Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
2024, Volume 30, Number 3, 230-241
DOI: 10.7546/nifs.2024.30.3.230-241

Intuitionistic fuzzy normal bi-topological space-approach of intuitionistic fuzzy open sets

Saikh Shahjahan Miah[1,*], Jewel Miah[2], Ruhul Amin[3] and Nigar Sultana[4,5]

[1] Department of Mathematics, Faculty of Science, Mawlana Bhashani Science and Technology University, Tangail-1902, Bangladesh
e-mail: skhshahjahan@gmail.com

[2] Department of Mathematics, Faculty of Science, Mawlana Bhashani Science and Technology University, Tangail-1902, Bangladesh
e-mail: ranajewel29889@gmail.com

[3] Department of Mathematics, Faculty of Science, Begum Rokeya University Rangpur-5404, Bangladesh
e-mail: ruhulamin@brur.ac.bd

[4] Department of Mathematics, Faculty of Science, Mawlana Bhashani Science and Technology University, Tangail-1902, Bangladesh

[5] Department of Mathematics, North Carolina State University Raleigh, NC, United States
e-mail: nsultan4@ncsu.edu

[*] *Corresponding Author*

Received: 30 August 2024 **Revised:** 14 October 2024
Accepted: 16 October 2024 **Online First:** 16 October 2024

Abstract: This paper provides nine newly proposed notions of intuitionistic fuzzy normal bi-topological spaces (IFNBTS) based on the concept of most explored field fuzzy bi-topological spaces using intuitionistic fuzzy open sets (IFOS). Further, the authors establish implications among the prescribed notions and show that these notions are good extensions of normal and

fuzzy normal bi-topological spaces. Finally, the authors study the image and pre-image of IFNBTS, demonstrating that they are also IFNBTS in the sense of IFOS.
Keywords: Intuitionistic fuzzy set (IFS), Intuitionistic fuzzy topological space (IFTS), Intuitionistic fuzzy bi-topological space, Intuitionistic fuzzy normal bi- topological space (IFNBTS).
2020 Mathematics Subject Classification: 03E72, 54A40, 94D05.

1 Introduction

Fuzzy topology [14], a significant area of research in fuzzy mathematics, was initially explored by Chang in 1968, building upon the concept of fuzzy sets [32]. In 1983, the concept of an intuitionistic fuzzy set [9] was later introduced by Atanassov. This concept considers both membership as well as non-membership degrees, with the condition that their combined sum is not more than 1. The foundational concepts of IFTS were developed by Coker with other colleagues [12, 13, 16, 18]. Following this, several researchers, including Saadati and Park [30], Singh and Srivastava [31], Lee and Lee [22], Ahmed et al. [1, 2], Mahabub et al. [24] further advanced the study of these spaces using IFSs. Specifically, there has been considerable focus on the exploration of intuitionistic fuzzy normal spaces [29]. More recently, Islam et al. [21] have been active in the field of fuzzy logic. Additionally, many fuzzy topologists [6, 8, 11, 28] have explored the separation axioms [5, 7, 23, 25] in fuzzy, intuitionistic fuzzy [26], fuzzy neutrosophic [27] topological spaces. Notably, Al-Qubati [3] introduced and examined new types of b-separation axioms (bT_i, $i = 0, 1, 2$) in the context of IFTS. Al-Qubati [4] also investigated various classes of normal spaces, including β^*-normal spaces, β-normal spaces, π-generalized β^*-normal spaces, and β^*-generalized normal spaces, within IFTS. Applications of intuitionistic fuzzy composite relation and context are used in the medical diagnostic process [19, 20]. However, IFNBTS has not yet been studied in the literature, despite their potential significance in fuzzy mathematics compared to intuitionistic fuzzy topological spaces (IFTS).

In this paper, we define and explore the concept of intuitionistic fuzzy α-normal bi-topological spaces in nine different ways using intuitionistic fuzzy open sets, and we investigate their properties. The remainder of the paper is structured as follows: Section 2 presents basic notions with preliminary results related to intuitionistic fuzzy types, IFS, and their operations with relevant notions that are needed to understand our work. Section 3 includes the nine new concepts of IFNBTS, explores the implications among these notions, and examines their various features and properties.

2 Perliminary discussion

An intuitionistic set A in a non-empty set X is an object of the form $A = (X, A_1, A_2)$ where M_1 and M_2 are subsets of X with $A_1 \cap A_2 = \varnothing$. The set A_1 is known as the set of members of A while A_2 is known as the set of non-members of A. In this article, instead of $A = (X, A_1, A_2)$

we use the simpler notation $A = (A_1, A_2)$ for an intuitionistic set [15], whereas an intuitionistic fuzzy set M in X is an object of the form $M = \{(x, \lambda_M(x), \eta_M(x)) : x \in X\}$, in which λ_M and η_M are fuzzy sets in X denoting respectively the degree of membership and non-membership with $\lambda_M(x) + \eta_M(x) \leq 1$. In this work, instead of $M = \{(x, \lambda_M(x), \eta_M(x)) : x \in X\}$, we use the simpler notation $M = (\lambda_M, \eta_M)$ for intuitionistic fuzzy sets [9].

On a nonempty set X, the intuitionistic fuzzy topology t [17] (in short, IFT) is a family of IFSs in X, such that:

(i) $0_\sim, 1_\sim \in t$,

(ii) $M \cap N \in t$, for all $M, N \in t$,

(iii) $\bigcup M_j \in t$ for any collection of family $\{M_j \in t, j \in J\}$, where J is an index set.

The pair (X, t) is called an IFTS, the members of IFTS are called intuitionistic fuzzy open sets (IFOS) in X, and their complements are said to be intuitionistic fuzzy closed sets (in short, IFCS) in X.

