Notes on Intuitionistic Fuzzy Sets
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New implication deviation and entropy measures in intuitionistic fuzzy sets

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Abstract: This paper presents two new concepts within the framework of intuitionistic fuzzy set (IFS). An implication deviation is introduced, formulated using the implication operator together with a universal quantifier in the framework of intuitionistic fuzzy set. In addition to being a crucial analytical tool, this measure emphasizes its theoretical significance in the study of intuitionistic fuzzy sets by making it easier to define basic notions like inclusion and similarity. In addition, the paper introduces two new entropy measures, referred to as the α-intuitionistic fuzzy entropy and the β-intuitionistic fuzzy entropy, to quantify uncertainty of intuitionistic fuzzy sets. The α-intuitionistic fuzzy entropy evaluates uncertainty in terms of the deviation from the extreme situations of complete inclusion and complete exclusion. In contrast, the β-intuitionistic fuzzy entropy reflects maximum uncertainty when degrees of membership and non-membership

are balanced or when the degree of hesitancy is maximal, while minimum entropy corresponds to the case, when complete inclusion or exclusion is there. Both entropy measures are constructed using the Hamming distance to ensure a consistent and intuitive representation of uncertainty.
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1 Introduction

In 1965, Zadeh [?] introduced the concept of fuzzy set theory, in which each element of a set is associated with a degree of membership in the unit interval $[0, 1]$. This framework allows for the modeling of vague or imprecise information, in which it is often unclear whether an element completely belongs to a set or not. However, in fuzzy set theory, only the degree of membership is defined, and the degree of non-membership is implicitly considered as its complement. This assumption may poorly represent the hesitation or indeterminacy present in many real-world scenarios.

To address this limitation, Atanassov [?] introduced intuitionistic fuzzy sets (IFSs) as a generalization of fuzzy sets. An IFS is characterized by three functions: the degree of membership, the degree of non-membership, and the degree of hesitancy, where the degree of hesitancy reflects the uncertainty arising from incomplete or insufficient information. Due to their ability to model multiple sources of uncertainty simultaneously, IFSs have been widely used in diverse domains.

The notion of entropy within fuzzy set theory arises from the necessity to quantify the extent of fuzziness or uncertainty inherent in a fuzzy set [?]. In such sets, entropy reflects the degree to which an element's membership diverges from absolute certainty-complete inclusion or exclusion. A crisp set exhibits zero entropy, whereas the highest entropy occurs when the membership value equals 0.5, representing total ambiguity. Over time, numerous formulations of fuzzy entropy have emerged, grounded in concepts such as distance measures [?], information theory [?], and similarity between a fuzzy set and its complement [?], etc.

Building on this foundation, intuitionistic fuzzy entropy broadens the concept of fuzziness by considering not only the degrees of membership and non-membership but also the degree of hesitation or indeterminacy inherent in IFSs, as introduced by Atanassov [?]. Given that IFSs provide a more expressive framework for representing uncertainty, the corresponding entropy measures must adequately account for this added complexity. Over time, several definitions of intuitionistic fuzzy entropy have been proposed, each reflecting a distinct viewpoint. Burillo and Bustince [?] developed one of the earliest axiomatic approaches to intuitionistic fuzzy entropy, relating the concept to the hesitation degree of an intuitionistic fuzzy set. Later, Szmidt [?] proposed a non-probabilistic entropy measure based on the ratio between the intuitionistic fuzzy cardinalities of $F \cap F^c$ and $F \cup F^c$. Gerstenkorn and Mańko [?] proposed three energy-based measures for intuitionistic fuzzy sets, namely arithmetic energy, logical energy, and intuitionistic fuzzy energy. These measures quantify uncertainty in intuitionistic fuzzy sets from different perspectives. Subsequently, Ban [?] established an axiomatic framework for intuitionistic fuzzy entropy using intuitionistic norm functions and demonstrated that entropy measures generated in this manner can, under suitable conditions, be regarded as intuitionistic fuzzy measures.

Further developments were made by Szmidt and Baldwin [?], who introduced non-probabilistic entropy measures applicable to both intuitionistic fuzzy sets and mass assignment theory. In a subsequent work, Szmidt and Kacprzyk [?] refined their earlier approach by defining entropy through the relative distances of an intuitionistic fuzzy set from its nearest and farthest crisp counterparts. The corresponding entropy values were then obtained using a normalized Hamming distance. Vlachos and Sergiadis [?] proposed an entropy measure for intuitionistic fuzzy sets based on the inner product between intuitionistic fuzzy vectors. Thereafter, Zhu and Li [?] proposed a new axiomatic definition of entropy for intuitionistic fuzzy sets, together with an explicit entropy formula that incorporates both the amount of information and the reliability of the intuitionistic fuzzy set. More recently, Montes et al. [?] proposed a unified divergence-based framework that encompasses and generalizes a broad class of existing intuitionistic fuzzy entropy measures. Collectively, these studies highlight the sustained effort to establish robust measures capable of capturing the intricate nature and informational richness of intuitionistic fuzzy sets. On the other hand, various types of negation operators are introduced in the literature of intuitionistic fuzzy sets. Some of they can be found in [?, ?].

While considerable efforts have been devoted to the construction and characterization of intuitionistic fuzzy implication operators [?,?,?,?,?], their potential for establishing and analyzing relationships among intuitionistic fuzzy sets has received comparatively less attention. Existing approaches for comparing intuitionistic fuzzy sets are predominantly based on distance measures [?], similarity measures [?, ?], and inclusion measures [?]. In contrast, the study of relationships between intuitionistic fuzzy sets from the perspective of logical implication operators remains relatively underexplored. Motivated by this gap, we introduce the notion of implication deviation for intuitionistic fuzzy sets, which provides a logical framework for quantifying the deviation between two intuitionistic fuzzy sets from the perspective of logical implication.

Several entropy measures for intuitionistic fuzzy sets available in the literature [?, ?, ?] are based on the idea of comparing a given intuitionistic fuzzy set with certain standard or references intuitionistic fuzzy values. Motivated by this philosophy and by the practical interpretation of these reference points, we propose the α-intuitionistic fuzzy entropy measure, which quantifies uncertainty through the deviation of an intuitionistic fuzzy set from these standard values. However, during our investigation, we observed that many existing entropy measures capture uncertainty from the perspective of information content. In particular, several existing entropy measures [?,?] treat the case $\langle 0.5, 0.5\rangle$ in the same manner as other situations where the degrees of membership and non-membership are merely equal, without recognizing its special significance as a state of maximum fuzziness. On the other hand, some entropy measures [?, ?] do not assign sufficiently high uncertainty to this point because they mainly associate uncertainty with hesitation. However, the point $\langle x, 0.5, 0.5\rangle$ represents a state of maximum ambiguity arising from the equal support for membership and non-membership. This uncertainty is inherent in the complete balance between the two opposing assessments and persists despite the absence of hesitation, since the degree of hesitation is zero. Consequently, the uncertainty associated with this type of fuzziness in the intuitionistic fuzzy set is not adequately captured by many existing entropy measures.

This observation motivated us to introduce a new set of entropy axioms that distinguish between uncertainty arising from hesitation and uncertainty arising from maximal fuzziness due to balanced membership and non-membership information. Based on these axioms, we develop the β-intuitionistic fuzzy entropy measure, which provides a more comprehensive representation of uncertainty in intuitionistic fuzzy sets by incorporating both sources of uncertainty.

In this paper, we propose the concept of implication deviation as a rigorous means to quantify how closely an IFS relation satisfies implication. Moreover, we design a new entropy measure that integrates the advantages of classical fuzzy entropy with the unique uncertainty introduced by hesitation. Collectively, these ideas seek to enhance the theoretical foundation of IFS while providing practical tools that more accurately represent the complexities of uncertain information encountered in real-world scenarios. In Section 2, we recall some basic definitions and preliminary results that are needed for the rest of the paper. These concepts provide the necessary background and will be used repeatedly in the subsequent sections. In this study, we advance the theoretical framework of IFSs in two principal directions. First, in Section 3, we present a measure that evaluates the extent to which an intuitionistic fuzzy relation diverges from ideal logical implication, thereby establishing a foundation for defining and analyzing essential relational properties. Second, in Section 4, we propose an entropy measure for IFS that not only aligns with the behaviour of an entropy measure of fuzzy sets but also integrates the additional uncertainty associated with hesitation. In Section 5, a comparison between the proposed entropy measures and existing entropy measures is presented. Together, these contributions aim to expand the analytical capabilities of IFSs and enhance their applicability in reasoning and decision-making processes under uncertainty.

2 Preliminaries

In this section, we briefly state some basic concepts, which will be used throughout this paper.

Definition 2.1. [?] *Let X be a universal set, then an intuitionistic fuzzy set A defined over X is denoted as $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$, where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1\ \forall x \in X$. Here, $\mu_A(x)$ is the degree of membership, $\nu_A(x)$ is the degree of non-membership of element $x \in X$ to A.*

The degree of hesitation, also referred to as the degree of indeterminacy, is denoted by $\pi_A(x)$ and is given by: $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$. This measure quantifies the uncertainty associated with the classification of x within A.

Let the collection of all intuitionistic fuzzy sets defined over a universal set X be denoted by $\mathrm{IFSs}(X)$.

Definition 2.2. [?] *Let $A, B \in \mathrm{IFSs}(X)$, where μ_A, ν_A and μ_B, ν_B represent the membership and non-membership functions of A and B, respectively. Then, the following set operations are defined for any $x \in X$:*

(i) $A \subseteq B \iff \mu_A(x) \leq \mu_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$,

(ii) $A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle \mid x \in X\}$,

(iii) $A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle \mid x \in X\}$,

(iv) $A \mid B = \{\, \langle x, \min(\mu_A(x), \nu_B(x)), \max(\mu_B(x), \nu_A(x))\rangle \mid x \in X\,\}$,

(v) $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$.

Definition 2.3. [?] *Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$ be an intuitionistic fuzzy set defined over a universe X. Then, A is called an intuitionistic fuzzy tautological set (IFTS) if and only if, for every element $x \in X$, $\mu_A(x) \geq \nu_A(x)$.*

The following are commonly used forms of Cartesian product operations for intuitionistic fuzzy sets A and B defined over X and Y, respectively, where μ_A, ν_A and μ_B, ν_B are the respective membership and non-membership functions:

$$R_1 : A \times_1 B = \{\langle (x, y), \mu_A(x) \cdot \mu_B(y),\ \nu_A(x) \cdot \nu_B(y)\rangle \mid x \in X,\ y \in Y\},$$

$$R_2 : A \times_2 B = \{\langle (x, y), \mu_A(x) + \mu_B(y) - \mu_A(x) \cdot \mu_B(y),\ \nu_A(x) \cdot \nu_B(y)\rangle \mid x \in X,\ y \in Y\},$$

$$R_3 : A \times_3 B = \{\langle (x, y), \mu_A(x) \cdot \mu_B(y),\ \nu_A(x) + \nu_B(y) - \nu_A(x) \cdot \nu_B(y)\rangle \mid x \in X,\ y \in Y\},$$

$$R_4 : A \times_4 B = \{\langle (x, y), \min(\mu_A(x), \mu_B(y)),\ \max(\nu_A(x), \nu_B(y))\rangle \mid x \in X,\ y \in Y\},$$

$$R_5 : A \times_5 B = \{\langle (x, y), \max(\mu_A(x), \mu_B(y)),\ \min(\nu_A(x), \nu_B(y))\rangle \mid x \in X,\ y \in Y\},$$

$$R_6 : A \times_6 B = \left\{\left\langle (x, y),\ \frac{\mu_A(x) + \mu_B(y)}{2},\ \frac{\nu_A(x) + \nu_B(y)}{2}\right\rangle \mid x \in X,\ y \in Y\right\}.$$

Definition 2.4. [?] *Let X and Y be arbitrary finite sets and let $\circ \in \{\times_1, \times_2, \ldots, \times_6\}$ be a fixed binary operation representing a generalized Cartesian product. An intuitionistic fuzzy relation under the operation $\circ$, denoted as a $\circ$-intuitionistic fuzzy relation, is an intuitionistic fuzzy set $R \subseteq X \times Y$ of the form $R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle \mid x \in X,\ y \in Y\}$, where $\mu_R : X \times Y \to [0, 1]$ and $\nu_R : X \times Y \to [0, 1]$ represent, respectively, the degree of membership and the degree of non-membership of the pair $(x, y) \in X \times Y$ in the relation R, satisfying: $0 \leq \mu_R(x, y) + \nu_R(x, y) \leq 1$ for any $(x, y) \in X \times Y$. The specific form of μ_R and ν_R are determined by the choice of the product operation $\circ$.*

Definition 2.5. [?] *Let $X = \{x_1, x_2, \ldots, x_n, \ldots\}$. An intuitionistic fuzzy relation R on X is called*

(i) reflexive, if and only if $R\langle x_i, x_i\rangle = \langle 1, 0\rangle$ holds for each $x_i \in X$,

(ii) symmetric, if and only if $R\langle x_i, x_j\rangle = R\langle x_j, x_i\rangle$ holds for each $x_i, x_j \in X$.

Definition 2.6. [?] *A real-valued function $I_{inc} : \mathrm{IFSs}(X) \times \mathrm{IFSs}(X) \to [0, 1]$ is called the inclusion measure of IFSs on universe X, if it satisfies the following properties:*

(i) $I_{inc}(\widetilde{X}, \widetilde{\phi}) = 0$, where $\widetilde{X} = \{\langle x, 1, 0\rangle \mid x \in X\}$ and $\widetilde{\phi} = \{\langle x, 0, 1\rangle \mid x \in X\}$,

(ii) $I_{inc}(A, B) = 1$ if and only if $A \subseteq B$,

(iii) if $A \subseteq B \subseteq C$, then $I_{inc}(C, A) \leq \min\{I_{inc}(B, A), I_{inc}(C, B)\}$.

