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Some ways and means to define addition and multiplication operations between intuitionistic fuzzy sets

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In this paper we introduce some operations on IFS [1]. P. Burillo and H. Bustince introduced T- and S- norms as follows:

$$P(A,B) = \{< x, T(\mu_A(x), \mu_B(x)), S(\nu_A(x), \nu_B(x)) > /x \in E\}$$

where

$$0 \leq T(\mu_A(x), \mu_B(x)) + S(\nu_B(x), \nu_A(x)) \leq 1$$

We shall define:

$$\overline{P}(A,B) = \{< x, S(\mu_A(x), \mu_B(x)), T(\nu_A(x), \nu_B(x)) > /x \in E\}$$

Therefore $\neg\overline{P} \equiv P$, where $\equiv$ is the ``equivalence" relation between operations.
For example, from De Morgan's law we have (see [1]):

$$\cap \equiv \overline{\cup}$$

$$+ \equiv \overline{\cdot}$$

When $T \equiv S$, then $P \equiv \overline{P}$.
For example ([1]):

$$@ \equiv \overline{@}$$

$$\bowtie \equiv \overline{\bowtie}$$

$$* \equiv \overline{*}$$

If $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in E\}$, then (see[1]):

$$\neg A = \{\langle x, \nu_A(x), \mu_A(x)\rangle / x \in E\}$$

We have that [1]

$$\neg\overline{P}(\neg A, \neg B) = P(A,B)$$

We see that T and S define P. Thus, it is very important to study the forms of T and S.

Below we demonstrate the conditions that T and S must satisfy in order for the operation $P(A, B)$ to have some desired properties (associativity or commutativity).

It is commutative when

$P(A, B) = P(B, A)$

$$\Leftrightarrow \begin{cases} T(\mu_A(x), \mu_B(x)) = T(\mu_B(x), \mu_A(x)) \\ S(\nu_A(x), \nu_B(x)) = S(\nu_B(x), \nu_A(x)) \end{cases}$$

It is associative when

$$P(P(A, B), C) = P(A, P(B, C)) \tag{1}$$

where

$$P(P(A, B), C) = \{\langle x, T(T(\mu_A(x), \mu_B(x)), \mu_C(x)), \\ S(S(\nu_A(x), \nu_B(x)), \nu_C(x))\rangle | x \in E\} \tag{2}$$

$$P(A, P(B, C)) = \{\langle x, T(\mu_A(x), T(\mu_B(x), \mu_C(x))), \\ S(\nu_A(x), S(\nu_B(x), \nu_C(x)))\rangle | x \in E\} \tag{3}$$

Replacing (2), (3) in (1), we get the equalities

$T(T(\mu_A(x), \mu_B(x)), \mu_C(x)) = T(\mu_A(x), T(\mu_B(x), \mu_C(x)))$

and $S(S(\nu_A(x), \nu_B(x)), \nu_C(x)) = S(\nu_A(x), S(\nu_B(x), \nu_C(x)))$

where

$$0 \leq T(T(\mu_A(x), \mu_B(x)), \mu_C(x)) + S(S(\nu_A(x), \nu_B(x)), \nu_C(x)) \leq 1$$

$$0 \leq T(\mu_A(x), T(\mu_B(x), \mu_C(x))) + S(\nu_A(x), S(\nu_B(x), \nu_C(x))) \leq 1$$

It is possible for $P(A, B)$ to be both commutative and associative only if T and S are solutions of the two systems

$$\begin{cases} T(T(\mu_A(x), \mu_B(x)), \mu_C(x)) = T(\mu_A(x), T(\mu_B(x), \mu_C(x))) \\ T(\mu_A(x), \mu_B(x)) = T(\mu_B(x), \mu_A(x)) \end{cases}$$

and

$$\begin{cases} S(S(\nu_A(x), \nu_B(x)), \nu_C(x)) = S(\nu_A(x), S(\nu_B(x), \nu_C(x))) \\ S(\nu_A(x), \nu_B(x)) = S(\nu_B(x), \nu_A(x)) \end{cases}$$

We will look for a solution of the above systems of the kind

$$\begin{cases} P(P(x, y), z) = P(x, P(y, z)) \\ P(x, y) = P(y, x) \end{cases}$$

where P is a polynomial with real coefficients and $\langle x, y, z\rangle \in I^3$, $\langle p, q, l\rangle \in I^3$ and $I = [0, 1]$. For this purpose we will prove the next lemma.

Lemma 1. If $R(x,y,z)$ and $Q(x,y,z)$ are two polynomials with real coefficients for which $R(x,y,z) = Q(x,y,z)$ for all $(x,y,z) \in I^3$ (where $I = [0,1]$) then $R(x,y,z) \equiv Q(x,y,z)$.

Proof: In the proof of the above statement, we will use the following well-known result from algebra (see [3]).

(*) *If k is a field and $T_1, T_2, \ldots, T_n$ are infinite sets from k and $f(x_1, \ldots, x_n)$ is a polynomial of n variables over k, then whenever $f(a_1, \ldots, a_n) = 0$ for all $a_i \in T_i (i = 1, .., n)$, we have that $f = 0$.*

Let $R(x,y,z) \not\equiv Q(x,y,z)$. From $R(x,y,z) = Q(x,y,z)$ there follows

$$W(x,y,z) = R(x,y,z) - Q(x,y,z) = 0$$

for $(x,y,z) \in I^3$(where $I = [0,1]$) where $W(x,y,z)$ is not a zero polynomial.