A function $f : X \to Y$ [10], with X and Y IFTS where $M = \{(x, \lambda_M(x), \eta_M(x)) : x \in X\}$ and $N = \{(y, \lambda_N(y), \eta_N(y)) : y \in Y\}$ are respectively IFSs in X and Y, then the pre-image [9] of N under f, denoted by $f^{-1}(N)$, is the IFS in X defined by

$$f^{-1}(N) = \{(x, (f^{-1}(\lambda_M))(x), (f^{-1}(\eta_M))(x)) : x \in X\}$$

$$= \{(x, \lambda_N(f(x)), \eta_N(f(x))) : x \in X\}.$$

And the image of M, denoted as $f(M)$, is the IFS in X expressed as

$$f(M) = \{(y, (f(\lambda_M))(y), (f(\eta_M))(y)) : y \in Y\},$$

where for each $y \in Y$,

$$f(\lambda_M)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \lambda_M(x), & \text{if } f^{-1}(y) \neq \varnothing, \\ 0, & \text{otherwise,} \end{cases}$$

$$f(\eta_M)(y) = \begin{cases} \inf_{x \in f^{-1}(y)} \eta_M(x), & \text{if } f^{-1}(y) \neq \varnothing, \\ 1, & \text{otherwise.} \end{cases}$$

The function $f : (X, t) \to (Y, t')$ is said to be continuous [17] if $f^{-1}(N) \in t$ for all $N \in t'$, and f is said to be open if $f(M) \in t'$ for all $M \in t$.

An intuitionistic topological space (X, T) is called normal [29] if for all closed sets F and G with $F \cap G = \varnothing_\sim$, there exist $M, N \in t$ such that $F \subset M$, $G \subset N$ with $M \cap N = \varnothing_\sim$. An IFTS (X, t) is said to be an intuitionistic fuzzy β-normal space [4] if for every pair of disjoint IFCS M and N, there exist two disjoint IFβOSs U and V such that $M \subseteq U$, $N \subseteq V$.

3 Intuitionistic fuzzy $\alpha-$normal bi-topological space (IFNBTS)

Definition 3.1. Suppose α be a non-negative number and $\alpha \in (0,1)$. An intuitionistic fuzzy bi-topological space (X,s,t) is said to be:

a. **IFNBTS (**$\alpha - i$**)** if for IFCSs $F = (\lambda_F, \eta_F)$, $G = (\lambda_G, \eta_G)$ with $(\lambda_F \cap \lambda_G)(x) = 0$, $\alpha < (\eta_F \cup \eta_G)(x)$ for all $x \in X$, then there exist IFOSs $M = (\lambda_M, \eta_M)$, $N = (\lambda_N, \eta_N) \in (s \cup t)$ such that $F \subset M$ and $G \subset N$ with $(\lambda_M \cap \lambda_N)(y) = 0$, $\alpha < (\eta_M \cup \eta_N)(y)$ for each $y \in X$.

b. **IFNBTS (**$\alpha - ii$**)** if for IFCSs $F = (\lambda_F, \eta_F)$, $G = (\lambda_G, \eta_G)$ with $(\lambda_F \cap \lambda_G)(x) = 0$, $\alpha < (\eta_F \cup \eta_G)(x)$ for all $x \in X$, then there exist IFOSs $M = (\lambda_M, \eta_M)$, $N = (\lambda_N, \eta_N) \in (s \cup t)$ such that $F \subset M$ and $G \subset N$ with $(\lambda_M \cap \lambda_N)(y) < \alpha < (\eta_M \cup \eta_N)(y)$ for each $y \in X$.

c. **IFNBTS (**$\alpha - iii$**)** if for IFCSs $F = (\lambda_F, \eta_F)$, $G = (\lambda_G, \eta_G)$ with $(\lambda_F \cap \lambda_G)(x) = 0$, $\alpha < (\eta_F \cup \eta_G)(x)$ for all $x \in X$, then there exist IFOSs $M = (\lambda_M, \eta_M)$, $N = (\lambda_N, \eta_N) \in (s \cup t)$ such that $F \subset M$ and $G \subset N$ with $(\lambda_M \cap \lambda_N) \subset (\eta_M \cup \eta_N)$ for each $y \in X$.

d. **IFNBTS (**$\alpha - iv$**)** if for IFCSs $F = (\lambda_F, \eta_F)$, $G = (\lambda_G, \eta_G)$ with $(\lambda_F \cap \lambda_G)(x) < \alpha < (\eta_F \cup \eta_G)(x)$ for all $x \in X$, then there exist IFOSs $M = (\lambda_M, \eta_M)$, $N = (\lambda_N, \eta_N) \in (s \cup t)$ such that $F \subset M$ and $G \subset N$ with $(\lambda_M \cap \lambda_N)(y) = 0$, $\alpha < (\eta_M \cup \eta_N)(y)$ for each $y \in X$.

e. **IFNBTS (**$\alpha - v$**)** if for IFCSs $F = (\lambda_F, \eta_F)$, $G = (\lambda_G, \eta_G)$ with $(\lambda_F \cap \lambda_G)(x) < \alpha < (\eta_F \cup \eta_G)(x)$ for all $x \in X$, then there exist IFOSs $M = (\lambda_M, \eta_M)$, $N = (\lambda_N, \eta_N) \in (s \cup t)$ such that $F \subset M$ and $G \subset N$ with $(\lambda_M \cap \lambda_M)(y) < \alpha < (\eta_M \cup \eta_N)(y)$ for each $y \in X$.

f. **IFNBTS (**$\alpha - vi$**)** if for IFCSs $F = (\lambda_F, \eta_F)$, $G = (\lambda_G, \eta_G)$ with $(\lambda_F \cap \lambda_G)(x) < \alpha < (\eta_F \cup \eta_G)(x)$ for all $x \in X$, then there exist IFOSs $M = (\lambda_M, \eta_M)$, $N = (\lambda_N, \eta_N) \in (s \cup t)$ such that $F \subset M$ and $G \subset N$ with $(\lambda_M \cap \lambda_N) \subset (\eta_M \cup \eta_N)$ for each $y \in X$.

