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A note on new distances between intuitionistic fuzzy sets

Peter Vassilev

Institute of Biophysics and Biomedical Engineering
Bulgarian Academy of Sciences
105 Acad. G. Bonchev Str., Sofia 1113, Bulgaria
e-mail: peter.vassilev@gmail.com

Abstract: In the present paper new distances between intuitionistic fuzzy sets are proposed. If the sets are fuzzy they agree with the well known distance defined over fuzzy sets.
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1 Introduction

To reflect better the uncertainty and vagueness inherent in real world, in 1965 L. Zadeh introduced the notion of fuzzy sets [6]. In 1983, K. Atanassov introduced the extended notion of intuitionistic fuzzy sets (cf. [1]).

We will briefly remind some basic definitions and notions.

Let X be a universe set, $A \subset X$, $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ are mappings reflecting the degree of membership and non-membership of the element $x \in X$ to the set A, respectively, such that for every x it is fulfilled that

$$\mu_A(x) + \nu_A(x) \leq 1 \tag{1}$$

Definition 1. Following [1], we call the set

$$A^* \stackrel{\text{def}}{=} \{x, \mu_A(x), \nu_A(x) | x \in E\}$$

an intuitionistic fuzzy set (IFS) and the mapping $\pi_A : X \to [0, 1]$, which is given in explicit form by

$$\pi_A(x) \stackrel{\text{def}}{=} 1 - \mu_A(x) - \nu_A(x), \tag{2}$$

is called **intuitionistic fuzzy index** (sometimes also: *hesitancy margin* or degree of *indeterminacy*) of the element x (cf. [4]).

Further we denote the class of all IFSs defined over a universe set X by IFS(X).

Definition 2 (cf. [1, p.134, (7.1)], [4, p.43, Definition 3.4]). For a given IFS $A \in$ IFS(X) the *degree of definiteness* of the element x is said to be:

$$\sigma_{1,A}(x) \stackrel{\text{def}}{=} \mu_A(x) + \nu_A(x) \tag{3}$$

This degree provides an intuitive measure of the certainty of the knowledge established for the element. Indeed, it is easy to see that it is directly related to *intuitionistic fuzzy index*, since for all $x \in X$, we have

$$\mu_A(x) + \nu_A(x) + \pi_A(x) = 1 \tag{4}$$

and hence

$$\sigma_{1,A}(x) = 1 - \pi_A(x).$$

Definition 3 (cf. [4, p. 39, Definition 3.1.]). We say that d : IFS$(X) \times$ IFS$(X) \to [0, +\infty)$ is a distance between intuitionistic fuzzy sets if the following conditions are fulfilled:

$$d(U,V) = 0 \Leftrightarrow U = V \tag{5}$$

$$d(U,V) = d(V,U) \tag{6}$$

$$d(U,V) + d(V,Q) \geq d(U,Q) \tag{7}$$

If a distance is such that d_N : IFS$(X) \times$ IFS$(X) \to [0,1]$, we say that d_N is a *normalized distance*.

Definition 4. If only the condition (5) from Definition 3 is not true for d^*, i.e. $d^*(U,V) = 0$ for some $U \neq V$, we say that d^* is a pseudodistance.

Remark 1 (cf. [2, p. 113]). *Pseudodistances are often used in practice because they can sometimes detect certain "similarities" better than a true distance. Therefore, for particular task it may be beneficial to construct a distance of the form:*

$$d' = d + d^*,$$

where d *is a proper distance and* d^* *is a pseudodistance. The fact that* d' *is a distance follows directly from the definitions.*

For simplicity we suppose further that X is discrete and $X = \{x_1, x_2, \ldots, x_n\}$. One of the most used distances used between intuitionistic fuzzy sets is the following (normalized) Hamming distance:

$$l(A,B) = \frac{1}{2n} \sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| \tag{8}$$

Szmidt and Kacprzyk [3] proposed the following three term distance:

$$l^1_{\text{IFS}}(A,B) = \frac{1}{2n} \sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)| \tag{9}$$

2 The proposed distances

In the present study we were looking into ways to incorporate the information about the element derived from *degree of definiteness* into the distance. Based on this idea we propose the following