Definition 2.7. [?] *A real-valued function* $S : \mathrm{IFSs}(X) \times \mathrm{IFSs}(X) \to [0, 1]$ *is called a similarity measure on IFSs*(X) *if it satisfies the following axiomatic conditions:*

(i) $0 \le S(A, B) \le 1$,
(ii) $S(A, B) = 1$ *if* $A = B$,
(iii) $S(A, B) = S(B, A)$,
(iv) if $A \subseteq B \subseteq C$, *then* $S(A, C) \le \min\{S(A, B), S(B, C)\}$.

Definition 2.8. [?] *Let* X *be a finite set. The function* $V : X \to [0, 1] \times [0, 1]$, *given by* $V(x) = \langle \mu(x), \nu(x)\rangle$, *assigns to each element* $x \in X$ *its intuitionistic fuzzy evaluation, where* $\mu(x)$ *and* $\nu(x)$ *denote the truth and falsity degrees of* x, *respectively.*

Definition 2.9. [?] *Let* X *be a finite set. Then, the pair* $\langle \mu(x), \nu(x)\rangle$, *that satisfies* $\mu(x), \nu(x) \in [0, 1]$ *and* $\mu(x) + \nu(x) \le 1$ *is called intuitionistic fuzzy pair associated with element* $x \in X$. *For two intuitionistic fuzzy pairs,* $\langle \mu_A(x), \nu_A(x)\rangle \le \langle \mu_B(x), \nu_B(x)\rangle$ *if and only if* $\mu_A(x) \le \mu_B(x)$ *and* $\nu_A(x) \ge \nu_B(x)$.

According to [?], for the construction of the quantifier $\forall_{139,3}$, the implication operator $\to_{139}$ together with the negation operator $\neg_1$ plays a crucial role, as these two operators generate the averaging-based quantifier structure used in our framework. Therefore, we include these specific operators in this section. In addition, we also consider the implication operator $\to_4$, which is considered as the standard implication operator [?], for establishing some results presented later.

Definition 2.10. [?, ?] *An intuitionistic fuzzy implication operator* $\to_{139}$: $[0, 1]^2 \times [0, 1]^2 \longrightarrow [0, 1]^2$ *is defined as* $\to_{139} (\langle a, b\rangle, \langle c, d\rangle) = \langle \frac{b+c}{2}, \frac{a+d}{2}\rangle$, *for any* $\langle a, b\rangle, \langle c, d\rangle \in [0, 1]^2$.

Definition 2.11. [?] *An intuitionistic fuzzy implication operator* $\to_4$: $[0, 1]^2 \times [0, 1]^2 \longrightarrow [0, 1]^2$ *is defined as* $\to_4 (\langle a, b\rangle, \langle c, d\rangle) = \langle \max\{b, c\}, \min\{a, d\}\rangle$, *for any* $\langle a, b\rangle, \langle c, d\rangle \in [0, 1]^2$.

Definition 2.12. [?] *An intuitionistic fuzzy negation operator* $\neg_1 : [0, 1]^2 \to [0, 1]^2$ *is defined as* $\neg_1\langle a, b\rangle = \langle b, a\rangle$, *for any* $\langle a, b\rangle \in [0, 1]^2$.

Definition 2.13. [?] *Let* $X = \{x_1, x_2, \ldots, x_n\}$ *be a finite set, and let* $P(x)$ *be a predicate with variable* x. *Let* $V(P(x)) = \langle \mu(P(x)), \nu(P(x))\rangle$. *The IF-interpretations of the (intuitionistic fuzzy) quantifiers "for all" (*$\forall$*) and "there exists" (*$\exists$*) are given by*

$$V(\forall x\, P(x)) = \langle \inf_{y\in X} \mu(P(y)),\ \sup_{y\in X} \nu(P(y))\rangle,$$
$$V(\exists x\, P(x)) = \langle \sup_{y\in X} \mu(P(y)),\ \inf_{y\in X} \nu(P(y))\rangle.$$

The quantifiers $\forall_{139,3}$ *and* $\exists_{139,3}$ *over* n *elements are defined as:*

$$V(\forall_{139,3}x\, P(x)) = V(\exists_{139,3}x\, P(x)) = \langle \frac{1}{n}\sum_{i=1}^{n} a_i,\ \frac{1}{n}\sum_{i=1}^{n} b_i\rangle,$$

where $V(P(x_i)) = \langle a_i, b_i\rangle$, *for each* $i = 1, 2, \ldots, n$.

3 Implication deviation

Let $X = \{x_1, x_2, x_3, \ldots, x_n\}$ represent a universal set for an intuitionistic fuzzy set defined over X. For real-world applications, we assume that X is a finite set. Now, let us consider the following definitions:

Definition 3.1. **[?, ?]** *An intuitionistic fuzzy implication operator is a function* $I_I : [0,1]^2 \times [0,1]^2 \to [0,1]^2$, *having the following properties:*

(i) if $\langle x, y\rangle \leq \langle x', y'\rangle$, *then* $I_I(\langle x, y\rangle, \langle z, t\rangle) \geq I_I(\langle x', y'\rangle, \langle z, t\rangle)$,*for all* $\langle z, t\rangle \in [0,1]^2$,

(ii) if $\langle z, t\rangle \leq \langle z', t'\rangle$, *then* $I_I(\langle x, y\rangle, \langle z, t\rangle) \leq I_I(\langle x, y\rangle, \langle z', t'\rangle)$,*for all* $\langle x, y\rangle \in [0,1]^2$,

(iii) $I_I(\langle 0,1\rangle, \langle 0,1\rangle) = \langle 1,0\rangle$,

(iv) $I_I(\langle 0,1\rangle, \langle 1,0\rangle) = \langle 1,0\rangle$,

(v) $I_I(\langle 1,0\rangle, \langle 1,0\rangle) = \langle 1,0\rangle$,

(vi) $I_I(\langle 1,0\rangle, \langle 0,1\rangle) = \langle 0,1\rangle$.

Many intuitionistic fuzzy implications satisfying Definition 3.1 have been proposed in the literature [**?**], including $\to_i$, where $i = 2-5,\ 8,\ 11-16,\ 18-20,\ 22-29,\ 32-35,\ 37,\ 40-45,\ 47-50,\ 52,\ 55-60,\ 62-63,\ 65,\ 68-70,\ 72-74,\ 76-81,\ 83-84,\ 87-90,\ 92-93,\ 96-99$, and so on. The theoretical developments presented in this paper are formulated in terms of Definition 3.1 and are therefore applicable to any intuitionistic fuzzy implication satisfying the required conditions of Definition 3.1.

We recall below a few axioms proposed by Klir and Yuan [**?**] that will be used in the subsequent results.

Axiom 1. $(\forall y)\ I(0, y) = 1$,

Axiom 2. $(\forall y)\ I(1, y) = y$,

Axiom 3. $(\forall x)\ I(x, x) = 1$,

Axiom 4. $(\forall x, y)\ I(x, y) = 1$ if and only if $x \leq y$,

Axiom 5. $(\forall x, y)\ I(x, y) = I(N(y), N(x))$, where N denotes a negation operator.

Definition 3.2. *Let* $X = \{x_1, x_2, \ldots, x_n\}$ *be a finite set, and let* $A, B \in \mathrm{IFSs}(X)$. *Then, the function* I_s : *IFSs*$(X) \times \mathrm{IFSs}(X) \to [0,1] \times [0,1]$ *is defined by*

$$I_s(A, B) = V(\forall_{139,3} x\, P_{AB}(x)) = \langle \frac{1}{n} \sum_{x \in X} r_{AB}(x),\ \frac{1}{n} \sum_{x \in X} s_{AB}(x)\rangle,$$

where $x \in X$ *and*

$$V(P_{AB}(x)) = \langle r_{AB}(x), s_{AB}(x)\rangle = I_I(\langle \mu_A(x), \nu_A(x)\rangle, \langle \mu_B(x), \nu_B(x)\rangle).$$

The quantifier $\forall_{139,3}$ is employed because it acts as a global quantifier (weight-center) [**?**]. Since the implication deviation is computed for each element individually, the arithmetic mean induced by $\forall_{139,3}$ provides a natural way to aggregate these local deviations into a single global measure. This quantifier gives equal importance to each element, preserves normalization, and yields an interpretable measure representing the average deviation of the intuitionistic fuzzy set. The present work adopts $\forall_{139,3}$ because the objective is to capture the overall average deviation of the set.

Definition 3.3. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and $A, B \in \mathrm{IFSs}(X)$. If $I_s(A, B) = \langle a, b\rangle$, then the implication deviation $I_D : \mathrm{IFSs}(X) \times \mathrm{IFSs}(X) \to [0, 1]$ is defined as $I_D(A, B) = \frac{|1-a|+|0-b|}{2}$.*

Here, $I_D(A, B)$ is constructed to measure how far an implication is from being a tautology in the intuitionistic fuzzy framework. In the setting of intuitionistic fuzzy sets, a tautology is represented by the pair $\langle 1, 0\rangle$, which denotes complete truth and zero falsity. If $I_s(A, B) = \langle a, b\rangle$, the deviation is calculated as the Hamming distance between $\langle a, b\rangle$ and $\langle 1, 0\rangle$, normalized by 2 so that $I_D \in [0, 1]$. The Hamming distance is one of the most widely used and well-established distance measures in the intuitionistic fuzzy set literature due to its simplicity and computational efficiency. Therefore, we have adopted the Hamming distance consistently throughout this paper. Moreover, the proposed implication deviation and the associated results are not restricted to the Hamming distance alone. Other standard distance measures, such as the Euclidean distance, may also be employed, and the validity of the proposed implication deviation and the derived results remain unaffected. The only exception is Theorem 15, which establishes a relationship between the implication deviation and the α-intuitionistic fuzzy entropy measure, for which the Hamming distance provides a more convenient mathematical formulation. Since this theorem is supplementary rather than central to the main contributions of the paper, the choice of the Hamming distance does not limit the general applicability of the proposed approach.

Example 3.1. *Let $X = \{x_1, x_2, x_3\}$ and consider two intuitionistic fuzzy sets $A, B \in \mathrm{IFSs}(X)$ defined as follows: $A = \{\langle x_1, 0.7, 0.2\rangle,\ \langle x_2, 0.4, 0.5\rangle,\ \langle x_3, 0.9, 0.1\rangle\}$ and $B = \{\langle x_1, 0.6, 0.3\rangle,\ \langle x_2, 0.2, 0.7\rangle,\ \langle x_3, 0.8, 0.1\rangle\}$.*

Now, we consider the implication $I_I(\langle a, b\rangle, \langle c, d\rangle) = \langle \min\{1, b + c\},\ \max\{0, a + d - 1\}\rangle$, and we get

Table 1. Evaluation of the intuitionistic fuzzy implication I_I

x	$I_I(\langle a, b\rangle, \langle c, d\rangle)$
x_1	$\langle 0.8, 0\rangle$
x_2	$\langle 0.7, 0.1\rangle$
x_3	$\langle 0.9, 0\rangle$

Thus, using Definition 3.2, we get $I_s(A, B) = \left\langle \frac{0.8+0.7+0.9}{3},\ \frac{0+0.1+0}{3} \right\rangle = \langle 0.8,\ 0.0333\rangle$. Then, the implication deviation is $I_D(A, B) = \frac{|1-0.8|+|0-0.0333|}{2} = \frac{0.2+0.0333}{2} = 0.11665$.

Lemma 1. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and $A, B, C, D \in$ IFSs(X). If $I_s(A, B) \leq I_s(C, D)$, then $I_D(A, B) \geq I_D(C, D)$.*

Proof. Assume that $I_s(A, B) = \langle a_1, b_1 \rangle$ and $I_s(C, D) = \langle a_2, b_2 \rangle$. Given that $I_s(A, B) \leq I_s(C, D)$, i.e., $\langle a_1, b_1 \rangle \leq \langle a_2, b_2 \rangle$. This gives $a_1 \leq a_2$ and $b_1 \geq b_2$. From $a_1 \leq a_2$, we obtain $1 - a_1 \geq 1 - a_2$. Therefore,

$$\frac{|1 - a_1| + |0 - b_1|}{2} \geq \frac{|1 - a_2| + |0 - b_2|}{2}.$$

Thus, using Definition 3.3, we get $I_D(A, B) \geq I_D(C, D)$. □

Theorem 1. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and $A, B, C \in \mathrm{IFSs}(X)$. Then, the following results hold:*

(i) if $A \subseteq B$, then $I_D(A, C) \leq I_D(B, C)$,

(ii) if $A \subseteq B$, then $I_D(C, A) \geq I_D(C, B)$,

(iii) if A is an intuitionistic fuzzy tautological set, then $I_D(A, C) \geq I_D(A^c, C)$.