g. **IFNBTS (**$\alpha - vii$**)** if for IFCSs $F = (\lambda_F, \eta_F)$, $G = (\lambda_G, \eta_G)$ with $(\lambda_F \cap \lambda_G) \subset (\eta_F \cup \eta_G)$ for all $x \in X$, then there exist IFOSs $M = (\lambda_M, \eta_M)$, $N = (\lambda_N, \eta_N) \in (s \cup t)$ such that $F \subset M$ and $G \subset N$ with $(\lambda_M \cap \lambda_N)(y) = 0$, $\alpha < (\eta_M \cup \eta_N)(y)$ for all $y \in X$.

h. **IFNBTS (**$\alpha - viii$**)** if for IFCSs $F = (\lambda_F, \eta_F)$, $G = (\lambda_G, \eta_G)$ with $(\lambda_F \cap \lambda_G) \subset (\eta_F \cup \eta_G)$ for all $x \in X$, then there exist IFOSs $M = (\lambda_M, \eta_M)$, $N = (\lambda_N, \eta_N) \in (s \cup t)$ such that $F \subset M$ and $G \subset N$ with $(\lambda_M \cap \lambda_M)(y) < \alpha < (\eta_M \cup \eta_N)(y)$ for all $y \in X$.

i. **IFNBTS (**$\alpha - ix$**)** if for IFCSs $F = (\lambda_F, \eta_F)$, $G = (\lambda_G, \eta_G)$ with $(\lambda_F \cap \lambda_G) \subset (\eta_F \cup \eta_G)$ for all $x \in X$, then there exist IFOSs $M = (\lambda_M, \eta_M)$, $N = (\lambda_N, \eta_N) \in (s \cup t)$ such that $F \subset M$ and $G \subset N$ with $(\lambda_M \cap \lambda_N) \subset (\eta_M \cup \eta_N)$ for all $y \in X$,
where λ and η are used to denote the grades of membership and non-membership of the intuitionistic fuzzy sets.

Theorem 3.1. *Suppose (X,s,t) is IFNBTS. Then we have the following implications where $\alpha \in (0,1)$, see Figure 1.*

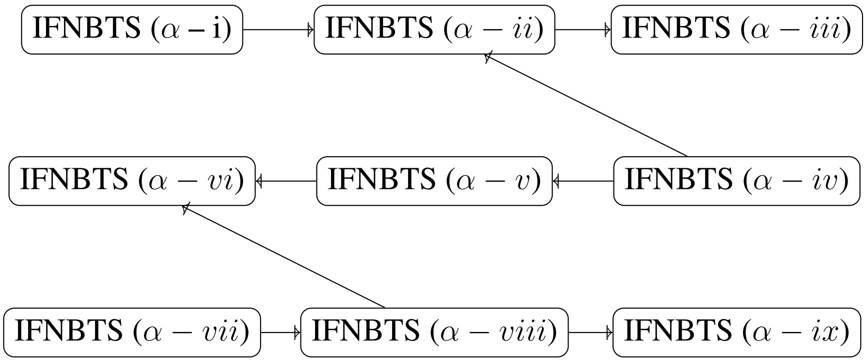

Figure 1. Implications among the IFNBTS notions

Proof. Let us consider (X, s, t) is IFNBTS $(\alpha - i)$. Let $F = (\lambda_F, \eta_F)$, $G = (\lambda_G, \eta_G)$ with $(\lambda_F \cap \lambda_G)(x) = 0$, $\alpha < (\eta_F \cup \eta_G)(x)$ for every $x \in X$. Since (X, s, t) is IFNBTS $(\alpha - i)$, there exist $M = (\lambda_M, \eta_M)$, $N = (\lambda_N, \eta_N) \in (s \cup t)$ such that $F \subset M$ and $G \subset N$ with

$$(\lambda_M \cap \lambda_N)(y) = 0, \quad \alpha < (\eta_M \cup \eta_N)(y) \text{ for all } y \in X \tag{1}$$

$$\Rightarrow (\lambda_M \cap \lambda_N)(y) < \alpha < (\eta_M \cup \eta_N)(y) \text{ for all } y \in X \tag{2}$$

$$\Rightarrow (\lambda_M \cap \lambda_N) \subset (\eta_M \cup \eta_N) \tag{3}$$

We see from (1), (2), and (3) that IFNBTS $(\alpha - i) \Rightarrow$ IFNBTS $(\alpha - ii) \Rightarrow$ IFNBTS $(\alpha - iii)$. Again, consider (X, s, t) is IFNBTS $(\alpha - iv)$. Let $F = (\lambda_F, \eta_F)$, $G = (\lambda_G, \eta_G)$ with $(\lambda_F \cap \lambda_G)(x) < \alpha < (\eta_F \cup \eta_G)(x)$ for each $x \in X$. Since (X, s, t) is IFNBTS $(\alpha - iv)$, there exist $M = (\lambda_M, \eta_M)$, $N = (\lambda_N, \eta_N) \in (s \cup t)$ such that $F \subset M$ and $G \subset N$ with the conditions like (1), (2) and (3).

From where we comment that IFNBTS $(\alpha - iv) \Rightarrow$ IFNBTS $(\alpha - v) \Rightarrow$ IFNBTS $(\alpha - vi)$. Moreover, assume that (X, s, t) is IFNBTS $(\alpha - vii)$. Let $F = (\lambda_F, \eta_F)$, $G = (\lambda_G, \eta_G)$ with $(\lambda_F \cap \lambda_G) \subset (\eta_F \cup \eta_G)$. Since (X, s, t) is IFNBTS $(\alpha - vii)$, there exist $M = (\lambda_M, \eta_M)$, $N = (\lambda_N, \eta_N) \in (s \cup t)$ such that $F \subset M$ and $G \subset N$ with identical conditions of (1), (2) and (3).

