Notes on Intuitionistic Fuzzy Sets
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On a new expanding modal-like operator on intuitionistic fuzzy sets

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Abstract: In the present paper we define a new operator similar to the operator $F_{\alpha,\beta}$, but having multiplicative nature. We study some of its properties and provide illustrative examples.
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1 Introduction

Intuitionistic fuzzy sets (IFSs) introduced in 1983 by K. Atanassov (see [1]) extended fuzzy sets (FS) (as defined by Zadeh in [4]) by introducing a non-membership degree $\nu_A(x)$ which reflects the extent to which an element does not belong to the considered set. The complement of the sum of the membership and non-membership degrees to 1 ($\pi_A(x)$) is called *hesitancy degree* or *index of indeterminacy*. The formal definition may be stated as follows.

Definition 1 ([1]). *Let X be a universe set, $A \subset X$. Then an intuitionistic fuzzy set generated by the set A is an object of the form:*

$$A^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}, \tag{1}$$

where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ are mappings, such that for any $x \in X$,

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1. \tag{2}$$

The two modal operators "possibility" ($\square$) and "necessity" ($\diamondsuit$), which transform an IFS to a FS, were introduced together with the concept of intuitionistic fuzzy sets, and show that IFSs are a richer extension than FS, as these operartors have no sense in the case of FS.

In 1988 in [2], these operators were generalized by the extended modal operator $F_{\alpha,\beta}$.

Definition 2 ([2]). *Let $\alpha, \beta, \alpha + \beta \in [0,1]$, then for any IFS A^*, we can define the operator:*

$$F_{\alpha,\beta}(A^*) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + \beta\pi_A(x)\rangle | x \in X\}. \tag{3}$$

Remark 1. We have $\square(A^*) = F_{0,1}(A^*)$ and $\diamondsuit(A^*) = F_{1,0}(A^*)$.

The operator $F_{\alpha,\beta}$ may be further extended as

Definition 3 ([3]). *Let A^*, B^* be IFSs, defined over the same universe X. Then we can define the operator:*

$$F_{B^*}(A^*) = \{\langle x, \mu_A(x) + \mu_B(x)\pi_A(x), \nu_A(x) + \nu_B(x)\pi_A(x)\rangle | x \in X\}. \tag{4}$$

The multiplicative operator $G_{\alpha,\beta}$ is defined as follows:

Definition 4 ([2]). *Let $\alpha, \beta, \in [0,1]$, then for any IFS A^*, we can define the operator:*

$$G_{\alpha,\beta}(A^*) = \{\langle x, \alpha\mu_A(x)+, \beta\nu_A(x)\rangle | x \in X\}. \tag{5}$$

In what follows, we will propose an operator which acts as $F_{\alpha,\beta}$, in the sense that it diminishes the hesitancy degree but is of multiplicative nature as $G_{\alpha,\beta}$.

2 The operator T_λ

Definition 5. *The operator $T_\lambda(A^*)$ for $\lambda \geq 0$ is defined by*

$$T_\lambda(A^*) = \{\langle x, \mu_A(x) z_{T_\lambda}(x), \nu_A(x) z_{T_\lambda}(x)\rangle | x \in X\}, \tag{6}$$

where $z_{T_\lambda}(x) = (1 - \mu_A(x))^{1+\lambda} + (1 - \nu_A(x))^{1+\lambda}$.

The fact that the operator T_λ transforms an IFS into an IFS may be stated as the following theorem.

Theorem 1. The operator $T_\lambda(A^*)$ for $\lambda \geq 0$ always produces an IFS.

Proof. Further we will make use of the fact that for any non-negative two numbers a, b, we have $\max(a, b) \leq a + b$. Obviously,

$$\mu_{T_\lambda(A^*)}(x) \geq 0 \text{ and } \nu_{T_\lambda(A^*)}(x) \geq 0.$$

Also, for any $\varepsilon > 0$

$$\mu_{T_\varepsilon(A^*)}(x) \leq \mu_{T_0(A^*)}(x), \text{ and } \nu_{T_\varepsilon(A^*)}(x) \leq \nu_{T_0(A^*)}(x).$$

Thus, it suffices to show that $\max(\mu_{T_0(A^*)}(x), \nu_{T_0(A^*)}(x)) \leq 1$.

A direct check shows that

$$\mu_{T_0(A^*)}(x) + \nu_{T_0(A^*)}(x) = (1 - \pi_A(x))(1 + \pi_A(x)) = 1 - \pi_A(x)^2 \leq 1.$$

Hence, $\max(\mu_{T_0(A^*)}(x), \nu_{T_0(A^*)}(x)) \leq \mu_{T_0(A^*)}(x) + \nu_{T_0(A^*)}(x) \leq 1$, and (2) is fulfilled. □

Remark 2. One can easily see that $T_0(A^*) = F_{A^*}(A^*)$.