Proof. (i) Since $A \subseteq B$, it follows for every $x \in X, \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, i.e., $\langle \mu_A(x), \nu_A(x) \rangle \leq \langle \mu_B(x), \nu_B(x) \rangle$. By condition (i) of Definition 3.1, we have

$$I_I(\langle \mu_A(x), \nu_A(x) \rangle, \langle \mu_C(x), \nu_C(x) \rangle) \geq I_I(\langle \mu_B(x), \nu_B(x) \rangle, \langle \mu_C(x), \nu_C(x) \rangle),$$

i.e., $\langle r_{AC}(x), s_{AC}(x) \rangle \geq \langle r_{BC}(x), s_{BC}(x) \rangle$. It implies

$$\langle \frac{1}{n} \sum_{x \in X} r_{AC}(x), \frac{1}{n} \sum_{x \in X} s_{AC}(x) \rangle \geq \langle \frac{1}{n} \sum_{x \in X} r_{BC}(x), \frac{1}{n} \sum_{x \in X} s_{BC}(x) \rangle,$$

i.e., $I_s(A, C) \geq I_s(B, C)$. Thus, by Lemma 1, we get $I_D(A, C) \leq I_D(B, C)$.

(ii) Since $A \subseteq B$, it follows for every $x \in X, \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, i.e., $\langle \mu_A(x), \nu_A(x) \rangle \leq \langle \mu_B(x), \nu_B(x) \rangle$. By condition (ii) of Definition 3.1, we have

$$I_I(\langle \mu_C(x), \nu_C(x) \rangle, \langle \mu_A(x), \nu_A(x) \rangle) \leq I_I(\langle \mu_C(x), \nu_C(x) \rangle, \langle \mu_B(x), \nu_B(x) \rangle),$$

i.e., $\langle r_{CA}(x), s_{CA}(x) \rangle \leq \langle r_{CB}(x), s_{CB}(x) \rangle$. It implies

$$\langle \frac{1}{n} \sum_{x \in X} r_{CA}(x), \frac{1}{n} \sum_{x \in X} s_{CA}(x) \rangle \leq \langle \frac{1}{n} \sum_{x \in X} r_{CB}(x), \frac{1}{n} \sum_{x \in X} s_{CB}(x) \rangle.$$

Consequently, this implies $I_s(C, A) \leq I_s(C, B)$. Thus, by Lemma 1, we get $I_D(C, A) \geq I_D(C, B)$.

(iii) Assume A is an intuitionistic fuzzy set, i.e., $\mu_A(x) \geq \nu_A(x)$, then $\langle \nu_A(x), \mu_A(x) \rangle \leq \langle \mu_A(x), \nu_A(x) \rangle$. Using condition (i) of Definition 3.1, we get

$$I_I(\langle \nu_A(x), \mu_A(x) \rangle, \langle \mu_C(x), \nu_C(x) \rangle) \geq I_I(\langle \mu_A(x), \nu_A(x) \rangle, \langle \mu_C(x), \nu_C(x) \rangle),$$

i.e., $\langle r_{A^cC}(x), s_{A^cC}(x)\rangle \geq \langle r_{AC}(x), s_{AC}(x)\rangle$ for any $x \in X$. It implies

$$\langle \frac{1}{n}\sum_{x\in X} r_{A^cC}(x), \frac{1}{n}\sum_{x\in X} s_{A^cC}(x)\rangle \geq \langle \frac{1}{n}\sum_{x\in X} r_{AC}(x), \frac{1}{n}\sum_{x\in X} s_{AC}(x)\rangle.$$

Then, using Definition 3.2, we have $I_s(A^c, C) \geq I_s(A, C)$. So, by Lemma 1, we get $I_D(A^c, C) \leq I_D(A, C)$. □

Corollary 1. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and $A, B \in \mathrm{IFSs}(X)$. For any $C \in \mathrm{IFSs}(X)$, the following results hold:*

(i) $I_D(A \cap B, C) \leq I_D(A, C)$ *and* $I_D(A \cap B, C) \leq I_D(B, C)$,

(ii) $I_D(A \mid B, C) \leq I_D(A, C)$,

(iii) $I_D(A, C) \leq I_D(A \cup B, C)$ *and* $I_D(B, C) \leq I_D(A \cup B, C)$.

Proof. The proof directly follows from the above theorem. □

Theorem 2. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set. If the implication operator I_I also satisfies Axiom 3, then for any $A \in \mathrm{IFSs}(X)$, $I_D(A, A) = 0$.*

Proof. By Axiom 3, we have $I_I(\langle \mu_A(x), \nu_A(x)\rangle, \langle \mu_A(x), \nu_A(x)\rangle) = \langle 1, 0\rangle$ for any $x \in X$. Then, $\langle r_{AA}(x), s_{AA}(x)\rangle = I_I(\langle \mu_A(x), \nu_A(x)\rangle, \langle \mu_A(x), \nu_A(x)\rangle) = \langle 1, 0\rangle$ for any $x \in X$. Then, by using Definition 3.2,

$$I_s(A, A) = \left\langle \frac{1}{n}\sum_{x\in X} r_{AA}(x),\ \frac{1}{n}\sum_{x\in X} s_{AA}(x) \right\rangle = \langle \frac{n}{n}, \frac{0}{n}\rangle = \langle 1, 0\rangle.$$

Therefore, by using Definition 3.3, $I_D(A, A) = \frac{|1-1|+|0-0|}{2} = 0$. □

Theorem 3. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and $A, B \in \mathrm{IFSs}(X)$. If R_1 be a reflexive relation between A and B, then $I_D(A, B) = 0$.*

Proof. Since R_1 is reflexive, it gives $\mu_{R_1}(x, x) = 1$ and $\nu_{R_1}(x, x) = 0$ for any $x \in X$. Let the relation R_1 be defined such that $\mu_{R_1}(x, x) = \mu_A(x) \cdot \mu_B(x)$ and $\nu_{R_1}(x, x) = \nu_A(x) \cdot \nu_B(x)$. Since, $\mu_A(x) \cdot \mu_B(x) = 1$, and we know that $0 \leq \mu_A(x) \leq 1$ and $0 \leq \mu_B(x) \leq 1$. So, it is obvious that $\mu_A(x) = \mu_B(x) = 1$. Hence, $\nu_A(x) = \nu_B(x) = 0$. Therefore, we have $\langle \mu_A(x), \nu_A(x)\rangle = \langle \mu_B(x), \nu_B(x)\rangle = \langle 1, 0\rangle$ for any $x \in X$. Hence, $I_D(A, B) = 0$. □

Theorem 4. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and $A, B \in \mathrm{IFSs}(X)$. If R_6 be a reflexive relation between A and B, then $I_D(A, B) = 0$.*

Proof. Since R_6 is reflexive, $\mu_{R_6}(x, x) = 1$ and $\nu_{R_6}(x, x) = 0$ for any $x \in X$. Then, $\frac{\mu_A(x)+\mu_B(x)}{2} = 1$ and $\frac{\nu_A(x)+\nu_B(x)}{2} = 0$. It gives $\mu_A(x) + \mu_B(x) = 2$ and $\nu_A(x) + \nu_B(x) = 0$. Hence, $\mu_A(x) = \mu_B(x) = 1$ and $\nu_A(x) = \nu_B(x) = 0$. Therefore, $\langle \mu_A(x), \nu_A(x)\rangle = \langle \mu_B(x), \nu_B(x)\rangle = \langle 1, 0\rangle$ for any $x \in X$. It gives $I_D(A, B) = \frac{|1-1+|0-0|}{2} = 0$. □

Theorem 5. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and $A, B \in \mathrm{IFSs}(X)$. If I_I satisfies Axiom 3 and R_6 is a symmetric relation between A and B such that $\mu_A(x_k) = \mu_B(x_k)$ and $\nu_A(x_k) = \nu_B(x_k)$ for some $x_k \in X$, then $I_D(A, B) = 0$.*

Proof. Let R_6 be a symmetric relation between A and B, we have $\mu_{R_6}(x_i, x_j) = \mu_{R_6}(x_j, x_i)$ and $\nu_{R_6}(x_i, x_j) = \nu_{R_6}(x_j, x_i)$ for any $x_i, x_j \in X$. It implies $\frac{\mu_A(x_i)+\mu_B(x_j)}{2} = \frac{\mu_A(x_j)+\mu_B(x_i)}{2}$. It gives, $(\forall x_i, x_j \in X)$ $(\mu_A(x_i) - \mu_B(x_i) = \mu_A(x_j) - \mu_B(x_j))$. Given that $\mu_A(x_k) = \mu_B(x_k)$ for some $x_k \in X$. Then, we get $(\forall i)$ $(\mu_A(x_i) = \mu_B(x_i))$. Similarly it holds for ν_A and ν_B. Thus, $\mu_A(x_i) = \mu_B(x_i)$ and $\nu_A(x_i) = \nu_B(x_i)$ for any $x_i \in X$. It implies $A = B$. Since, I_I satisfies Axiom 3, then by Theorem 2, we get $I_D(A, B) = 0$. □

Theorem 6. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and $A, B \in \mathrm{IFSs}(X)$. If I_I satisfies Axiom 5, when N is taken as $\neg_1$, then $I_D(A, B) = I_D(B^c, A^c)$.*

Proof. By Axiom 5, we have

$$I_I(\langle \mu_A(x), \nu_A(x)\rangle, \langle \mu_B(x), \nu_B(x)\rangle) = I_I(N(\langle \mu_B(x), \nu_B(x)\rangle), N(\langle \mu_A(x), \nu_A(x)\rangle)),$$

i.e., $I_I(\langle \mu_A(x), \nu_A(x)\rangle, \langle \mu_B(x), \nu_B(x)\rangle) = I_I(\langle \nu_B(x), \mu_B(x)\rangle, \langle \nu_A(x), \mu_A(x)\rangle)$. Then, $(\forall x \in X)$ $(\langle r_{AB}(x), s_{AB}(x)\rangle = \langle r_{B^cA^c}(x), s_{B^cA^c}(x)\rangle)$. It implies

$$\langle \frac{1}{n}\sum_{x\in X} r_{AB}(x), \frac{1}{n}\sum_{x\in X} s_{AB}(x)\rangle = \langle \frac{1}{n}\sum_{x\in X} r_{B^cA^c}(x), \frac{1}{n}\sum_{x\in X} s_{B^cA^c}(x)\rangle.$$

So, $I_s(A, B) = I_s(B^c, A^c)$. Hence, $I_D(A, B) = I_D(B^c, A^c)$. □

Theorem 7. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set. If I_I satisfies Axiom 1 and if there exists $A \in \mathrm{IFSs}(X)$ such that $A = \{\langle x, 0, 1\rangle | x \in X\}$, then for any $B \in \mathrm{IFSs}(X)$, $I_D(A, B) = 0$.*

Proof. We have $A = \{\langle x, 0, 1\rangle | x \in X\}$. By using Axiom 1, for any $x \in X$ and for any $B \in \mathrm{IFSs}(X)$, we have $I_I(\langle \mu_A(x), \nu_A(x)\rangle, \langle \mu_B(x), \nu_B(x)\rangle) = \langle 1, 0\rangle$. Now, using Definition 3.2, we get $I_s(A, B) = \left\langle \frac{1}{n}\sum_{x\in X} r_{AB}(x), \frac{1}{n}\sum_{x\in X} s_{AB}(x)\right\rangle = \langle \frac{n}{n}, \frac{0}{n}\rangle = \langle 1, 0\rangle$. Therefore, $I_D(A, B) = \frac{|1-1|+|0-0|}{2} = 0$. □

Theorem 8. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and $A, B, C \in \mathrm{IFSs}(X)$. If $A \subseteq B \subseteq C$, then:*

(i) $I_D(A, C) \le \min\{I_D(A, B), I_D(B, C)\}$,

(ii) $I_D(C, A) \ge \max\{I_D(C, B), I_D(B, A)\}$.

Proof. (i) Given $A \subseteq B \subseteq C$. Then, using condition (ii) of Theorem 1 for $B \subseteq C$, we get $I_D(A, B) \ge I_D(A, C)$. Similarly, using condition (i) of Theorem 1 for $A \subseteq B$, we have $I_D(A, C) \le I_D(B, C)$. Hence, $I_D(A, C) \le \min\{I_D(A, B), I_D(B, C)\}$.

(ii) Given $A \subseteq B \subseteq C$. Then, using condition (i) of Theorem 1 for $B \subseteq C$, we get $I_D(B, A) \le I_D(C, A)$. Similarly, using condition (ii) of Theorem 1 for $A \subseteq B$, we have $I_D(C, A) \ge I_D(C, B)$. Hence, $I_D(C, A) \ge \max\{I_D(C, B), I_D(B, A)\}$. □

The following theorem provides a geometric interpretation of the proposed implication deviation. It shows that all I_s values represented by ordered pairs lying on a line with slope 1 and vertical intercept $2k-1$ have the same implication deviation k.