From where we see that IFNBTS $(\alpha - vii) \Rightarrow$ IFNBTS $(\alpha - viii) \Rightarrow$ IFNBTS $(\alpha - ix)$. Also, consider that (X, s, t) is IFNBTS $(\alpha - iv)$. Assume IFCSs $F = (\lambda_F, \eta_F)$, $G = (\lambda_G, \eta_G)$ with $(\lambda_F \cap \lambda_G)(x) = 0$, $\alpha < (\eta_F \cup \eta_G)(x)$ for all $x \in X$. But $(\lambda_F \cap \lambda_G)(x) = 0$, $\alpha < (\eta_F \cup \eta_G)(x) \Rightarrow (\lambda_F \cap \lambda_G)(x) < \alpha < (\eta_F \cup \eta_G)(x)$ for all $x \in X$. Since (X, s, t) is IFNBTS $(\alpha - iv)$, there exist $M = (\lambda_M, \eta_M)$, $N = (\lambda_N, \eta_N) \in (s \cup t)$ such that $F \subset M$ and $G \subset N$ with $(\lambda_M \cap \lambda_N)(y) = 0$, $\alpha < (\eta_M \cup \eta_N)(y)$ for all $y \in X$. But $(\lambda_M \cap \lambda_N)(y) = 0$, $\alpha < (\eta_M \cup \eta_N)(y) \Rightarrow (\lambda_M \cap \lambda_N)(y) < \alpha < (\eta_M \cup \eta_N)(y)$ for all $y \in X$. Therefore, (X, s, t) is IFNBTS $(\alpha - ii)$. Similarly, we can prove that IFNBTS $(\alpha - viii) \Rightarrow$ IFNBTS $(\alpha - vi)$. This completes the proof of the theorem. □

Theorem 3.2. *If (X, S, T) is a bi-topological space and (X, s, t) is the corresponding IFBTS where $s = \{(1_{M_j}, 1_{M_j^c}), j \in J : M_j \in (S \cup T)\}$ and $t = \{(1_{N_j}, 1_{N_j^c}), j \in J : N_j \in (S \cup T)\}$, then (X, S, T) is Normal $\Leftrightarrow$ (X, s, t) is IF-Normal $(\alpha - k)$ for any $k = i, ii, iii, ..., ix$.*

Proof. We shall verify this for $k = i$. Suppose (X, S, T) is a normal space. Let us assume $x \in X$ and $(1_F, 1_{F^c})$, $(1_G, 1_{G^c})$ are closed in (X, s, t) with $(1_F \cap 1_G)(x) = 0$, $\alpha < (1_{F^c} \cup 1_{G^c})(x)$.

Now $(1_F \cap 1_G)(x) = 0$ for all $x \in X \Rightarrow F \cap G = \varnothing$ and by definition of (s, t), clearly F, G are closed in (X, s, t).

Again, since (X, S, T) is Normal, then there exist $M, N \in (s \cup t)$ such that $F \subset M$ and $G \subset N$ with $M \cap N = \varnothing$. By definition of (s, t), clearly $(1_M, 1_{M^c})$, $(1_N, 1_{N^c}) \in (s \cup t)$. Also, it is clear that $(1_F, 1_{F^c}) \subset (1_M, 1_{M^c})$ and $(1_G, 1_{G^c}) \subset (1_N, 1_{N^c})$ as $F \subset M$ and $G \subset N$. Since $M \cap N = \varnothing$, thus for any $y \in X$, $(1_M \cap 1_N)(y) = 0$ and $(1_{M^c} \cup 1_{N^c})(y) = 1 \Rightarrow (1_M \cap 1_N)(y) = 0$, $\alpha < (1_{M^c} \cup 1_{N^c})(y)$. So, (X, s, t) is IF-Normal $(\alpha - i)$.

Conversely, let (X, s, t) be IF-Normal $(\alpha - i)$. Suppose $x \in X$ and F, G are closed in (X, s, t) and $F \cap G = \varnothing$. Obviously, $(1_F, 1_{F^c})$, $(1_G, 1_{G^c})$ are closed in (X, s, t) with $(1_F \cap 1_G)(x) = 0$, $(1_{F^c} \cup 1_{G^c})(x) = 1$ for each $x \in X \Rightarrow (1_F \cap 1_G)(x) = 0$, $\alpha < (1_{F^c} \cup 1_{G^c})(x)$ for all $x \in X$. Since (X, s, t) is IF-Normal $(\alpha - i)$, then there exist $(1_M, 1_{M^c})$, $(1_N, 1_{N^c}) \in (s \cup t)$ such that $(1_F, 1_{F^c}) \subset (1_M, 1_{M^c})$ and $(1_G, 1_{G^c}) \subset (1_N, 1_{N^c})$ with $(1_M \cap 1_N)(y) = 0$, $\alpha < (1_{M^c} \cup 1_{N^c})(y)$ for all $y \in X$.

By definition of intuitionistic bi-topology (X, s, t), it is clear that $M, N \in (S \cup T)$ and clearly $(1_M \cap 1_N)(y) = 0$ for all $y \in X$ as $1_M, 1_N$ are characteristic functions. Which implies $M \cap N = \varnothing$. Hence (X, S, T) is normal. □

Theorem 3.3. *If (X,S,T) is an intuitionistic bi-topological space and (X,s,t) is the corresponding IFBTS where $s = \{(1_{M_{j1}}, 1_{M_{j2}}), j \in J : M_j \in (S \cup T)\}$ and $t = \{(1_{N_{j1}}, 1_{N_{j2}}), j \in J : N_j \in (S \cup T)\}$, then (X, S, T) is Normal $\Leftrightarrow$ (X, s, t) is IF-Normal $(\alpha - k)$ for any $k = i, ii, iii, ..., ix$.*

Proof. We shall prove (X, S, T) is normal if and only if (X, s, t) is IF-Normal $(\alpha - iv)$. Suppose (X, S, T) is normal. Assume $(1_{F_1}, 1_{F_2})$ and $(1_{G_1}, 1_{G_2})$ are closed in (X, s, t) with $(1_{F_1} \cap 1_{G_1})(x) < \alpha < (1_{F_2} \cup 1_{G_2})(x)$ for all $x \in X$. Since $1_{F_1}, 1_{F_2}, 1_{G_1}$, and 1_{G_2} are characteristic functions and $\alpha \in (0, 1)$, thus $(1_{F_1} \cap 1_{G_1})(x) = 0$ and $(1_{F_2} \cup 1_{G_2})(x) = 1$, for all $x \in X$.