Theorem 1. Let $X = \{x_1, x_2, \ldots, x_n\}$. Then $d_\sigma : \text{IFS}(X) \times \text{IFS}(X) \to [0, 1]$ given for any $A, B \in \text{IFS}(X)$ by:

$$d_\sigma(A, B) = \frac{1}{2n}\sum_{i=1}^{n} |\sigma_{1,A}(x_i)\mu_A(x_i) - \sigma_{1,B}(x_i)\mu_B(x_i)| + |\sigma_{1,A}(x_i)\nu_A(x_i) - \sigma_{1,B}(x_i)\nu_B(x_i)| \quad (10)$$

is a distance.

Proof. The fact that d_σ satisfies (6) is obvious. Let us consider (5). We shall prove that it is satisfied. If $A = B$ it is obvious that $d_\sigma(A, B) = 0$. It remains to prove that if $d_\sigma(A, B) = 0$, then $A = B$.

It is well known that (see [5, p.11]):

$$|a| + |b| \geq |a + b| \quad (11)$$

Denoting for brevity:

$$\begin{aligned} a_i &= \sigma_{1,A}(x_i)\mu_A(x_i) - \sigma_{1,B}(x_i)\mu_B(x_i); \\ b_i &= \sigma_{1,A}(x_i)\nu_A(x_i) - \sigma_{1,B}(x_i)\nu_B(x_i); \\ c_i &= a_i + b_i = (\sigma_{1,A}(x_i))^2 - (\sigma_{1,B}(x_i))^2 \end{aligned} \quad (12)$$

Using (11) and (10) we obtain:

$$d_\sigma(A, B) = \frac{1}{2n}\sum_{i=1}^{n} |a_i| + |b_i| \geq \frac{1}{2n}\sum_{i=1}^{n} |c_i| \quad (13)$$

Hence, for $d_\sigma(A, B) = 0$ it is required that for all i:

$$|c_i| = 0.$$

However, this is only possible when $\sigma_{1,A}(x_i) = \sigma_{1,B}(x_i)$ for all i. But in such case it is easy to see that $|a_i| = |b_i| = 0$ if and only if $\mu_A(x_i) = \mu_B(x_i)$ and $\nu_A(x_i) = \nu_B(x_i)$ for all i. Hence, $d_\sigma(A, B) = 0$ implies $A = B$. Therefore, $d_\sigma(A, B)$ satisfies (5).

The validity of (7) follows directly from (11). □

Corollary 1. Let f be a continuous monotounously increasing function such that $f(0) = 0$ and $f(1) = 1$. Then

$$\begin{aligned} d_{f(\sigma)}(A, B) = &\frac{1}{2n}\sum_{i=1}^{n}(|f(\sigma_{1,A}(x_i))\mu_A(x_i) - f(\sigma_{1,B}(x_i))\mu_B(x_i)| \\ &+ |f(\sigma_{1,A}(x_i))\nu_A(x_i) - f(\sigma_{1,B}(x_i))\nu_B(x_i)|) \end{aligned} \quad (14)$$

is a distance.

Proof. The fact that $d_{f(\sigma)}$ satisfies (6) is obvious. The validity of (7) follows directly from (11). Let us consider (5). Using again (11), we obtain that a necessary condition for $d_{f(\sigma)}(A,B) = 0$, is $|f(\sigma_{1,A}(x_i))\sigma_{1,A}(x_i) - f(\sigma_{1,B}(x_i))\sigma_{1,B}(x_i)| = 0$. But this is only possible if $\sigma_{1,A}(x_i) = \sigma_{1,B}(x_i)$, since f is monotonously increasing. Hence, $f(\sigma_{1,A}(x_i)) = f(\sigma_{1,B}(x_i))$, i.e. $d_{f(\sigma)}(A,B) = 0$ only when $\mu_A(x) = \mu_B(x)$, $\nu_A(x) = \nu_B(x)$, that is when $A = B$. □

Corollary 2. Let

$$\sigma(A) \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=1}^{n}\sigma_{1,A}(x_i).$$

Then

$$d_{\sigma_{\text{avg}}}(A,B) = \frac{1}{2n}\sum_{i=1}^{n}(|\sigma(A)\mu_A(x_i) - \sigma(B)\mu_B(x_i)| + |\sigma(A)\nu_A(x_i) - \sigma(B)\nu_B(x_i)|) \tag{15}$$

is a distance.