Therefore, when $n = 3$, (*) contradicts (**).

Lemma 2. If $P(x,y)$ is a polynomial with real coefficients which is a solution of the system

$$\begin{cases} P(x,y) = P(y,x) \\ P(P(x,y),z) = P(x,P(y,z)) \end{cases} \tag{4}$$

then $P(x,y) = a(x+y) + b(xy)$, where $a = 1$ or $a = 0$

Proof: From (4) it follows that $\Rightarrow P(x,y) = ax^n y^k + Q(x,y) + ax^k y^n$ where n is the maximal degree of x and y in $P(x,y)$ and the maximal degree of x and y is less than n in $Q(x,y)$.

Let $n > 1$. Then we have two possibilities

$$P(x,y) = ax^n y^k + Q(x,y) + ax^k y^n \text{ when } n \neq k \tag{5}$$

$$P(x,y) = ax^n y^n + Q(x,y) \text{ when } n = k \tag{6}$$

From (5) it follows that

$$P(P(x,y),z) = a(ax^n y^k + Q(x,y) + ax^k y^n)^n z^k$$

$$+Q(ax^n y^k + Q(x,y) + ax^k y^n, z) + a(ax^n y^k + Q(x,y) + ax^k y^n)^k z^n \tag{7}$$

$$P(x,P(y,z)) = ax^n(ay^n z^k + Q(y,z) + ay^k z^n)^k$$

$$+Q(x, ay^n z^k + Q(y,z) + ay^k z^n) + ax^k(ay^n z^k + Q(y,z) + ay^k z^n)^n \tag{8}$$

If we want the second equation of (4) to hold, then by Lemma 1, $P(P(x,y),z)$ should coincide with $P(x,P(y,z))$. The maximal degree of x on the right side on (7) is n^2, but in (8) it is n. Therefore case (5) is impossible.

$$P(P(x,y),z) = a(ax^n y^n + Q(x,y))^n z^n + Q(ax^n y^n + Q(x,y),z) \tag{9}$$

$$P(x,P(y,z)) = ax^n(ay^n z^n + Q(y,z))^n + Q(x, ay^n z^n + Q(y,z)) \tag{10}$$

Again, if we want the second equation of (4) to hold, then by Lemma 1, $P(P(x,y),z)$ should coincide with $P(x,P(y,z))$.

On the right side on (6) the addend with maximal degree of x is $a^2x^{n^2}y^{n^2}z^n$, but in (10), it is $a^2x^ny^{n^2}z^{n^2}$.

Therefore case (6) is impossible.

Thus $n = 1$. We have therefore that $P(x,y) = a(x+y) + bxy$. It is also true that $P(x,y) = P(y,x)$.

$$P(P(x,y),z) = a(a(x+y) + bxy + z) + b(a(x+y) + bxy)z =$$
$$= a^2x + a^2y + abxy + az + baxz + bayz + b^2xyz$$
$$P(x,P(y,z)) = a(x + a(y+z) + byz) + bx(a(y+z) + byz) =$$
$$= ax + a^2y + a^2z + abyz + abxy + abxz + b^2xyz$$

We want system (4) to hold, but this is only possible when $a^2x + a^2y + abxy + az + baxz + bayz + b^2xyz = ax + a^2y + a^2z + abyz + abxy + abxz + b^2xyz$

$a^2x + a^2y + az = ax + a^2y + a^2z$ $(x,y,z) \in I^3$

$\Leftrightarrow a^2 = a \Leftrightarrow a(a-1) = 0 \Leftrightarrow a = 0$ and $a = 1$

Theorem. From all polynomials with real coefficients that can participate in the construction of the operation

$P(A,B) = \{\langle x, T(\mu_A(x), \mu_B(x)), S(\nu_A(x), \nu_B(x))\rangle | x \in E\}$

(where A,B are IFS)

in place of T and S, so that P is commutative and associative, only $a(x+y) + bxy$ are such that

$$a = \begin{cases} 1; b \in [-2,-1] \\ 0; b \in [-1,-1] \end{cases}$$

Proof: We have from (7) and Lemma 2 that every such polynomial have the form $a(x+y) + bxy$ where $a = 1$ or $a = 0$. Let us consider cases 1) $a = 1$ and 2) $a = 0$

1) $a = 1$ from Lemma 2 and $0 \leq P(P(x,y),z) \leq 1$ we have $x + y + z + b(xy + xz + yz) + b^2xyz \leq 1$ but $< x,y,z > \in I^3$ then $3 + 3b + b^2 \leq 1$. We solve this inequality and obtain that $b \in [-2,-1]$

2) $a = 0$ from Lemma 2 and $0 \leq P(P(x,y),z) \leq 1$ we have $b^2xyz \leq 1$ where $< x,y,z > \in I^3$ then $b^2 \leq 1$. We solve this inequality and obtain that $b \in [-1,1]$.

References

[1] Atanassov, K., Intuitionistic Fuzzy Sets, Physica Verlag, 1999.

[2] Lang, S., Algebra, Mir, Moscow, 1968 (Russian translation).