By definition of intuitionistic bi-topology (X, s, t), it is obvious that $F = (F_1, F_2)$, $G = (G_1, G_2)$ are closed in (X, S, T). Now $F_1 \cap F_2 = \varnothing$ and $F_1 \cup F_2 = X$ as $(1_{F_1} \cap 1_{G_1})(x) = 0$ and $(1_{F_2} \cup 1_{G_2})(x) = 1$, for all $x \in X \Rightarrow F \cap G = (\varnothing, X) = \varnothing_\sim$. Since (X, S, T) is normal, we have $M = (M_1, M_2)$, $N = (N_1, N_2) \in (S \cup T)$ such that $F \subset M$ and $G \subset N$ with $M \cap N = \varnothing_\sim$.

Now $F \subset M$ and $G \subset N \Rightarrow F_1 \subset M_1, F_2 \supset M_2$ and $G_1 \subset N_1, G_2 \supset N_2$. Also $M \cap N = \varnothing_\sim \Rightarrow M_1 \cap N_1 = \varnothing$ and $M_2 \cup N_2 = X$. By definition of intuitionistic bi-topology (X, s, t), $(1_{M_1}, 1_{M_2})$, $(1_{N_1}, 1_{N_2}) \in (s \cup t)$. Clearly $(1_{M_1} \cap 1_{N_1})(x) = 0$ and $(1_{M_2} \cup 1_{N_2})(x) = 1$, for all $x \in X$ as $M_1 \cap N_1 = \varnothing$ and $M_2 \cup N_2 = X$.

$$\Rightarrow (1_{M_1} \cap 1_{N_1})(x) = 0, \quad \alpha < (1_{M_2} \cup 1_{N_2})(x), \text{ for all } x \in X.$$

Also, clearly $(1_{F_1}, 1_{F_2}) \subset (M_1, M_2)$ and $(1_{G_1}, 1_{G_2}) \subset (N_1, N_2)$ since $F_1 \subset M_1, F_2 \supset M_2$ and $G_1 \subset N_1, G_2 \supset N_2$. Hence (X, s, t) is IF-Normal $(\alpha - iv)$.

Conversely, let (X, s, t) be IF-Normal $(\alpha - iv)$. Let $F = (F_1, F_2)$, $G = (G_1, G_2)$ be closed in (X, S, T) and $F \cap G = (\varnothing, X) = \varnothing_\sim$.

By intuitionistic bi-topology (X, s, t), clearly $(1_{F_1}, 1_{F_2})$ and $(1_{G_1}, 1_{G_2})$ are closed in (X, s, t). Also $(1_{F_1} \cap 1_{G_1})(x) = 0$ and $(1_{F_2} \cup 1_{G_2})(x) = 1$, for all $x \in X$.

$$\Rightarrow (1_{F_1} \cap 1_{G_1})(x) < \alpha < (1_{F_2} \cup 1_{G_2})(x) \text{ for all } x \in X \text{ as } \alpha \in (0, 1).$$

Since (X, s, t) is IF-Normal $(\alpha - iv)$, there exist $(1_{M_1}, 1_{M_2})$, $(1_{N_1}, 1_{N_2}) \in (s \cup t)$ such that $(1_{F_1}, 1_{F_2}) \subset (M_1, M_2)$ and $(1_{G_1}, 1_{G_2}) \subset (N_1, N_2)$ with $(1_{M_1} \cap 1_{N_1})(x) = 0$, $\alpha < (1_{M_2} \cup 1_{N_2})(x)$, for all $x \in X$ as $M_1 \cap N_1 = \varnothing$ and $M_2 \cup N_2 = X$.
Hence $(1_{M_1} \cap 1_{N_1})(x) = 0$, $(1_{M_2} \cup 1_{N_2})(x) = 1$ as $1_{M_1}, 1_{M_2}, 1_{N_1}, 1_{N_2}$ are characteristic functions.

By definition of intuitionistic bi-topology (X, s, t), (M_1, M_2), $(N_1, N_2) \in (S \cup T)$.

Now $(M_1, M_2) \cap (N_1, N_2) = (M_1 \cap N_1, M_2 \cup N_2) = (\varnothing, X) = \varnothing_\sim$ as $(1_{M_1} \cap 1_{N_1})(x) = 0$ and $(1_{M_2} \cup 1_{N_2})(x) = 1,$ for all $x \in X]$.

Again $(1_{F_1}, 1_{F_2}) \subset (M_1, M_2) \Rightarrow F_1 \subset M_1, F_2 \supset M_2 \Rightarrow (F_1, F_2) \subset (M_1, M_2)$. In the same manner, we have $(G_1, G_2) \subset (N_1, N_2)$. Hence (X, S, T) is normal. □

Theorem 3.4. *If (X, s, t) is an IFBTS and $\alpha, \beta \in (0, 1)$ with $\alpha > \beta$, then (X, s, t) is IFNBTS $(\beta - iii) \Rightarrow (X, s, t)$ is IFNBTS $(\alpha - iii)$ and (X, s, t) is IFNBTS $(\alpha - vii) \Rightarrow (X, s, t)$ is IFNBTS $(\beta - vii)$.*

Proof. Let (X, s, t) be IFNBTS $(\beta - iii)$ and $\alpha > \beta$. Suppose $F = (\lambda_F, \eta_F)$, $G = (\lambda_G, \eta_G)$ are closed in (X, s, t) and $(\lambda_M \cap \lambda_N)(x) = 0$, $\alpha < (\eta_M \cup \eta_N)(x)$ for all $x \in X$.

