcal{G}$ be a finite group which acts on $\mathcal{M}$ and $\mathcal{M}^{'}$. Let $f : \mathcal{M} \to \mathcal{M}^{'}$ be a $\mathcal{G}$-epimorphism. If P is an intuitionistic fuzzy $\mathcal{G}$-semiprimary ideal which is constant on* $\operatorname{Ker} f$ *of $\mathcal{M}$, then $f(P)$ is an intuitionistic fuzzy $\mathcal{G}$-semiprimary ideal of $\mathcal{M}^{'}$.*

6 Conclusion

In this study, we have undertaken a comprehensive investigation of the influence of group actions on intuitionistic fuzzy ideals within the framework of a Γ-ring $\mathcal{M}$. Our primary focus has been on establishing and elucidating the structural relationships between intuitionistic fuzzy $\mathcal{G}$-primary ($\mathcal{G}$-semiprimary) ideals and the corresponding intuitionistic fuzzy primary (semiprimary) ideals of $\mathcal{M}$. Notably, we demonstrated that the largest $\mathcal{G}$-invariant intuitionistic fuzzy ideal contained within an intuitionistic fuzzy primary (semiprimary) ideal is itself an intuitionistic fuzzy $\mathcal{G}$-primary ($\mathcal{G}$-semiprimary) ideal of $\mathcal{M}$. Conversely, if Q is an intuitionistic fuzzy $\mathcal{G}$-primary ($\mathcal{G}$-semiprimary) ideal of $\mathcal{M}$, then there exists an intuitionistic fuzzy primary (semiprimary) ideal P of $\mathcal{M}$ such that $P^{\mathcal{G}} = Q$.

Furthermore, we explored the conditions under which intuitionistic fuzzy $\mathcal{G}$-semiprimary ideals are connected to the radical of intuitionistic fuzzy ideals, thereby deepening our understanding of the ideal structure under group actions. The study also included an analysis of level cut sets associated with intuitionistic fuzzy $\mathcal{G}$-primary ($\mathcal{G}$-semiprimary) ideals, highlighting their behavior under group action. Additionally, we provided a robust characterization of these ideals in terms of intuitionistic fuzzy points of $\mathcal{M}$, offering an alternative yet insightful perspective on their structure.

Finally, we examined the behavior of intuitionistic fuzzy $\mathcal{G}$-primary ($\mathcal{G}$-semiprimary) ideals under $\mathcal{G}$-homomorphisms, thereby extending our findings to a broader algebraic context. Overall, the results obtained contribute to a deeper and more unified understanding of the interplay between group actions and intuitionistic fuzzy ideal theory in Γ-rings, paving the way for further explorations in this area.

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