We will now extend the operator $T_\lambda(A^*)$ by analogy with the operator $G_{\alpha,\beta}$.

Definition 6. *The operator* $T_{\lambda,\alpha,\beta}(A^*)$ *for* $\lambda \geq 0$ *and* $\alpha, \beta \in [0, 1]$ *is defined by*

$$T_{\lambda,\alpha,\beta}(A^*) = \{\langle x, \alpha\mu_A(x)z_{T_\lambda}(x), \beta\nu_A(x)z_{T_\lambda}(x)\rangle | x \in X\}, \quad (7)$$

where $z_{T_\lambda}(x) = (1 - \mu_A(x))^{1+\lambda} + (1 - \nu_A(x))^{1+\lambda}$.

Proof. In the same manner as in the proof of Theorem 1, we have:

$$\mu_{T_{\lambda,\alpha,\beta}(A^*)}(x) \geq 0 \text{ and } \nu_{T_{\lambda,\alpha,\beta}(A^*)}(x) \geq 0$$

and

$$\begin{aligned} \max(\mu_{T_{\lambda,\alpha,\beta}(A^*)}(x), \nu_{T_{\lambda,\alpha,\beta}(A^*)}(x)) &\leq \mu_{T_{0,\alpha,\beta}(A^*)}(x) + \nu_{T_{0,\alpha,\beta}(A^*)}(x) \\ &\leq \max(\alpha, \beta)(1 - (\pi_A(x))^2) \\ &\leq (1 - (\pi_A(x))^2) \leq 1 \end{aligned}$$

□

Further we consider several cases of operators consecutively applied to their result for illustrating what the proposed operators achieve in geometrical terms.

Example 1. The result of several consecutive applications of the operator $T_{0,1,1} = T_0$ to the IFS $A = \{\langle x, 0.3, 0.15\rangle\}$ is presented on the next Figure 1.

This operator straightforwardly moves the point along the line passing through the point $(0, 0)$ and $\mu_A(x), \nu_A(x)$ towards the hypotenuse.

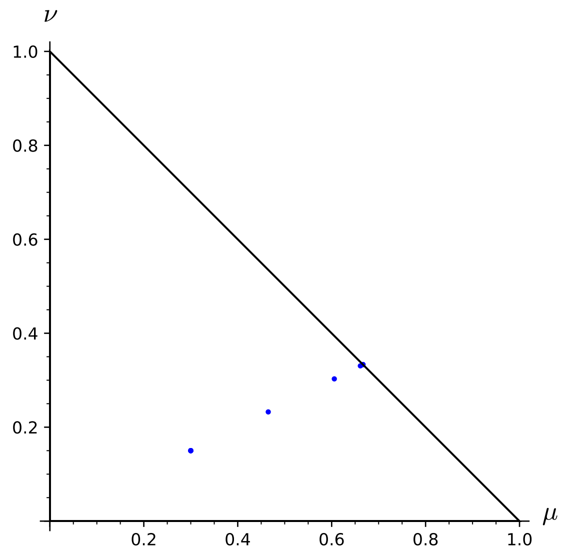

Figure 1. Result of the application of T_0 to the IFS $A = \{\langle x, 0.3, 0.15\rangle\}$ several consecutive times

Example 2. The result of several consecutive applications of the operator $T_{1,0.9,0.6}$ to the IFS $A = \{\langle x, 0.3, 0.15\rangle\}$ is shown on the next Figure 2.

In this case the greater value of λ and the lower value of β results in gradual drop towards the μ-axis.

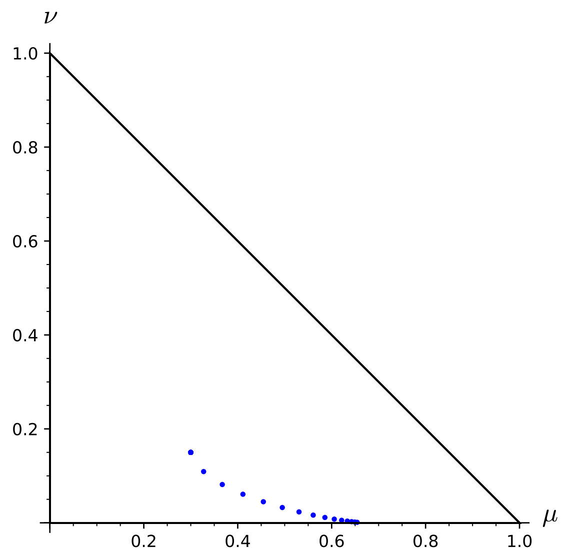

Figure 2. Application of $T_{1,0.9,0.6}$ to the IFS $A = \{\langle x, 0.3, 0.15\rangle\}$ several consecutive times.