Now $(\lambda_M \cap \lambda_N)(x) = 0$, $\alpha < (\eta_M \cup \eta_N)(x)$ for all $x \in X \Rightarrow (\lambda_M \cap \lambda_N)(x) = 0$, $\beta < (\eta_M \cup \eta_N)(x)$ for all $x \in X$ as $\alpha > \beta$. Since (X, s, t) is IFNBTS $(\beta - iii)$, there exist $M = (\lambda_M, \eta_M)$, $N = (\lambda_N, \eta_N) \in (s \cup t)$ such that $F \subset M$ and $G \subset N$ with $(\lambda_M \cap \lambda_N) \subset (\eta_M \cup \eta_N)$, i.e., if for any closed $F = (\lambda_F, \eta_F)$, $G = (\lambda_G, \eta_G)$ in (X, s, t) with $(\lambda_F \cap \lambda_G)(x) = 0$, $\alpha < (\eta_F \cup \eta_G)(x)$ for all $x \in X$, then there exist $M = (\lambda_M, \eta_M)$, $N = (\lambda_N, \eta_N) \in (s \cup t)$ such that $F \subset M$ and $G \subset N$ with $(\lambda_M \cap \lambda_N) \subset (\eta_M \cup \eta_N)$. Hence (X, s, t) is IFNBTS $(\alpha - iii)$.

In the same way, we can show that (X, s, t) is IFNBTS $(\alpha - vii) \Rightarrow (X, s, t)$ is IFNBTS $(\beta - vii)$. □

Theorem 3.5. *If (X, s, t), (Y, s', t') are IFBTSs with $f : X \to Y$ being bijective, closed, and continuous, then we have that (Y, s', t') is IFNBTS $(\alpha - k) \Rightarrow (X, s, t)$ is IFNBTS $(\alpha - k)$ for $k = i, ii, iii, ..., ix$.*

Proof. Assume (Y, s', t') is IFNBTS $(\alpha - i)$ and $F = (\lambda_F, \eta_F)$, $G = (\lambda_G, \eta_G)$ are IFCS in (X, s, t) with $(\lambda_F \cap \lambda_G)(x) = 0$, $\alpha < (\eta_F \cup \eta_G)(x)$ for all $x \in X$. Now $f(F) = (f(\lambda_F), f(\eta_F))$, $f(G) = (f(\lambda_G), f(\eta_G))$ are closed in (Y, s', t') as f is closed. Since f is bijective, then there are some unique $x \in X$, with $f(x) = y$, i.e., $f^{-1}(y) = x$.

Now, for each $y \in Y$, we have

$$(f(F) \cap f(G))(y) = ((f(\lambda_F) \cap f(\lambda_G))(y), (f(\eta_F) \cup f(\eta_G))(y)).$$

However,

$$\begin{aligned}(f(\lambda_F) \cap f(\lambda_G))(y) &= \min(f(\lambda_F)(y), f(\lambda_G)(y)) \\ &= \min\left(\sup_{q \in f^{-1}(y)} \lambda_F(q), \sup_{q \in f^{-1}(y)} \lambda_G(q)\right) \\ &= \min(\lambda_F(x), \lambda_G(x)) \\ &= (\lambda_F \cap \eta_G)(x) \\ &= 0\end{aligned}$$

and

$$\begin{aligned}(f(\eta_F) \cup f(\eta_G))(y) &= \max(f(\eta_F)(y), f(\eta_G)(y)) \\ &= \max\left(\inf_{q \in f^{-1}(y)} \eta_F(q), \inf_{q \in f^{-1}(y)} \eta_G(q)\right) \\ &= \max(\eta_F(x), \eta_G(x)) \\ &= (\eta_F \cup \eta_G)(x) \\ &> \alpha.\end{aligned}$$

Thus, $(f(\lambda_F) \cap f(\lambda_G))(y) = 0$, $\alpha < (f(\eta_F) \cup f(\eta_G))(y)$ for any $y \in Y$.

As (Y, s', t') is IFNBTS ($\alpha - i$), there exist $M = (\lambda_M, \eta_M)$, $N = (\lambda_N, \eta_N) \in (s' \cup t')$ such that $f(F) \subset M$ and $f(G) \subset N$ with $(\lambda_M \cap \lambda_N)(y) = 0$, $\alpha < (\eta_M \cup \eta_N)(y)$ for all $y \in Y$. Now $f(F) \subset M \Rightarrow f^{-1}(f(F)) \subset f^{-1}(M)$. But $F = f^{-1}(f(F))$ as f is injective. So $F \subset f^{-1}(M)$ and likewise, $G \subset f^{-1}(N)$. Also $f^{-1}(M), f^{-1}(N) \in (s \cup t)$, as f is continuous. Again, for each $x \in X$,

$$\begin{aligned}(f^{-1}(M) \cap f^{-1}(N))(x) &= \left((f^{-1}(\lambda_M), f^{-1}(\eta_M)) \cap (f^{-1}(\lambda_N), f^{-1}(\eta_N))\right)(x) \\ &= \left((f^{-1}(\lambda_M) \cap f^{-1}(\lambda_N))(x), (f^{-1}(\eta_M) \cup f^{-1}(\eta_N))(x)\right).\end{aligned}$$

However, we get

$$\begin{aligned}(f^{-1}(\lambda_M) \cap f^{-1}(\lambda_N))(x) &= \min(f^{-1}(\lambda_M)(x), f^{-1}(\lambda_N)(x)) \\ &= \min(\lambda_M(f(x)), \lambda_N(f(x))) \\ &= (\lambda_M \cap \eta_N)(f(x)) \\ &= 0,\end{aligned}$$

since $(\lambda_M \cap \lambda_N)(y) = 0$ for every $y \in Y$, and

$$\begin{aligned}(f^{-1}(\eta_M) \cup f^{-1}(\eta_N))(x) &= \max((f^{-1}(\eta_M))(x), (f^{-1}(\eta_N))(x)) \\ &= \max(\eta_M(f(x)), \eta_N(f(x))) \\ &= (\eta_M \cup \eta_N)(f(x)) \\ &> \alpha,\end{aligned}$$

since $\alpha < (\eta_M \cup \eta_N)(y)$ for every $y \in Y$.