Theorem 9. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and $A, B, C, D \in \text{IFSs}(X)$. If $I_s(A,B) = \langle a_1, b_1\rangle$, $I_s(C,D) = \langle a_2, b_2\rangle$ and $b_t = a_t + (2k-1)$, for $t = 1, 2$ and some $k \in [0,1]$, then $I_D(A,B) = I_D(C,D) = k$.*

Proof. We have $I_s(A,B) = \langle a_1, b_1\rangle$. By Definition 3.3, we get $I_D(A,B) = \frac{|1-a_1|+|0-b_1|}{2}$. Similarly, if $I_s(C,D) = \langle a_2, b_2\rangle$, then $I_D(C,D) = \frac{|1-a_2|+|0-b_2|}{2}$. Using the condition $b_t = a_t + (2k-1)$ for $t = 1, 2$, we obtain $|1-a_t| + |0-b_t| = |1-a_t| + |a_t + (2k-1)| = 2k$. Thus, $I_D(A,B) = I_D(C,D) = k$, where $k \in [0,1]$. □

Example 3.2. *Let $I_s(A,B) = \langle 0.5, 0.1\rangle$ and $I_s(C,D) = \langle 0.6, 0.2\rangle$. Choose $k = 0.3$. Then, $\langle a_1, b_1\rangle = \langle 0.5, 0.1\rangle$ and $\langle a_2, b_2\rangle = \langle 0.6, 0.2\rangle$ are such that both ordered pairs lie on the line $b_t = a_t + (2k-1) = a_t - 0.4$, for $t = 1, 2$. Now, $I_D(A,B) = \frac{|1-0.5|+|0-0.1|}{2} = \frac{0.5+0.1}{2} = 0.3$, and $I_D(C,D) = \frac{|1-0.6|+|0-0.2|}{2} = \frac{0.4+0.2}{2} = 0.3$. Therefore, $I_D(A,B) = I_D(C,D) = 0.3 = k$.*

Theorem 10. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and $A, B \in \text{IFSs}(X)$. If I_I satisfies Axiom 4, then $I_{inc}(A,B) = 1 - I_D(A,B)$ satisfies the following conditions:*

(i) $I_{inc}(\widetilde{X}, \widetilde{\phi}) = 0$, where $\widetilde{X} = \{\langle x, 1, 0\rangle \mid x \in X\}$ and $\widetilde{\phi} = \{\langle x, 0, 1\rangle \mid x \in X\}$,

(ii) $I_{inc}(A,B) = 1$ if and only if $A \subseteq B$,

(iii) if $A \subseteq B \subseteq C$, then $I_{inc}(C,A) \le \min\{I_{inc}(B,A), I_{inc}(C,B)\}$.

Proof. (i) Since, for the intuitionistic fuzzy set $\widetilde{X}$, $\langle \mu_{\widetilde{X}}(x), \nu_{\widetilde{X}}(x)\rangle = \langle 1, 0\rangle$ and for the intuitionistic fuzzy set $\widetilde{\phi}$, $\langle \mu_{\widetilde{\phi}}(x), \nu_{\widetilde{\phi}}(x)\rangle = \langle 0, 1\rangle$. By condition (vi) of Definition 3.1, $I_I(\langle \mu_{\widetilde{X}}(x), \nu_{\widetilde{X}}(x)\rangle, \langle \mu_{\widetilde{\phi}}(x), \nu_{\widetilde{\phi}}(x)\rangle) = \langle 0, 1\rangle$ for any $x \in X$. Therefore, $I_D(\widetilde{X}, \widetilde{\phi}) = \frac{|1-0|+|0-1|}{2} = 1$. Hence, $I_{inc}(\widetilde{X}, \widetilde{\phi}) = 1 - I_D(\widetilde{X}, \widetilde{\phi}) = 1 - 1 = 0$.

(ii) Let $I_{inc}(A,B) = 1$. Then, $I_D(A,B) = 0$. It implies $I_s(A,B) = \langle 1, 0\rangle$.
Now, $I_s(A,B) = \left\langle \frac{1}{n}\sum_{x\in X} r_{AB}(x),\ \frac{1}{n}\sum_{x\in X} s_{AB}(x)\right\rangle = \langle 1, 0\rangle$. It gives $\frac{1}{n}\sum_{x\in X} r_{AB}(x) = 1$ and $\frac{1}{n}\sum_{x\in X} s_{AB}(x) = 0$, i.e, $\sum_{x\in X} r_{AB}(x) = n$ and $\sum_{x\in X} s_{AB}(x) = 0$. Then, $r_{AB}(x) = 1$ and $s_{AB}(x) = 0$ for any $x \in X$. $I_I(\langle \mu_A(x), \nu_A(x)\rangle, \langle \mu_B(x), \nu_B(x)\rangle) = \langle r_{AB}(x), s_{AB}(x)\rangle$ implies $I_I(\langle \mu_A(x), \nu_A(x)\rangle, \langle \mu_B(x), \nu_B(x)\rangle) = \langle 1, 0\rangle$ for any $x \in X$. Since I_I satisfies Axiom 4, we have $I_I(\langle \mu_A(x), \nu_A(x)\rangle, \langle \mu_B(x), \nu_B(x)\rangle) = \langle 1, 0\rangle$ if and only if $\langle \mu_A(x), \nu_A(x)\rangle \le \langle \mu_B(x), \nu_B(x)\rangle$ for any $x \in X$. It implies, $A \subseteq B$.
Conversely, when $A \subseteq B$, then $\langle \mu_A(x), \nu_A(x)\rangle \le \langle \mu_B(x), \nu_B(x)\rangle$ for any $x \in X$. Since I_I satisfies Axiom 4, thus $I_I(\langle \mu_A(x), \nu_A(x)\rangle, \langle \mu_B(x), \nu_B(x)\rangle) = \langle 1, 0\rangle$. So, we have $I_s(A,B) = \langle 1, 0\rangle$. Therefore, $I_D(A,B) = \frac{|1-1|+|0-0|}{2} = 0$. Hence, $I_{inc}(A,B) = 1 - I_D(A,B) = 1 - 0 = 1$.

(iii) Given, $A \subseteq B \subseteq C$. Then, using condition (ii) of Theorem 8, we get $I_D(C,A) \ge \max\{I_D(C,B), I_D(B,A)\}$. Hence, $I_{inc}(C,A) \le I_{inc}(B,A)$ and $I_{inc}(C,A) \le I_{inc}(C,B)$. Therefore, $I_{inc}(C,A) \le \min\{I_{inc}(B,A), I_{inc}(C,B)\}$. □

Theorem 11. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and $A, B, C \in \mathrm{IFSs}(X)$. If I_I satisfies Axiom 3, then $S(A,B) = 1 - \max\{I_D(A,B), I_D(B,A)\}$ satisfies the following conditions:*

(i) $0 \leq S(A,B) \leq 1$,

(ii) $S(A,B) = 1$ *if* $A = B$,

(iii) $S(A,B) = S(B,A)$,

(iv) if $A \subseteq B \subseteq C$, *then* $S(A,C) \leq \min\{S(A,B), S(B,C)\}$.

Proof. (i) We have $I_D(A,B) \in [0,1]$ and $I_D(B,A) \in [0,1]$. Thus, $S(A,B) = 1 - \max\{I_D(A,B), I_D(B,A)\} \in [0,1]$. Hence, $S(A,B) \in [0,1]$.

(ii) Since I_I satisfies Axiom 3, then using Theorem 2, we get $I_D(A,A) = 0$. It implies $S(A,A) = 1 - \max\{I_D(A,A), I_D(A,A)\}$, i.e, $S(A,A) = 1 - 0 = 1$.

(iii) We have $S(A,B) = 1 - \max\{I_D(A,B), I_D(B,A)\} = 1 - \max\{I_D(B,A), I_D(A,B)\} = S(B,A)$.

(iv) Let $A \subseteq B \subseteq C$. Since $A \subseteq C$, it follows from condition (ii) of Theorem 1 that $I_D(A,C) \leq I_D(A,A)$. Again, by condition (i) of Theorem 1, we get $I_D(A,A) \leq I_D(C,A)$. Thus, we have $\max\{I_D(A,C), I_D(C,A)\} = I_D(C,A)$. It gives $S(A,C) = 1 - I_D(C,A)$. Similarly, for $A \subseteq B$ and $B \subseteq C$, we have $\max\{I_D(A,B), I_D(B,A)\} = I_D(B,A)$ and $\max\{I_D(B,C), I_D(C,B)\} = I_D(C,B)$, giving $S(A,B) = 1 - I_D(B,A)$ and $S(B,C) = 1 - I_D(C,B)$, respectively. Moreover, since $A \subseteq B \subseteq C$, condition (ii) of Theorem 8 gives $I_D(C,A) \geq \max\{I_D(C,B), I_D(B,A)\}$. Hence, it follows that $S(A,C) = 1 - I_D(C,A) \leq 1 - I_D(C,B) = S(B,C)$, and $S(A,C) \leq 1 - I_D(B,A) = S(A,B)$. Therefore, we conclude that $S(A,C) \leq \min\{S(A,B), S(B,C)\}$. □

4 Two new intuitionistic fuzzy entropy measures

After examining the existing studies on intuitionistic fuzzy entropy [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**], we propose two intuitionistic fuzzy entropy measures that reflect distinct perspectives on uncertainty. The α-intuitionistic fuzzy entropy measure is constructed based on a previously established set of axioms [**?**] and primarily emphasizes the roles of membership and non-membership degrees in quantifying uncertainty associated with an intuitionistic fuzzy set. In contrast, the β-intuitionistic fuzzy entropy measure is developed using a newly introduced set of axioms and extends this viewpoint by explicitly incorporating the hesitation degree. By simultaneously accounting for membership, non-membership, and hesitation, the proposed β-intuitionistic fuzzy entropy provides a more comprehensive characterization of uncertainty. Each measure possesses its own significance, being conceptually robust and better suited to capturing uncertainty in real-world situations.

Definition 4.1. [**?**] *An entropy on IFSs(X) is a mapping E : IFSs$(X) \to [0,1]$ that satisfies the following axioms:*

(1) $E(A) = 0$ *if and only if* A *is a crisp set,*

(2) $E(A) = 1$ *if and only if* $\mu_A(x) = \nu_A(x)$ *for all* $x \in X$*,*

(3) $E(A) \leq E(B)$ *if* A *is less fuzzy than* B*, i.e.,* $\mu_A(x) \leq \mu_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$ *for* $\mu_B(x) \leq \nu_B(x)$*, or* $\mu_A(x) \geq \mu_B(x)$ *and* $\nu_A(x) \leq \nu_B(x)$ *for* $\mu_B(x) \geq \nu_B(x)$*,*

(4) $E(A) = E(A^c)$*, where* A^c *is the intuitionistic fuzzy complement of* A*.*

Throughout the rest of this paper, the symbols α and β are used solely as notation. Specifically, they appear as superscripts in Definitions 4.2 and 4.5, and as subscripts in Definitions 4.3 and 4.6, without any intrinsic mathematical significance.

Definition 4.2. *Let* X *be a finite set and* $A \in$ *IFSs*(X)*. We define the* α*-local entropy measure of an element* $x \in X$*, which is given by the mapping* $e_A^\alpha : X \to [0,1]$ *and defined as follows:*

$$e_A^\alpha(x) = \begin{cases} |1 - \mu_A(x)| + |0 - \nu_A(x)|, & \text{if } \mu_A(x) > \nu_A(x), \\ |0 - \mu_A(x)| + |1 - \nu_A(x)|, & \text{if } \mu_A(x) \leq \nu_A(x). \end{cases}$$

Here, $\mu_A(x)$ *and* $\nu_A(x)$ *represent the degree of membership and degree of non-membership of* x *in the intuitionistic fuzzy set* A*, respectively.*

Definition 4.3. *Let* X *be a finite set with* $|X| = n$*. The* α*-intuitionistic fuzzy entropy is a mapping* $E_\alpha : \mathrm{IFSs}(X) \to [0,1]$ *defined by* $E_\alpha(A) = \frac{1}{n}\sum_{x\in X} e_A^\alpha(x)$*.*

In Figure 1, the proposed α-intuitionistic fuzzy entropy measure focuses on the points $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$, which describe the two extreme situations of an element completely belonging to a set and not belonging to it at all. Every element of an intuitionistic fuzzy set lies within the triangular region shown in Figure 1, which is divided into two subregions, namely region I and region II.

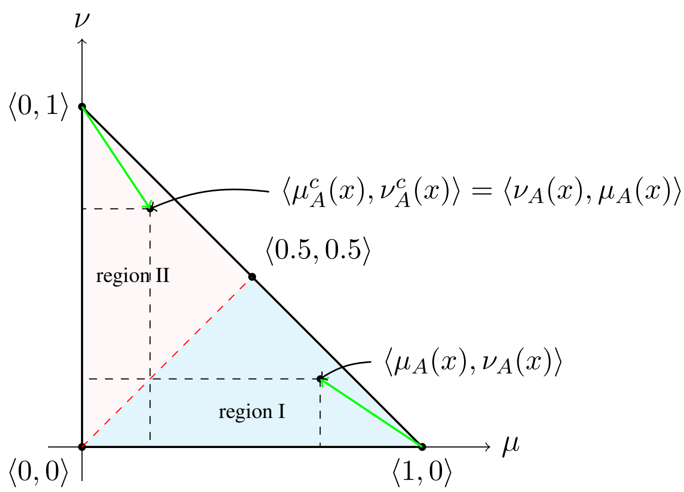

Figure 1. Geometric representation of the α-intuitionistic fuzzy entropy E_α.

The main idea behind this measure is that the entropy of an element depends on where it is located in this triangle. If the element lies in region I, its entropy is determined by how far it is from the point $\langle 1, 0\rangle$; if it lies in region II, the entropy is determined by how far it is from the point $\langle 0, 1\rangle$.