Proof. The fact that $d_{\sigma_{\text{avg}}}(A,B)$ satisfies (6) is obvious. The validity of (7) follows directly from (11). Let us consider (5). Using (11), we obtain that a necessary condition for $d_{\sigma_{\text{avg}}}(A,B) = 0$, is $|\sigma(A)\sigma_{1,A}(x_i) - \sigma(B)\sigma_{1,B}(x_i)| = 0$. Without loss of generality assume that $\sigma(A) > \sigma(B)$. Then there exists at least one x_0, such that $\sigma_{1,A}(x_0) > \sigma_{1,B}(x_0)$, i.e. $|\sigma(A)\sigma_{1,A}(x_0) - \sigma(B)\sigma_{1,B}(x_0)| > 0$, hence, $d_{\sigma_{\text{avg}}}(A,B) > 0$. Thus we must have $\sigma(A) = \sigma(B)$, then if the necessary condition is to hold $\sigma_{1,A}(x_i) = \sigma_{1,B}(x_i)$, for all i and consequently, we conclude (as in the proofs above) that $d_{\sigma_{\text{avg}}}(A,B) = 0$ only when $A = B$. □

Corollary 3. We can define a three-term analogue of the above analogues to equation (9). In other words the following are normalized distances.

$$d'_{\sigma}(A,B) = d_{\sigma}(A,B) + \frac{1}{2n}\sum_{i=1}^{n}|\sigma_{1,A}(x_i)\pi_A(x_i) - \sigma_{1,B}(x_i)\pi_B(x_i)| \tag{16}$$

$$d'_{f(\sigma)}(A,B) = d_{f(\sigma)}(A,B) + \frac{1}{2n}\sum_{i=1}^{n}(|f(\sigma_{1,A}(x_i))\pi_A(x_i) - f(\sigma_{1,B}(x_i))\pi_B(x_i)| \tag{17}$$

$$d'_{\sigma_{\text{avg}}}(A,B) = d_{\sigma_{\text{avg}}}(A,B) + \frac{1}{2n}\sum_{i=1}^{n}|\sigma(A)\pi_A(x_i) - \sigma(B)\pi_B(x_i)| \tag{18}$$

Proof. The fact that all are distances follows from Remark 1 and Definitions 3 and 4. We will show that all are normalized, i.e. that they cannot obtain value greater than 1. Using the fact that (see [5, p.11]):

$$|a| - |b| \leq |a+b|,$$

and having in mind (4), we obtain, respectively:

$$d'_{\sigma}(A,B) \leq \frac{1}{2n}\sum_{i=1}^{n}(\sigma_{1,A}(x_i) + \sigma_{1,B}(x_i)) \leq 1$$

$$d'_{f(\sigma)}(A,B) \leq \frac{1}{2n}\sum_{i=1}^{n}(f(\sigma_{1,A}(x_i)) + f(\sigma_{1,B}(x_i))) \leq 1$$

$$d'_{\sigma_{\text{avg}}}(A,B) \leq \frac{1}{2}(\sigma(A)+\sigma(B)) \leq 1.$$

□

Remark 2. *The proposed here distances coincides with the distance between fuzzy sets given by:*

$$d(A,B) = \frac{1}{n}\sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)|$$

since for fuzzy sets $|\mu_A(x_i) - \mu_B(x_i)| = |\nu_A(x_i) - \nu_B(x_i)|$ *and* $\sigma_{1,A}(x_i) = \sigma_{1,B}(x_i) = 1$.

3 Conclusion

In the present paper we have proposed new distances between intuitionistic fuzzy sets which utilize membership, non-membership and the degree of definiteness. We showed that these distances coincide with the distance between fuzzy sets. In future work we will study more of their properties.

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