Thus, $(f^{-1}(\lambda_M) \cap f^{-1}(\lambda_N))(x) = 0$, $\alpha < (f^{-1}(\eta_M) \cup f^{-1}(\eta_N))(x)$ for all $x \in X$. Hence (X, s, t) is IFNBTS ($\alpha - i$). In the same way, this theorem holds for $k = ii, iii, ..., ix$. □

Theorem 3.6. *Let (X, s, t), (Y, s', t') be IFNBTSs with $f : X \to Y$ being injective and continuous. Then (X, s, t) is IFNBTS $(\alpha - k)$ $\Rightarrow$ (Y, s', t') is IFNBTS $(\alpha - k)$ for $k = i, ii, iii, ..., ix$.*

Proof. Assume (X, s, t) is IFNBTS $(\alpha - ii)$. Let $F = (\lambda_F, \eta_F)$ and $G = (\lambda_G, \eta_G)$ be closed in (Y, s', t') with $(\lambda_F \cap \lambda_G)(y) = 0$, $\alpha < (\eta_F \cup \eta_G)(y)$ for all $y \in Y$. As f is continuous, thus $f^{-1}(F)$ and $f^{-1}(G)$ are closed in (X, s, t).

Again, for each $x \in X$, we get

$$\begin{aligned}(f^{-1}(F) \cap f^{-1}(G))(x) &= \left((f^{-1}(\lambda_F), f^{-1}(\eta_F)) \cap (f^{-1}(\lambda_G), f^{-1}(\eta_G))\right)(x) \\ &= \left((f^{-1}(\lambda_F) \cap f^{-1}(\lambda_G))(x), (f^{-1}(\eta_F) \cup f^{-1}(\eta_G))(x)\right).\end{aligned}$$

But we have

$$\begin{aligned}(f^{-1}(\lambda_M) \cap f^{-1}(\lambda_N))(x) &= \min\left((f^{-1}(\lambda_M))(x), (f^{-1}(\lambda_N))(x)\right) \\ &= \min(\lambda_M(f(x)), \lambda_N(f(x))) \\ &= (\lambda_M \cap \eta_N)(f(x)) \\ &= 0,\end{aligned}$$

since $(\lambda_M \cap \lambda_N)(y) = 0$ for each $y \in Y$, and

$$\begin{aligned}(f^{-1}(\eta_M) \cup f^{-1}(\eta_N))(x) &= \max\left((f^{-1}(\eta_M))(x), (f^{-1}(\eta_N))(x)\right) \\ &= \max(\eta_M(f(x)), \eta_N(f(x))) \\ &= (\eta_M \cup \eta_N)(f(x)) \\ &> \alpha,\end{aligned}$$

since $\alpha < (\eta_M \cup \eta_N)(y)$ for all $y \in Y$.

Thus, $(f^{-1}(\lambda_M) \cap f^{-1}(\lambda_N))(x) = 0$, $\alpha < (f^{-1}(\eta_M) \cup f^{-1}(\eta_N))(x)$ for all $x \in X$. Since (X, s, t) is IFNBTS $(\alpha - ii)$, there exist $M = (\lambda_M, \eta_M)$, $N = (\lambda_N, \eta_N) \in (s \cup t)$ such that $f^{-1}(F) \subset M$ and $f^{-1}(G) \subset N$ with $(\lambda_M \cap \lambda_N)(x) < \alpha < (\eta_M \cup \eta_N)(x)$ for all $x \in X$. Since f is open, thus $f(M), f(N) \in (s' \cup t')$. Also $f^{-1}(F) \subset M \Rightarrow f(f^{-1}(F)) \subset f(M) \Rightarrow F \subset f(M)$ as f is onto. In the same manner, $G \subset f(N)$. Since f is bijective, then for any $w \in Y$, there exists $z \in X$ with $f(z) = w$, i.e., $f^{-1}(w) = \{z\}$.
Now, for each $w \in Y$, we have

$$(f(F) \cap f(G))(w) = ((f(\lambda_M) \cap f(\lambda_N))(w), (f(\eta_M) \cup f(\eta_N))(w)).$$

However,

$$\begin{aligned}(f(\lambda_M) \cap f(\lambda_N))(w) &= \min(f(\lambda_M)(w), f(\lambda_N)(w)) \\ &= \min\left(\sup_{q \in f^{-1}(w)} \lambda_M(q), \sup_{q \in f^{-1}(w)} \lambda_N(q)\right) \\ &= \min(\lambda_M(x), \lambda_N(z)) \\ &= (\lambda_M \cap \lambda_N)(z) \\ &< \alpha,\end{aligned}$$

and

$$
\begin{aligned}
(f(\eta_M) \cup f(\eta_N))(w) &= \max(f(\eta_M)(w), f(\eta_N)(w)) \\
&= \max\left(\inf_{q \in f^{-1}(w)} \eta_M(q), \inf_{q \in f^{-1}(w)} \eta_N(q) \right) \\
&= \max(\eta_M(z), \eta_N(z)) \\
&= (\eta_M \cup \eta_N)(z) \\
&> \alpha.
\end{aligned}
$$

Thus, $(f(\lambda_M) \cap f(\lambda_N))(w) < \alpha < (f(\eta_M) \cup f(\eta_N))(w)$ for any $w \in Y$. Hence (Y, s', t') is IFNBTS ($\alpha - ii$). In the same way, we can prove this theorem for $k = i, iii, iv, ..., ix$. □

4 Conclusion

In this article, nine new notions of intuitionistic fuzzy α-normal bi-topological space have been defined, and some relationships among them have been established, followed by a good extension property which showed the accuracy of the prescribed notions of IFNBTS. We have detected that Theorems 3.5 and 3.6 represent that our notions bear bitopological property in the context of IFNBTS. Furthermore, numerous future researches may be inspired by the normal separation axiom of intuitionistic fuzzy bi-topological spaces. Properties of being order-preserving, hereditary using the sense of quasi-coincidence, can be considered as future investigations following this work.

Acknowledgements

The authors extend their heartfelt thanks to the referees and the editor for their thorough review of the article and their valuable comments that enhanced the quality of this paper.

References

[1] Ahmed, E., Hossain, M. S., & Ali, D. M. (2014). On intuitionistic fuzzy T_0 spaces. *Journal of Bangladesh Academy of Sciences*, 38(2), 197–207.

[2] Ahmed, E., Hossain, M. S., & Ali, D. M. (2015). On intuitionistic fuzzy R_1-spaces. *Journal of Mathematical and Computational Science*, 5(5), 681–693.