To characterize entropy in the framework of intuitionistic fuzzy sets, we propose the following four axioms that an entropy measure should satisfy:

Definition 4.4. *A Γ-entropy on IFSs(X) is a mapping E' : IFSs$(X) \to [0, 1]$ that satisfies the following axioms:*

(1) $E'(A) = 0$ *if and only if A is a crisp set, i.e., $\mu_A(x) = 1$ and $\nu_A(x) = 0$ or $\mu_A(x) = 0$ and $\nu_A(x) = 1$ for any $x \in X$,*

(2) $E'(A) = 1$ *if and only if $\mu_A(x) = \nu_A(x) = 0$ or $\mu_A(x) = \nu_A(x) = 0.5$ for any $x \in X$,*

(3) $E'(A) \leq E'(B)$ *if $\mu_A(x) + \nu_A(x) < 0.5$, $\mu_B(x) + \nu_B(x) < 0.5$ and $\pi_A(x) \leq \pi_B(x)$ or $\mu_A(x) + \nu_A(x) \geq 0.5$, $\mu_B(x) + \nu_B(x) \geq 0.5$ and $|0.5 - \mu_A(x)| + |0.5 - \nu_A(x)| \geq |0.5 - \mu_B(x)| + |0.5 - \nu_B(x)|$ for any $x \in X$,*

(4) $E'(A) = E'(A^c)$.

The significance of the above axioms can be summarized as follows.

(i) The axiom (1) states that entropy attains its minimum value only for crisp sets, where every element is either completely included or completely excluded from the set. Since such sets contain no uncertainity, they represent complete certainty and therefore have zero entropy.

(ii) The axiom (2) states that the entropy attains its maximum value only when $\pi_A(x) = 1$ for all $x \in X$, or equivalently, when $\mu_A(x) = \nu_A(x) = 0$ for all $x \in X$. This is consistent with the intuitive notion of uncertainty in intuitionistic fuzzy sets. When $\pi_A(x) = 1$, both the degrees of membership and non-membership are zero, indicating complete hesitation regarding the status of x. Similarly, when $\mu_A(x) = \nu_A(x) = 0.5$ for all $x \in X$, then $\pi_A(x) = 0$ for all $x \in X$. In that case, there is no preference toward either membership or non-membership, resulting in maximum ambiguity. Thus, both situations represent the highest possible uncertainty and consequently the maximum entropy.

(iii) Unlike traditional entropy axioms, axiom (3) distinguishes between two different origins of uncertainty in intuitionistic fuzzy sets. When $\mu(x) + \nu(x) < 0.5$ for an element x of any intuitionistic fuzzy set, the degree of hesitation is dominant. It indicates that the available information is insufficient. In this situation, uncertainty is mainly attributed to ignorance, and entropy increases with the degree of hesitation $\pi(x)$. On the other hand, when $\mu(x) + \nu(x) \geq 0.5$ for an element x of any intuitionistic fuzzy set, sufficient information about the element is available and hesitation is relatively small. In this case, entropy increases as $\langle \mu(x), \nu(x)\rangle$ approaches $\langle 0.5, 0.5\rangle$, since $\langle 0.5, 0.5\rangle$ represents a state of maximal ambiguity arising due to degrees of membership and non-membership despite the absence of hesitation. Therefore, the proposed axiom enables the entropy measure to provide a more realistic characterization of uncertainty by distinguishing between uncertainty due to hesitancy and uncertainty due to ambiguity.

(iv) The axiom (4) expresses the symmetry of uncertainty with respect to membership and non-membership values. Since the complement of an intuitionistic fuzzy set only reverses the roles of these two values without changing the amount of uncertainty present, the entropy remains unchanged under the complement.

Definition 4.5. *Let X be a finite set and $A \in$ IFSs(X). We define the β-local entropy measure of an element $x \in X$, which is given by the mapping $e_A^\beta : X \to [0,1]$ and defined as follows:*

$$e_A^\beta(x) = \begin{cases} |1-\mu_A(x)| + |0-\nu_A(x)|, & \text{if } \mu_A(x) > 0.5, \\ |0-\mu_A(x)| + |1-\nu_A(x)|, & \text{if } \nu_A(x) > 0.5, \\ 1-(|0-\mu_A(x)| + |0-\nu_A(x)|), & \text{if } \mu_A(x)+\nu_A(x) < 0.5, \\ 1-(|0.5-\mu_A(x)| + |0.5-\nu_A(x)|), & \text{if } \mu_A(x) \le 0.5,\ \nu_A(x) \le 0.5, \\ & \quad \mu_A(x)+\nu_A(x) \ge 0.5, \end{cases}$$

where $\mu_A(x)$ and $\nu_A(x)$ represent the degree of membership and degree of non-membership of x in the intuitionistic fuzzy set A, respectively.

Definition 4.6. *Let X be a finite set with $|X| = n$. The β-intuitionistic fuzzy entropy is a mapping $E_\beta : \mathrm{IFSs}(X) \to [0,1]$ defined by $E_\beta(A) = \frac{1}{n}\sum_{x\in X} e_A^\beta(x)$.*

In Figure 2, the proposed β-intuitionistic fuzzy entropy identifies the points $\langle 0.5, 0.5\rangle$ and $\langle 0, 0\rangle$ as representing the state of maximum entropy. The point $\langle 0.5, 0.5\rangle$ represents a situation in which the degrees of membership and non-membership are equal, leading to a high level of uncertainty due to this balance, even though the degree of hesitation is zero. On the other hand, at $\langle 0, 0\rangle$, the degree of hesitation reaches its maximum value 1, which also contributes to complete uncertainty of the intuitionistic fuzzy set. Thus, both these points represent states of maximal entropy.Whereas, the points $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ correspond to the extreme cases of an element fully belonging to a set and not belonging to it at all, respectively, and therefore yield minimal entropy. For this reason, the four points $\langle 0.5, 0.5\rangle$, $\langle 0, 0\rangle$, $\langle 1, 0\rangle$, and $\langle 0, 1\rangle$ are regarded as the extremal points in our construction.

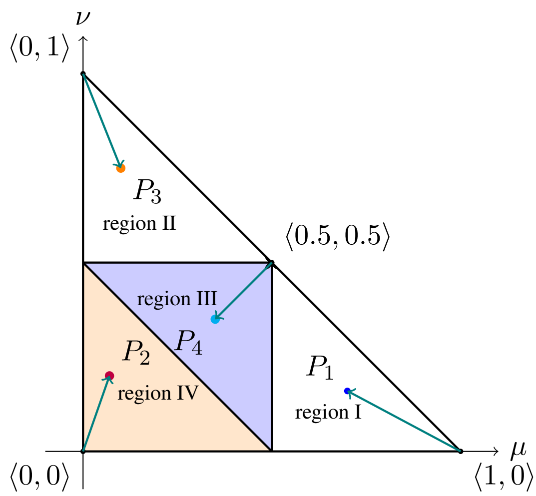

Figure 2. Geometric representation of the β-intuitionistic fuzzy entropy E_β.

Since every element of an intuitionistic fuzzy set lies within the triangular region shown in Figure 2, this region is further partitioned into four congruent triangular subregions, each associated with a distinct extremal point. In particular, regions I and II correspond to the extremal points $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$, whereas regions III and IV are associated with $\langle 0.5, 0.5\rangle$ and $\langle 0, 0\rangle$, respectively. When an intuitionistic fuzzy set is considered, each element is first located within this triangular domain and assigned to the appropriate subregion. The element is then compared with the extremal point linked to that subregion, and the distance between them is evaluated. In particular, the Hamming distance is employed for this purpose. The β-local entropy of an element is defined as its distance from the nearest extremal point; moreover, for elements located near the points of maximal entropy, the corresponding Hamming distance is subtracted from 1 to ensure that the entropy behaves consistently across the entire region. Accordingly, we consider four representative points P_1, P_2, P_3, and P_4, each lying in one of the four subregions, to illustrate the proposed β-local entropy within the respective regions.

Theorem 12. *The α-intuitionistic fuzzy entropy E_α satisfies all the axioms of entropy.*

Proof. (1) Suppose A is a crisp set. Then, for each $x \in X$, either $\mu_A(x) = 1$ and $\nu_A(x) = 0$, or $\mu_A(x) = 0$ and $\nu_A(x) = 1$. In both the cases, using Definition 4.2, we get $e^\alpha_A(x) = 0$ $\forall x \in X$. Hence, $E_\alpha(A) = 0$.

Conversely, if $E_\alpha(A) = 0$, then $\frac{1}{n}\sum_{x\in X} e^\alpha_A(x) = 0$. It implies $e^\alpha_A(x) = 0\ \forall x \in X$.

Case 1: If $\mu_A(x) > \nu_A(x)$, then we have $e^\alpha_A(x) = |1-\mu_A(x)| + |0-\nu_A(x)| = 0$. It implies $1 - \mu_A(x) = 0$ and $\nu_A(x) = 0$. Here, we get $\mu_A(x) = 1$ and $\nu_A(x) = 0$ for any $x \in X$.

Case 2: If $\mu_A(x) \leq \nu_A(x)$, then we have $e^\alpha_A(x) = |0 - \mu_A(x)| + |1 - \nu_A(x)| = 0$. So, $\nu_A(x) = 1$ and $\mu_A(x) = 0$ for any $x \in X$. Therefore, A must be a crisp set in both the cases.

(2) Suppose $E_\alpha(A) = 1$. Then, $\frac{1}{n}\sum_{x\in X} e^\alpha_A(x) = 1$ implies $\sum_{x\in X} e^\alpha_A(x) = n$, i.e., $e^\alpha_A(x) = 1$ $\forall x \in X$.

Case 1: If $\mu_A(x) > \nu_A(x)$, then $e^\alpha_A(x) = |1 - \mu_A(x)| + |0 - \nu_A(x)| = 1$. It gives $\nu_A(x) = \mu_A(x)$, a contradiction.

Case 2: If $\mu_A(x) \leq \nu_A(x)$, then $e^\alpha_A(x) = |0 - \mu_A(x)| + |1 - \nu_A(x)| = 1$. It gives $\mu_A(x) = \nu_A(x)$. Therefore, $E_\alpha(A) = 1$ if $\mu_A(x) = \nu_A(x)\ \forall x \in X$.

Conversely, when $(\forall x \in X)\ (\mu_A(x) = \nu_A(x))$, then $e^\alpha_A(x) = |0-\mu_A(x)|+|1-\nu_A(x)| = 1$. Therefore, using Definition 4.3, we get $E_\alpha(A) = 1$.

(3) We consider the following cases to prove that $E_\alpha(A) \leq E_\alpha(B)$.
Case 1: If $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for $\mu_B(x) \leq \nu_B(x)$, it follows that $\mu_A(x) \leq \nu_A(x)$ and $\mu_B(x) \leq \nu_B(x)$. So, we have $e^\alpha_A(x) = |0-\mu_A(x)|+|1-\nu_A(x)| = \mu_A(x)+1-\nu_A(x)$ and $e^\alpha_B(x) = |0-\mu_B(x)|+|1-\nu_B(x)| = \mu_B(x)+1-\nu_B(x)$. Since $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, so we get $1-\nu_A(x) \leq 1-\nu_B(x)$. Thus, we obtain $\mu_A(x)+1-\nu_A(x) \leq \mu_B(x) + 1 - \nu_B(x)$, i.e., $e^\alpha_A(x) \leq e^\alpha_B(x)$ for any $x \in X$. Hence, $E_\alpha(A) \leq E_\alpha(B)$.

Case 2: If $\mu_A(x) \geq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$ for $\mu_B(x) \geq \nu_B(x)$, it follows that $\mu_A(x) \geq \nu_A(x)$ and $\mu_B(x) \geq \nu_B(x)$. This case can be divided into four subcases: $\mu_A(x) > \nu_A(x)$ and $\mu_B(x) > \nu_B(x)$; $\mu_A(x) > \nu_A(x)$ and $\mu_B(x) = \nu_B(x)$; $\mu_A(x) = \nu_A(x)$ and $\mu_B(x) > \nu_B(x)$; and $\mu_A(x) = \nu_A(x)$ and $\mu_B(x) = \nu_B(x)$.

- **(i) Subcase 1:** If $\mu_A(x) > \nu_A(x)$ and $\mu_B(x) > \nu_B(x)$, then $e^{\alpha}_A(x) = |1 - \mu_A(x)| + |0 - \nu_A(x)| = 1 - \mu_A(x) + \nu_A(x)$ and $e^{\alpha}_B(x) = |1 - \mu_B(x)| + |0 - \nu_B(x)| = 1 - \mu_B(x) + \nu_B(x)$. Since $\mu_A(x) \geq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$, it implies $1 - \mu_A(x) \leq 1 - \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$. So, we have $1 - \mu_A(x) + \nu_A(x) \leq 1 - \mu_B(x) + \nu_B(x)$, i.e., $e^{\alpha}_A(x) \leq e^{\alpha}_B(x)$ for any $x \in X$. Hence, $E_\alpha(A) \leq E_\alpha(B)$.
- **(ii) Subcase 2:** If $\mu_A(x) > \nu_A(x)$ and $\mu_B(x) = \nu_B(x)$, then $e^{\alpha}_A(x) = |1 - \mu_A(x)| + |0 - \nu_A(x)| = 1 - \mu_A(x) + \nu_A(x)$ and $e^{\alpha}_B(x) = |0 - \mu_B(x)| + |1 - \nu_B(x)| = 1 - \nu_B(x) + \nu_B(x) = 1$. Since $e^{\alpha}_A(x) \in [0, 1]$, it gives $e^{\alpha}_A(x) \leq e^{\alpha}_B(x)$. Hence, $E_\alpha(A) \leq E_\alpha(B)$.
- **(iii) Subcase 3:** If $\mu_A(x) = \nu_A(x)$ and $\mu_B(x) > \nu_B(x)$, together with $\mu_A(x) \geq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$, we obtain $\mu_A(x) \geq \mu_B(x) > \nu_B(x) \geq \nu_A(x)$. But, this is a contradiction. Hence, the given conditions cannot hold simultaneously.
- **(iv) Subcase 4:** If $\mu_A(x) = \nu_A(x)$ and $\mu_B(x) = \nu_B(x)$, then we obtain $e^{\alpha}_A(x) = 1$ and $e^{\alpha}_B(x) = 1$. It implies $E_\alpha(A) = E_\alpha(B)$.