Example 3. Finally, the result of several consecutive aplications of the operator $T_{0,0.7,1}$ to the IFS $A = \{\langle x, 0.3, 0.15\rangle\}$ is given on the Figure 3 below.

Here, initially the trajectory starts towards the hypotenuse, however, the contracting value of α gradually pushes it towards the ν-axis.

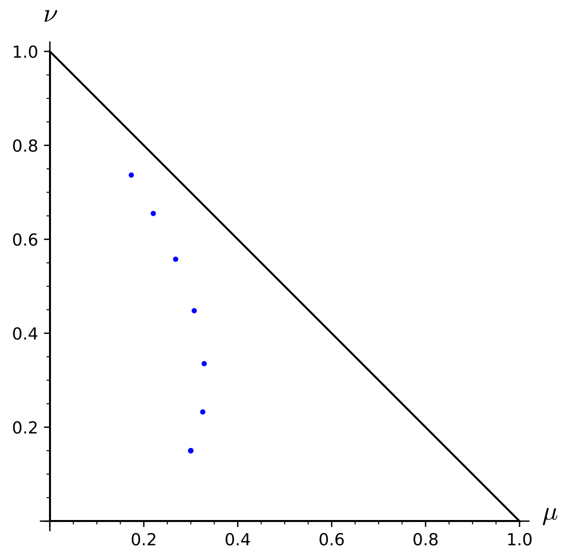

Figure 3. Application of $T_{0,0.7,1}$ to the IFS $A = \{\langle x, 0.3, 0.15\rangle\}$ several consecutive times.

Since we consider a single point as the object on which the operator acts, we must note that the following statement is true.

Proposition 1. *For any intuitionistic fuzzy point $\langle a, b\rangle$, such that $0 \leq a + b < 1$, we can find a point $\langle a^*, b^*\rangle$, such that $T_0(\langle a^*, b^*\rangle) = \langle a, b\rangle$.*

Proof. First let $a = 0$. Then we must have:

$$\begin{cases} a^*(2 - a^* - b^*) = 0 \\ b^*(2 - a^* - b^*) = b \end{cases}$$

This is only possible for $a^* = 0$. Solving further we obtain:

$$b^* = 1 - \sqrt{1 - b}$$

Analogously, if $b = 0$, we will obtain

$$a^* = 1 - \sqrt{1 - a}$$

Further, we will assume both $a, b > 0$, thus:

$$\begin{cases} a^*(2 - a^* - b^*) = a \\ b^*(2 - a^* - b^*) = b \end{cases}$$

Without loss of generality, let us assume $\max(a, b) = a$. Then $a = tb$ for $t = \frac{a}{b} > 1$, i.e.,

$$\begin{cases} a^*(2 - a^* - b^*) = tb \\ b^*(2 - a^* - b^*) = b \end{cases}$$

Evidently, the same proportion must be present in the left-hand side, i.e.,

$$\begin{cases} tb^*(2 - tb^* - b^*) = tb \\ b^*(2 - tb^* - b^*) = b \end{cases}$$

Solving it, after substituting back we finally obtain:

$$\begin{cases} a^* = \dfrac{a}{a+b}(1 - \sqrt{1-a-b}) \\ b^* = \dfrac{b}{a+b}(1 - \sqrt{1-a-b}) \end{cases}$$

Therefore, in all cases we find a point $\langle a^*, b^* \rangle$, that the operator T_0 transforms to $\langle a, b \rangle$. This completes the proof. □

From the above it is clear that for the extended operator $T_{\lambda,\alpha,\beta}(A^*)$ at least one such point must exist for a suitable choice of the parameters λ, α, β. In general, there may be a whole set of points being transformed to a given point $\langle a, b \rangle$. For instance, for the point $u = \langle 0.3, 0.3 \rangle$, we have

$$u = T_{0,\frac{6}{11},\frac{15}{22}}\left(\left\langle \frac{1}{2}, \frac{2}{5} \right\rangle\right) = T_{0,\frac{5}{8},\frac{5}{8}}\left(\left\langle \frac{2}{5}, \frac{2}{5} \right\rangle\right) = T_{\frac{1}{2},\frac{41472}{43435},\frac{23328}{31025}}\left(\left\langle \frac{7}{16}, \frac{5}{9} \right\rangle\right)$$

The way to describe the set of all points being mapped to a given point for different choices of the parameters λ, α, β will be an object of further study.

3 Conclusion

In the present paper we introduced a new modal-like operator over IFSs, similar to operator $F_{\alpha,\beta}$ but of multiplicative nature. We investigated some of its properties established some results regarding it.

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