[3] Al-Qubati, A. A. Q. (2015). On b-separation axioms in intuitionistic fuzzy topological spaces. *International Journal of Mathematics Trends and Technology*, 21(2), 83–93.

[4] Al-Qubati, A. A. Q. (2018). On intuitionistic fuzzy β and β^*-normal spaces. *International Journal of Mathematical Analysis*, 12(11), 517–531.

[5] Amin, R., Hossain, M. S., & Miah, S. S. (2020). Fuzzy pairwise regular bitopological spaces in quasi-coincidence sense.*J. Bangladesh Acad. Sci.*, 44(2), 139–143.

[6] Amin, R., Islam, R., Shaha, S. K., & Miah, S. S. (2022). Some properties of T_0 fuzzy soft topological spaces in quasi-coincidence sense. *Journal of Mechanics of Continua and Mathematical Sciences*, 17(4), 8–20.

[7] Atanassov, K. T. (1983). Intuitionistic fuzzy sets. *Proceedings of VII ITKR Session*, Sofia, 20–23 June 1983 (Deposed in Centr. Sci.-Techn. Library of the Bulg. Acad. of Sci., 1697/84) (in Bulgarian). Reprinted: *Int. J. Bioautomation*, 2016, 20(S1), S1–S6.

[8] Atanassov, K. T. (1988). Review and new results on intuitionistic fuzzy sets. *Mathematical Foundations of Artificial Intelligence Seminar*, Sofia, Preprint IM-MFAIS1-88. Reprinted: *Int. J. Bioautomation*, 2016, 20(S1), S7–S16.

[9] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1), 87–96.

[10] Atanassov K. (1992). Remarks on the intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 51(1), 117–118.

[11] Atanassov, K. T., & Stoeva, S. (1983). Intuitionistic fuzzy sets. In: *Polish Symposium on Interval & Fuzzy Mathematics*, Poznan, 1983, 23–26.

[12] Bayhan, S., & Çoker, D. (1996). On fuzzy separation axioms in intuitionistic fuzzy topological spaces. *BUSEFAL*, 67, 77–87.

[13] Bayhan, S., & Çoker, D. (2005). Pairwise separation axioms in intuitionistic topological spaces. *Hacettepe Journal of Mathematics and Statistics*, 34S, 101–114.

[14] Chang, C. L. (1968). Fuzzy topological spaces. *Journal of Mathematical Analysis and Applications*, 24(1), 182–190.

[15] Çoker, D. (1996). A note on intuitionistic sets and intuitionistic points. *Turkish Journal of Mathematics*, 20(3), 343–351.

[16] Çoker, D. (1996). An introduction to fuzzy subspace in intuitionistic fuzzy topological spaces. *Journal of Fuzzy Mathematics*, 4, 749–764.

[17] Çoker, D. (1997). An introduction to intuitionistic fuzzy topological spaces. *Fuzzy Sets and Systems*, 88(1), 81–89.

[18] Çoker, D., & Bayhan, S. (2001). On separation axioms in intuitionistic topological spaces. *International Journal of Mathematics and Mathematical Sciences*, 27(10), 621–630.

[19] Ejegwa, P. A., & Onasanya, B. O. (2018). Improved intuitionistic fuzzy composite relation and its application to medical diagnostic process. *Notes on Intuitionistic Fuzzy Sets*, 25(1), 43–58.

[20] George, J. B., & Jose, S. (2019). Medical diagnosis in intuitionistic fuzzy context. *Notes on Intuitionistic Fuzzy Sets*, 25(1), 59–63.

[21] Islam, R., Hossain, M. S., & Hoque, M. F. (2020). A study on intuitionistic L-fuzzy T_1 spaces. *Notes on Intuitionistic Fuzzy Sets*, 26(3), 33–42.

[22] Lee, S. J., & Lee, E. P. (2000). The category of intuitionistic fuzzy topological spaces. *Bulletin of the Korean Mathematical Society*, 37(1), 63–76.

[23] Lupianez, F. G. (2004). Separation in intuitionistic fuzzy topological spaces. *International Journal of Pure and Applied Mathematics*, 17(1), 29–34.

[24] Mahbub, M. A., Hossain, M. S., & Hossain, M. A. (2021). Connectedness concept in intuitionistic fuzzy topological spaces. *Notes on Intuitionistic Fuzzy Sets*, 27(1), 72–82.

[25] Miah, S. S., & Jumur, B. N. (2024). Intuitionistic fuzzy R_0 bitopological space: An in-depth exploration. *GANIT: Journal of Bangladesh Mathematical Society*, 44(1), 23–28.

[26] Miah, S. S., & Kabir, H. (2024). Study on intuitionistic fuzzy Hausdorff (T_2) bitopological spaces: Theoretical insights. *J. Bangladesh Acad. Sci.*, 48(1), 51–59.

[27] Miah, S. S., Lily, F. T., Amin, M. R., Karim, R., & Akbar, M. A. (2024). Separation axioms on fuzzy neutrosophic bitopological spaces using fuzzy neutrosophic open sets. *Journal of Umm Al-Qura University for Applied Sciences*, DOI: https://doi.org/10.1007/s43994-024-00173-7.

[28] Miah, S. S., Ronjon, R., & Sultana, S. (2024). An in-depth exploration of intuitionistic fuzzy T_0 in the context of bitopology. *Notes on Intuitionistic Fuzzy Sets*, 30(1), 66–76.

[29] Prova, T. T., & Hossain, M. S. (2020). Intuitionistic fuzzy based regular and normal spaces. *Notes on Intuitionistic Fuzzy Sets*, 26(4), 53–63.

[30] Saadati, R., & Park, J. H. (2006). On the intuitionistic fuzzy topological spaces. *Chaos, Solitons & Fractals*, 27(2), 331–344.

[31] Singh, A. K., & Srivastava, R. (2012). Separation axioms in intuitionistic fuzzy topological spaces. *Advances in Fuzzy Systems*, 2012, Article ID 604396.

[32] Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, 8, 338–353.