(4) By condition (v) of Definition 2.2, we have $\mu_{A^c}(x) = \nu_A(x)$ and $\nu_{A^c}(x) = \mu_A(x)$ for any $x \in X$. Thus, we have the following two cases:

Case 1: If $\mu_A(x) > \nu_A(x)$, then $e^{\alpha}_A(x) = |1 - \mu_A(x)| + |0 - \nu_A(x)|$. Thus, for A^c, we have $\mu_{A^c}(x) = \nu_A(x) < \mu_A(x) = \nu_{A^c}(x)$. It implies $\mu_{A^c}(x) < \nu_{A^c}(x)$. In this case, $e^{\alpha}_{A^c}(x) = |0 - \mu_{A^c}(x)| + |1 - \nu_{A^c}(x)| = |0 - \nu_A(x)| + |1 - \mu_A(x)|$. Therefore, $e^{\alpha}_A(x) = e^{\alpha}_{A^c}(x)$.

Case 2: If $\mu_A(x) \leq \nu_A(x)$, then $e^{\alpha}_A(x) = |0 - \mu_A(x)| + |1 - \nu_A(x)|$. Now, $\mu_{A^c}(x) = \nu_A(x) \geq \mu_A(x) = \nu_{A^c}(x)$, it gives $\mu_{A^c}(x) \geq \nu_{A^c}(x)$.

- **(i) Subcase 1:** When $\mu_{A^c}(x) > \nu_{A^c}(x)$, we have $e^{\alpha}_{A^c}(x) = |1 - \mu_{A^c}(x)| + |0 - \nu_{A^c}(x)| = |1 - \nu_A(x)| + |0 - \mu_A(x)|$. Therefore, $e^{\alpha}_A(x) = e^{\alpha}_{A^c}(x)$ for any $x \in X$.
- **(ii) Subcase 2:** When $\mu_{A^c}(x) = \nu_{A^c}(x)$, it gives $e^{\alpha}_{A^c}(x) = 1$. Also, $\mu_A(x) = \nu_{A^c}(x) = \mu_{A^c}(x) = \nu_A(x)$ gives $e^{\alpha}_A(x) = 1$. Hence, $e^{\alpha}_A(x) = e^{\alpha}_{A^c}(x)$ for any $x \in X$. So, $E_\alpha(A) = E_\alpha(A^c)$. □

Theorem 13. *The β-intuitionistic fuzzy entropy E_β satisfies all the axioms of Γ-entropy.*

Proof. (1) Let us assume that A is a crisp set. Then, for each $x \in X$, either $\mu_A(x) = 1$ and $\nu_A(x) = 0$, or $\mu_A(x) = 0$ and $\nu_A(x) = 1$.

Case 1: When $\mu_A(x) > 0.5$, we have $e^{\beta}_A(x) = |1 - \mu_A(x)| + |0 - \nu_A(x)| = |1 - 1| + |0 - 0| = 0$.

Case 2: When $\nu_A(x) > 0.5$, we have $e^{\beta}_A(x) = |0 - \mu_A(x)| + |1 - \nu_A(x)| = |0 - 0| + |1 - 1| = 0$. Therefore, in both the cases, $(\forall x \in X)$ $(e^{\beta}_A(x) = 0)$. Hence, $E_\beta(A) = \frac{1}{n} \sum_{x \in X} e^{\beta}_A(x) = 0$.

Conversely, suppose $E_\beta(A) = 0$. By Definition 4.5, $e_A^\beta(x) \geq 0$ for any $x \in X$. Thus, it follows that $e_A^\beta(x) = 0$ for any $x \in X$. Now, we consider the following four cases under which $e_A^\beta(x) = 0$:

Case 1: If $\mu_A(x) > 0.5$, then $e_A^\beta(x) = |1 - \mu_A(x)| + |0 - \nu_A(x)| = 0$ gives $\mu_A(x) = 1$ and $\nu_A(x) = 0$.

Case 2: If $\nu_A(x) > 0.5$, then $e_A^\beta(x) = |0 - \mu_A(x)| + |1 - \nu_A(x)| = 0$ gives $\nu_A(x) = 1$ and $\mu_A(x) = 0$.

Case 3: If $\mu_A(x) + \nu_A(x) < 0.5$, then $e_A^\beta(x) = 1 - (|0 - \mu_A(x)| + |0 - \nu_A(x)|) = 0$ gives $\mu_A(x) + \nu_A(x) = 1$, which is a contradiction.

Case 4: If $\mu_A(x) \leq 0.5$, $\nu_A(x) \leq 0.5$, and $\mu_A(x) + \nu_A(x) \geq 0.5$, then $e_A^\beta(x) = 1 - (|0.5 - \mu_A(x)| + |0.5 - \nu_A(x)|) = 0$ gives $(0.5 - \mu_A(x) + 0.5 - \nu_A(x)) = 1$. It means $\mu_A(x) + \nu_A(x) = 0$, which is a contradiction.

Therefore, in all the cases where $e_A^\beta(x) = 0$, we must have either $\mu_A(x) = 1$, $\nu_A(x) = 0$, or $\mu_A(x) = 0$, $\nu_A(x) = 1$. Therefore, A is a crisp set. Thus, we get A is a crisp set if and only if $E_\beta(A) = 0$.

(2) Suppose $E_\beta(A) = 1$. Then, $E_\beta(A) = \frac{1}{n}\sum_{x\in X} e_A^\beta(x) = 1$ gives $\sum_{x\in X} e_A^\beta(x) = n$. It implies $(\forall x \in X)$ $(e_A^\beta(x) = 1)$. Now, we have the following four cases:

Case 1: If $\mu_A(x) > 0.5$, then $e_A^\beta(x) = |1-\mu_A(x)|+|0-\nu_A(x)| = 1$ implies $\mu_A(x) = \nu_A(x)$, which is a contradiction.

Case 2: If $\nu_A(x) > 0.5$, then $e_A^\beta(x) = |0-\mu_A(x)|+|1-\nu_A(x)| = 1$ implies $\mu_A(x) = \nu_A(x)$, which is a contradiction.

Case 3: If $\mu_A(x) + \nu_A(x) < 0.5$, then $e_A^\beta(x) = 1 - (|0 - \mu_A(x)| + |0 - \nu_A(x)|) = 1$ implies $\mu_A(x) + \nu_A(x) = 0$, i.e., $\mu_A(x) = \nu_A(x) = 0$.

Case 4: If $\mu_A(x) \leq 0.5$, $\nu_A(x) \leq 0.5$ and $\mu_A(x) + \nu_A(x) \geq 0.5$, then $e_A^\beta(x) = 1 - (|0.5 - \mu_A(x)| + |0.5 - \nu_A(x)|) = 1$ implies $\mu_A(x) + \nu_A(x) = 1$. It gives $\mu_A(x) = \nu_A(x) = 0.5$.

Hence, $e_A^\beta(x) = 1$ when either $\mu_A(x) = \nu_A(x) = 0$ or $\mu_A(x) = \nu_A(x) = 0.5$. Thus, if $E_\beta(A) = 1$, the pair $\langle \mu_A(x), \nu_A(x)\rangle$ must be either $\langle 0, 0\rangle$ or $\langle 0.5, 0.5\rangle$ for any $x \in X$.

Conversely, let $\langle \mu_A(x), \nu_A(x)\rangle = \langle 0, 0\rangle$ or $\langle 0.5, 0.5\rangle$ for any $x \in X$. In both cases, we get $e_A^\beta(x) = 1$. Thus, using Definition 4.6, we get $E_\beta(A) = \frac{1}{n}\sum_{x\in X} e_A^\beta(x) = 1$.

(3) We consider the following cases to prove that $E_\beta(A) \leq E_\beta(B)$.

Case 1: If $\mu_A(x)+\nu_A(x) < 0.5$, $\mu_B(x)+\nu_B(x) < 0.5$, and $\pi_A(x) \leq \pi_B(x)$ for any $x \in X$, i.e., $1-\mu_A(x)-\nu_A(x) \leq 1-\mu_B(x)-\nu_B(x)$, then we get $1-(|0-\mu_A(x)|+|0-\nu_A(x)|) \leq 1-(|0-\mu_B(x)|+|0-\nu_B(x)|)$. By Definition 4.5, we obtain $e_A^\beta(x) \leq e_B^\beta(x)$ for any $x \in X$. Therefore, $E_\beta(A) \leq E_\beta(B)$.

Case 2: If $\mu_A(x)+\nu_A(x) \geq 0.5$, $\mu_B(x)+\nu_B(x) \geq 0.5$, and $|0.5-\mu_A(x)|+|0.5-\nu_A(x)| \geq |0.5 - \mu_B(x)| + |0.5 - \nu_B(x)|$. Then, we have the following three cases:

(a) Suppose $\mu_A(x) > 0.5$. Then, the following three subcases may arise:

(iii) If $\mu_A(x) > 0.5$ and $\mu_B(x) > 0.5$, then $e_A^\beta(x) = |1-\mu_A(x)| + |0-\nu_A(x)| = 1-\mu_A(x)+\nu_A(x)$, and $e_B^\beta(x) = |1-\mu_B(x)|+|0-\nu_B(x)| = 1-\mu_B(x)+\nu_B(x)$. The given condition $|0.5-\mu_A(x)|+|0.5-\nu_A(x)| \geq |0.5-\mu_B(x)|+|0.5-\nu_B(x)|$ implies $\mu_A(x)-\nu_A(x) \geq \mu_B(x)-\nu_B(x)$, i.e., $1-\mu_A(x)+\nu_A(x) \leq 1-\mu_B(x)+\nu_B(x)$. Hence $e_A^\beta(x) \leq e_B^\beta(x)$ for any $x \in X$. Thus, $E_\beta(A) \leq E_\beta(B)$.

(ii) If $\mu_A(x) > 0.5$ and $\nu_B(x) > 0.5$, then $e_A^\beta(x) = 1-\mu_A(x)+\nu_A(x)$ and $e_B^\beta(x) = 1-\nu_B(x)+\mu_B(x)$. The given condition $|0.5-\mu_A(x)|+|0.5-\nu_A(x)| \geq |0.5-\mu_B(x)|+|0.5-\nu_B(x)|$ implies $\mu_A(x)-\nu_A(x) \geq \nu_B(x)-\mu_B(x)$. It implies $1-\mu_A(x)+\nu_A(x) \leq 1-\nu_B(x)+\mu_B(x)$. Hence $e_A^\beta(x) \leq e_B^\beta(x)$ for any $x \in X$. So, $E_\beta(A) \leq E_\beta(B)$.

(iii) If $\mu_A(x) > 0.5$ and $\mu_B(x) \leq 0.5, \nu_B(x) \leq 0.5$ with $\mu_B(x)+\nu_B(x) \geq 0.5$, then $e_A^\beta(x) = 1-\mu_A(x)+\nu_A(x)$ and $e_B^\beta(x) = 1-|0.5-\mu_B(x)|-|0.5-\nu_B(x)| = \mu_B(x)+\nu_B(x)$. The given condition $|0.5-\mu_A(x)|+|0.5-\nu_A(x)| \geq |0.5-\mu_B(x)|+|0.5-\nu_B(x)|$ implies $\mu_A(x)-\nu_A(x) \geq 1-(\mu_B(x)+\nu_B(x))$. It gives $1-\mu_A(x)+\nu_A(x) \leq \mu_B(x)+\nu_B(x)$. Hence, $e_A^\beta(x) \leq e_B^\beta(x)$ for any $x \in X$. Thus, $E_\beta(A) \leq E_\beta(B)$.

(b) Suppose $\nu_A(x) > 0.5$. Then, the following three subcases may arise:

(i) If $\nu_A(x) > 0.5$ and $\mu_B(x) > 0.5$, then $e_A^\beta(x) = 1-\nu_A(x)+\mu_A(x)$ and $e_B^\beta(x) = 1-\mu_B(x)+\nu_B(x)$. The given condition $|0.5-\mu_A(x)|+|0.5-\nu_A(x)| \geq |0.5-\mu_B(x)|+|0.5-\nu_B(x)|$ implies $\nu_A(x)-\mu_A(x) \geq \mu_B(x)-\nu_B(x)$, i.e., $1-\nu_A(x)+\mu_A(x) \leq 1-\mu_B(x)+\nu_B(x)$. Therefore, $e_A^\beta(x) \leq e_B^\beta(x)$ for any $x \in X$. Hence, $E_\beta(A) \leq E_\beta(B)$.

(ii) If $\nu_A(x) > 0.5$ and $\nu_B(x) > 0.5$, then $e_A^\beta(x) = 1-\nu_A(x)+\mu_A(x)$ and $e_B^\beta(x) = 1-\nu_B(x)+\mu_B(x)$. The given condition $|0.5-\mu_A(x)|+|0.5-\nu_A(x)| \geq |0.5-\mu_B(x)|+|0.5-\nu_B(x)|$ implies $\nu_A(x)-\mu_A(x) \geq \nu_B(x)-\mu_B(x)$, i.e., $1-\nu_A(x)+\mu_A(x) \leq 1-\nu_B(x)+\mu_B(x)$. Hence, $e_A^\beta(x) \leq e_B^\beta(x)$ for any $x \in X$. Thus, $E_\beta(A) \leq E_\beta(B)$.

(iii) If $\nu_A(x) > 0.5$ and $\mu_B(x), \nu_B(x) \leq 0.5$ with $\mu_B(x)+\nu_B(x) \geq 0.5$, then $e_A^\beta(x) = 1-\nu_A(x)+\mu_A(x)$ and $e_B^\beta(x) = \mu_B(x)+\nu_B(x)$. The given condition $|0.5-\mu_A(x)|+|0.5-\nu_A(x)| \geq |0.5-\mu_B(x)|+|0.5-\nu_B(x)|$ implies $\nu_A(x)-\mu_A(x) \geq 1-(\mu_B(x)+\nu_B(x))$, i.e., $1-\nu_A(x)+\mu_A(x) \leq \mu_B(x)+\nu_B(x)$. Therefore, $e_A^\beta(x) \leq e_B^\beta(x)$ for any $x \in X$. Hence, $E_\beta(A) \leq E_\beta(B)$.

(c) Suppose $\mu_A(x) \leq 0.5$, $\nu_A(x) \leq 0.5$, and $\mu_A(x)+\nu_A(x) \geq 0.5$. Then, the following three subcases may arise:

(i) If $\mu_B(x) > 0.5$, then $e_A^\beta(x) = 1-(|0.5-\mu_A(x)|+|0.5-\nu_A(x)|) = \mu_A(x)+\nu_A(x)$ and $e_B^\beta(x) = 1-\mu_B(x)+\nu_B(x)$. The given condition $|0.5-\mu_A(x)|+|0.5-\nu_A(x)| \geq |0.5-\mu_B(x)|+|0.5-\nu_B(x)|$ implies $\mu_A(x)+\nu_A(x) \leq 1-\mu_B(x)+\nu_B(x)$. Hence, $e_A^\beta(x) \leq e_B^\beta(x)$ for any $x \in X$. Thus, $E_\beta(A) \leq E_\beta(B)$.

(ii) If $\nu_B(x) > 0.5$, then $e_A^\beta(x) = \mu_A(x)+\nu_A(x)$ and $e_B^\beta(x) = 1-\nu_B(x)+\mu_B(x)$. The given condition $|0.5-\mu_A(x)|+|0.5-\nu_A(x)| \geq |0.5-\mu_B(x)|+|0.5-\nu_B(x)|$ implies $\mu_A(x)+\nu_A(x) \leq 1-\nu_B(x)+\mu_B(x)$. Hence, $e_A^\beta(x) \leq e_B^\beta(x)$ for any $x \in X$. Thus, $E_\beta(A) \leq E_\beta(B)$.

(iii) If $\mu_B(x) \leq 0.5$, $\nu_B(x) \leq 0.5$, and $\mu_B(x) + \nu_B(x) \geq 0.5$, then $e_A^\beta(x) = 1 - (|0.5 - \mu_A(x)| + |0.5 - \nu_A(x)|)$ and $e_B^\beta(x) = 1 - (|0.5 - \mu_B(x)| + |0.5 - \nu_B(x)|)$. Since $|0.5 - \mu_A(x)| + |0.5 - \nu_A(x)| \geq |0.5 - \mu_B(x)| + |0.5 - \nu_B(x)|$, it follows that $e_A^\beta(x) \leq e_B^\beta(x)$ for any $x \in X$. Thus, $E_\beta(A) \leq E_\beta(B)$.

(4) **Case 1:** If $\mu_A(x) > 0.5$, then $e_A^\beta(x) = |1-\mu_A(x)|+|0-\nu_A(x)| = 1-\mu_A(x)+\nu_A(x)$. From (v) of Definition 2.2, we have $\mu_{A^c}(x) = \nu_A(x)$ and $\nu_{A^c}(x) = \mu_A(x)$. Then, the condition $\mu_A(x) > 0.5$ becomes $\nu_{A^c}(x) > 0.5$. Again, we have $e_{A^c}^\beta(x) = |0-\mu_{A^c}(x)|+|1-\nu_{A^c}(x)| = \mu_{A^c}(x) + 1 - \nu_{A^c}(x) = \nu_A(x) + 1 - \mu_A(x)$. Thus, $e_A^\beta(x) = e_{A^c}^\beta(x)$ for any $x \in X$.

Case 2: If $\nu_A(x) > 0.5$, then $e_A^\beta(x) = |0-\mu_A(x)|+|1-\nu_A(x)| = \mu_A(x)+1-\nu_A(x)$. Now, $\nu_A(x) > 0.5$ implies $\mu_{A^c}(x) > 0.5$. Hence, we have $e_{A^c}^\beta(x) = |1-\mu_{A^c}(x)|+|0-\nu_{A^c}(x)| = 1 - \mu_{A^c}(x) + \nu_{A^c}(x) = 1 - \nu_A(x) + \mu_A(x)$. Thus, $e_A^\beta(x) = e_{A^c}^\beta(x)$ for any $x \in X$.

Case 3: If $\mu_A(x)+\nu_A(x) < 0.5$, then $e_A^\beta(x) = 1-(|0-\mu_A(x)|+|0-\nu_A(x)|) = 1-\mu_A(x)-\nu_A(x)$. From (v) of Definition 2.2, we have $\mu_{A^c}(x) = \nu_A(x)$ and $\nu_{A^c}(x) = \mu_A(x)$. Hence, the condition $\mu_A(x) + \nu_A(x) < 0.5$ becomes $\mu_{A^c}(x) + \nu_{A^c}(x) < 0.5$. Using Definition 4.5, we obtain $e_{A^c}^\beta(x) = 1 - (|0 - \mu_{A^c}(x)| + |0 - \nu_{A^c}(x)|) = 1 - \mu_{A^c}(x) - \nu_{A^c}(x) = 1 - \nu_A(x) - \mu_A(x)$. Thus, $e_A^\beta(x) = e_{A^c}^\beta(x)$ for any $x \in X$.

Case 4: If $\mu_A(x) \leq 0.5$, $\nu_A(x) \leq 0.5$, and $\mu_A(x) + \nu_A(x) \geq 0.5$, we have $e_A^\beta(x) = 1 - (|0.5 - \mu_A(x)| + |0.5 - \nu_A(x)|) = \mu_A(x) + \nu_A(x)$. Again, using (v) of Definition 2.2, we have $\mu_{A^c}(x) = \nu_A(x)$ and $\nu_{A^c}(x) = \mu_A(x)$. The given conditions then reduces to $\mu_{A^c}(x) \leq 0.5$, $\nu_{A^c}(x) \leq 0.5$, and $\mu_{A^c}(x)+\nu_{A^c}(x) \geq 0.5$. Applying Definition 4.5, we have $e_{A^c}^\beta(x) = 1 - (|0.5 - \mu_{A^c}(x)| + |0.5 - \nu_{A^c}(x)|) = \mu_{A^c}(x) + \nu_{A^c}(x) = \nu_A(x) + \mu_A(x)$. Therefore, $(\forall x \in X)$ $(e_A^\beta(x) = e_{A^c}^\beta(x))$.

Thus, in all possible cases, $e_A^\beta(x) = e_{A^c}^\beta(x)$ for any $x \in X$. Hence, $E_\beta(A) = E_\beta(A^c)$. □

Below, we present several examples illustrating different situations, namely a highly hesitant intuitionistic fuzzy set, a nearly crisp intuitionistic fuzzy set, two intuitionistic fuzzy sets having the same degrees of membership or non-membership balance but different degree of hesitation, and two intuitionistic fuzzy sets for which E_α and E_β yield significantly different rankings. These examples demonstrate the behavior of the proposed entropy measures under various circumstances.

Example 4.1 (Highly hesitant intuitionistic fuzzy set). *Let* $X = \{x_1, x_2, x_3, x_4, x_5\}$. *We define an intuitionistic fuzzy set* $A = \{\langle x_1, 0.1, 0.1\rangle, \langle x_2, 0, 0\rangle, \langle x_3, 0.2, 0.3\rangle, \langle x_4, 0.5, 0.5\rangle, \langle x_5, 0.4, 0.4\rangle\}$. *Then, by Definition* 4.2, *we obtain the* α*-local entropy measures of* x_1, x_2, x_3, x_4 *and* x_5 *as shown below:*

$$e_A^\alpha(x_1) = |0 - 0.1| + |1 - 0.1| = 1,$$
$$e_A^\alpha(x_2) = |0 - 0| + |1 - 0| = 1,$$
$$e_A^\alpha(x_3) = |0 - 0.2| + |1 - 0.3| = 0.9,$$
$$e_A^\alpha(x_4) = |0 - 0.5| + |1 - 0.5| = 1,$$
$$e_A^\alpha(x_5) = |0 - 0.4| + |1 - 0.4| = 1.$$

Thus, the α*-intuitionistic fuzzy entropy is* $E_\alpha(A) = \frac{1}{5}(1+1+0.9+1+1) = \frac{4.9}{5} = 0.98$.

Similarly, by Definition 4.5*, we obtain the* β*-local entropy measures of* x_1, x_2, x_3, x_4 *and* x_5 *as shown below:*

$$\begin{aligned}
e^\beta_A(x_1) &= 1 - \{|0-0.1| + |0-0.1|\} = 0.8,\\
e^\beta_A(x_2) &= 1 - \{|0-0| + |0-0|\} = 1,\\
e^\beta_A(x_3) &= 1 - \{|0.5-0.2| + |0.5-0.3|\} = 0.5,\\
e^\beta_A(x_4) &= 1 - \{|0.5-0.5| + |0.5-0.5|\} = 1,\\
e^\beta_A(x_5) &= 1 - \{|0.5-0.4| + |0.5-0.4|\} = 0.8.
\end{aligned}$$

Thus, the β*-intuitionistic fuzzy entropy is* $E_\beta(A) = \frac{1}{5}(0.8+1+0.5+1+0.8) = \frac{4.1}{5} = 0.82$.

Example 4.2 (Nearly crisp intuitionistic fuzzy set)**.** *Let* $X = \{x_1, x_2, x_3, x_4, x_5\}$*. We define an intuitionistic fuzzy set* $B = \{\langle x_1, 1, 0\rangle, \langle x_2, 0, 1\rangle, \langle x_3, 0, 0.9\rangle, \langle x_4, 0.8, 0.1\rangle, \langle x_5, 0.8, 0\rangle\}$*. Then, by Definition* 4.2*, we obtain the* α*-local entropy measures of* x_1, x_2, x_3, x_4, x_5 *as shown below:* $e^\alpha_B(x_1) = |1-1| + |0-0| = 0$, $e^\alpha_B(x_2) = |0-0| + |1-1| = 0$, $e^\alpha_B(x_3) = |0-0| + |1-0.9| = 0.1$, $e^\alpha_B(x_4) = |1-0.8| + |0-0.1| = 0.3$, $e^\alpha_B(x_5) = |1-0.8| + |0-0| = 0.2$. *Thus, the* α*-intuitionistic fuzzy entropy is* $E_\alpha(B) = \frac{1}{5}(0+0+0.1+0.3+0.2) = \frac{0.6}{5} = 0.12$.

Again, by Definition 4.5*, we obtain the* β*-local entropy measures of* x_1, x_2, x_3, x_4, x_5 *as shown below:* $e^\beta_A(x_1) = |1-1|+|0-0| = 0$, $e^\beta_B(x_2) = |0-0|+|1-1| = 0$, $e^\beta_B(x_3) = |0-0|+|1-0.9| = 0.1$, $e^\beta_B(x_4) = |1-0.8| + |0-0.1| = 0.3$, $e^\beta_B(x_5) = |1-0.8| + |0-0| = 0.2$. *Thus, the* β*-intuitionistic fuzzy entropy is* $E_\beta(B) = \frac{1}{5}(0+0+0.1+0.3+0.2) = \frac{0.6}{5} = 0.12$.

Example 4.3 (Two intuitionistic fuzzy sets with the same degree of membership but different degree of hesitancy)**.** *Let* $X = \{x_1, x_2, x_3, x_4, x_5\}$ *and consider the intuitionistic fuzzy sets* $C = \{\langle x_1, 0.5, 0\rangle, \langle x_2, 0, 0.2\rangle, \langle x_3, 0.2, 0.6\rangle, \langle x_4, 0.1, 0.1\rangle, \langle x_5, 0.3, 0.2\rangle\}$ *and* $D = \{\langle x_1, 0.5, 0.2\rangle, \langle x_2, 0, 0.8\rangle, \langle x_3, 0.2, 0.3\rangle, \langle x_4, 0.1, 0\rangle, \langle x_5, 0.3, 0.5\rangle\}$*. Then, by Definition* 4.2*, we obtain the* α*-local entropy measures of* x_1, x_2, x_3, x_4 *and* x_5 *for C as shown below:* $e^\alpha_C(x_1) = |1-0.5| + |0-0| = 0.5$, $e^\alpha_C(x_2) = |0-0| + |1-0.2| = 0.8$, $e^\alpha_C(x_3) = |0-0.2|+|1-0.6| = 0.6$, $e^\alpha_C(x_4) = |0-0.1|+|1-0.1| = 1$, $e^\alpha_C(x_5) = |1-0.3|+|0-0.2| = 0.9$. *Thus, the* α*-intuitionistic fuzzy entropy of C is* $E_\alpha(C) = \frac{1}{5}(0.5+0.8+0.6+1+0.9) = \frac{3.8}{5} = 0.76$.

Similarly, we obtain the α*-local entropy measures of* x_1, x_2, x_3, x_4 *and* x_5 *for D as shown below:* $e^\alpha_D(x_1) = |1-0.5| + |0-0.2| = 0.7$, $e^\alpha_D(x_2) = |0-0| + |1-0.8| = 0.2$, $e^\alpha_D(x_3) = |0-0.2|+|1-0.3| = 0.9$, $e^\alpha_D(x_4) = |1-0.1|+|0-0| = 0.9$, $e^\alpha_D(x_5) = |0-0.3|+|1-0.5| = 0.8$. *Thus, the* α*-intuitionistic fuzzy entropy of D is* $E_\alpha(D) = \frac{1}{5}(0.7+0.2+0.9+0.9+0.8) = \frac{3.5}{5} = 0.70$. *Thus, the* α*-intuitionistic fuzzy entropy measures of C and D indicate that C is slightly more uncertain than D, since* $E_\alpha(C) = 0.76 > E_\alpha(D) = 0.70$.

Using Definition 4.5*, the corresponding* β*-local entropy values for the elements of C are computed as follows:* $e^\beta_C(x_1) = 1 - \big(|0.5-0.5| + |0.5-0|\big) = 0.5$, $e^\beta_C(x_2) = 1 - \big(|0-0| + |0-0.2|\big) = 0.8$, $e^\beta_C(x_3) = |0-0.2| + |1-0.6| = 0.6$, $e^\beta_C(x_4) = 1 - \big(|0-0.1| + |0-0.1|\big) = 0.8$, $e^\beta_C(x_5) = 1 - \big(|0-0.3| + |0-0.2|\big) = 0.5$. *Therefore, the* β*-intuitionistic fuzzy entropy of C is* $E_\beta(C) = \frac{1}{5}(0.5+0.8+0.6+0.8+0.5) = \frac{3.2}{5} = 0.64$. *Similarly, for the intuitionistic fuzzy set D, we obtain* $e^\beta_D(x_1) = 1 - \big(|0.5-0.5| + |0.5-0.2|\big) = 0.7$, $e^\beta_D(x_2) = |0-0| + |1-0.8| = 0.2$,

$e_D^\beta(x_3) = 1 - \big(|0.5-0.2| + |0.5-0.3|\big) = 0.5$, $e_D^\beta(x_4) = 1 - \big(|0-0.1| + |0-0|\big) = 0.9$, $e_D^\beta(x_5) = 1 - \big(|0.5-0.3| + |0.5-0.5|\big) = 0.8$. *Hence, the β-intuitionistic fuzzy entropy of D is* $E_\beta(D) = \frac{1}{5}(0.7+0.2+0.5+0.9+0.8) = \frac{3.1}{5} = 0.62$. *Thus, the β-intuitionistic fuzzy entropy measures of C and D indicate that C is slightly more uncertain than D, since* $E_\beta(C) = 0.64 > E_\beta(D) = 0.62$.

5 Comparison with some existing entropy measures

In this section, we compare the proposed entropy measure with several existing entropy measures for intuitionistic fuzzy sets to illustrate its effectiveness and consistency.

The entropy measure $E_{BB}(A)$, introduced by Burillo and Bustince [**?**], is:

$$E_{BB}(A) = \frac{1}{n}\sum_{i=1}^{n} \pi_A(x_i).$$

The entropy measure $E_{SK}(A)$, introduced by Szmidt and Kacprzyk [**?**], is:

$$E_{SK}(A) = \frac{1}{n}\sum_{i=1}^{n} \frac{\mathrm{maxCount}\big(A(x_i) \cap A^c(x_i)\big)}{\mathrm{maxCount}\big(A(x_i) \cup A^c(x_i)\big)},$$

where

$$\mathrm{maxCount}(A(x_i)) = \mu_A(x_i) + \pi_A(x_i),\ x_i \in X.$$

The entropy measure $E_{VS}(A)$, introduced by Vlachos and Sergiadis [**?**], is:

$$E_{VS}(A) = \frac{1}{n}\sum_{i=1}^{n} \frac{2\mu_A(x_i)\nu_A(x_i) + \pi_A^2(x_i)}{\mu_A^2(x_i) + \nu_A^2(x_i) + \pi_A^2(x_i)}.$$

The entropy measure $E_{ZL}(A)$, introduced by Zhu and Li [**?**], is:

$$E_{ZL}(A) = \frac{1}{4n}\sum_{i=1}^{n} \big(1 + \pi_A(x_i)\big)\big(1 - |\mu_A(x_i) - \nu_A(x_i)| + \pi_A(x_i)\big).$$

Following the methodologies and examples adopted in the existing literature [**?**, **?**, **?**], we also consider a single element universal set for the comparison of entropy measures.

Example 5.1. *Let* $X = \{x\}$ *and let* $A = \{\langle x, 0.5, 0.5\rangle\}$, $B = \{\langle x, 0.3, 0.3\rangle\}$, $C = \{\langle x, 0.4, 0.4\rangle\}$, $D = \{\langle x, 0, 0\rangle\}$. *Then,* $E_\beta(A) = 1$, $E_\beta(B) = 0.6$, $E_\beta(C) = 0.8$, $E_\beta(D) = 1$. *Moreover,* $E_{VS}(A) = E_{VS}(B) = E_{VS}(C) = E_{VS}(D) = 1$, *and* $E_{SK}(A) = E_{SK}(B) = E_{SK}(C) = E_{SK}(D) = 1$.

The above example shows that the proposed entropy E_β provides greater discriminative ability than existing entropy measures. Although the intuitionistic fuzzy sets A, B, C, and D represent different uncertainty situations, both E_{VS} and E_{SK} assign the same maximum entropy value to all of them, thereby failing to distinguish their uncertainty levels. In contrast, the proposed entropy distinguishes these intuitionistic fuzzy sets by assigning $E_\beta(B) < E_\beta(C) < E_\beta(A) = E_\beta(D)$, which is consistent with their respective levels of uncertainty.

Further, the Burillo–Bustince entropy measure assigns $E_{BB}(A) = 0, E_{BB}(B) = 0.4, E_{BB}(C) = 0.2, E_{BB}(D) = 1$. Although E_{BB} distinguishes these intuitionistic fuzzy sets, assign zero entropy to the highly ambiguous intuitionistic fuzzy value $\langle 0.5, 0.5\rangle$, since its entropy depends only on the hesitation degree. This contradicts the intuitive notion that $\langle 0.5, 0.5\rangle$ represents maximum ambiguity. Similarly, the Zhu–Li entropy measure yields $E_{ZL}(A) = 0.25, E_{ZL}(B) = 0.49, E_{ZL}(C) = 0.36, E_{ZL}(D) = 1$. Here, the entropy at $\langle 0.5, 0.5\rangle$ is even smaller than those of $\langle 0.3, 0.3\rangle$ and $\langle 0.4, 0.4\rangle$, indicating that it does not adequately capture the maximum uncertainty associated with the balanced intuitionistic fuzzy value. In contrast, the proposed intuitionistic fuzzy entropy measure E_β correctly assigns the maximum entropy to both $\langle 0.5, 0.5\rangle$ and $\langle 0, 0\rangle$, while preserving meaningful discrimination among the remaining intuitionistic fuzzy sets.

Example 5.2. *Let $X = \{x\}$ and $A = \{\langle x, 0.6, 0.4\rangle\}$, $B = \{\langle x, 0.6, 0.3\rangle\}$, $C = \{\langle x, 0.7, 0.3\rangle\}$, $D = \{\langle x, 0.7, 0.2\rangle\}$. Then, $E_\alpha(A) = 0.8$, $E_\alpha(B) = 0.7$, $E_\alpha(C) = 0.6, E_\alpha(D) = 0.5$. On the other hand, the existing entropy measures produces $E_{BB}(A) = 0, E_{BB}(B) = 0.1, E_{BB}(C) = 0, E_{BB}(D) = 0.1; E_{SK}(A) = 0.67, E_{SK}(B) = 0.57, E_{SK}(C) = 0.42, E_{SK}(D) = 0.375; E_{VS}(A) = 0.92, E_{VS}(B) = 0.80, E_{VS}(C) = 0.72, E_{VS}(D) = 0.53$ and $E_{ZL}(A) = 0.20, E_{ZL}(B) = 0.22, E_{ZL}(C) = 0.15, E_{ZL}(D) = 0.165$.*

This example further illustrates the ability of the proposed intuitionistic fuzzy entropy measure E_α to discriminate between different levels of uncertainity in intuitionistic fuzzy sets. Since $A = \langle x, 0.6, 0.4\rangle$ is closest to the maximally ambiguous state $\langle 0.5, 0.5\rangle$, it attains the largest entropy value. As the pair $\langle \mu, \nu\rangle$ moves progressively away from this state through B, C, and D, the corresponding entropy values decrease, yielding $E_\alpha(A) > E_\alpha(B) > E_\alpha(C) > E_\alpha(D)$. In contrast, the entropy measure E_{BB} depends mainly on hesitation and therefore assigns identical entropy values to A and C, as well as to B and D, failing to distinguish their different degrees of uncertainity. But, E_{SK} and E_{VS} successfully preserves the ordering among the four intuitionistic fuzzy sets. Moreover, E_{ZL} assigns a larger entropy value to B than to A, despite both intuitionistic fuzzy sets having the same degree of membership and A having a higher degree of non-membership. Similar observations can be made for C and D. Thus, the proposed entropy E_α provides a more consistent assessment of uncertainty.

Now, we provide a theorem below that explains the relationship between the α-intuitionistic fuzzy entropy and implication deviation I_D. They are based on Definitions 4.3 and 3.3, respectively.

Theorem 14. *Let $A, B \in \mathrm{IFSs}(X)$. Assume that the intuitionistic implication operator I_I is defined by $I_I(\langle a, b\rangle, \langle c, d\rangle) = \langle \max\{b, c\}, \min\{a, d\}\rangle$. If $B = \{\langle x, 0, 1\rangle \mid x \in X\}$, then the α-intuitionistic fuzzy entropy of A is given by*

$$E_\alpha(A) = \begin{cases} 2I_D(A^c, B), & \text{if } \mu_A(x) > \nu_A(x) \text{ for all } x \in X, \\ 2I_D(A, B), & \text{if } \mu_A(x) \leq \nu_A(x) \text{ for all } x \in X. \end{cases}$$

Proof. For any $x \in X$, we get $I_I(\langle \mu_A(x), \nu_A(x)\rangle, \langle 0, 1\rangle) = \langle \max(\nu_A(x), 0), \min(\mu_A(x), 1)\rangle = \langle \nu_A(x), \mu_A(x)\rangle$. Hence, by Definition 3.2, we get $I_s(A, B) = \langle \frac{1}{n}\sum_{x\in X} \nu_A(x), \frac{1}{n}\sum_{x\in X} \mu_A(x)\rangle$.

Using Definition 3.3, the corresponding implication deviation is

$$\begin{aligned} I_D(A,B) &= \frac{\left|1-\frac{1}{n}\sum_{x\in X}\nu_A(x)\right| + \left|0-\frac{1}{n}\sum_{x\in X}\mu_A(x)\right|}{2} \\ &= \frac{1-\frac{1}{n}\sum_{x\in X}\nu_A(x) + \frac{1}{n}\sum_{x\in X}\mu_A(x)}{2}. \end{aligned}$$

Similarly, for the intuitionistic fuzzy set A^c, we have

$$I_D(A^c,B) = \frac{1-\frac{1}{n}\sum_{x\in X}\mu_A(x) + \frac{1}{n}\sum_{x\in X}\nu_A(x)}{2}.$$

Now, by Definition 4.2, the α-local entropy of $x \in X$ in A is given by:

$$e_A^\alpha(x) = \begin{cases} |1-\mu_A(x)| + |0-\nu_A(x)|, & \text{if } \mu_A(x) > \nu_A(x), \\ |0-\mu_A(x)| + |1-\nu_A(x)|, & \text{if } \mu_A(x) \le \nu_A(x). \end{cases}$$

So, Definition 4.3 yields

$$E_\alpha(A) = \begin{cases} \frac{1}{n}\sum_{x\in X}\big(1-\mu_A(x)+\nu_A(x)\big), & \text{if } \mu_A(x) > \nu_A(x), \\ \frac{1}{n}\sum_{x\in X}\big(1-\nu_A(x)+\mu_A(x)\big), & \text{if } \mu_A(x) \le \nu_A(x). \end{cases}$$

This simplifies to

$$\begin{aligned} E_\alpha(A) &= \begin{cases} 1-\frac{1}{n}\sum_{x\in X}\mu_A(x) + \frac{1}{n}\sum_{x\in X}\nu_A(x), & \text{if } \mu_A(x) > \nu_A(x), \\ 1-\frac{1}{n}\sum_{x\in X}\nu_A(x) + \frac{1}{n}\sum_{x\in X}\mu_A(x), & \text{if } \mu_A(x) \le \nu_A(x). \end{cases} \\ &= \begin{cases} 2I_D(A^c,B), & \text{if } \mu_A(x) > \nu_A(x) \text{ for all } x \in X, \\ 2I_D(A,B), & \text{if } \mu_A(x) \le \nu_A(x) \text{ for all } x \in X. \end{cases} \end{aligned}$$ □

Example 5.3. *Let $X = \{x_1, x_2, x_3\}$. We define an intuitionistic fuzzy set $A = \{\langle x_1, 0.8, 0.1\rangle, \langle x_2, 0.7, 0.2\rangle, \langle x_3, 0.6, 0.3\rangle\}$. Then, by Definition 4.2, we obtain the α-local entropy measures of x_1, x_2 and x_3 as shown below:*

$$e_A^\alpha(x_1) = |1-0.8| + |0-0.1| = 0.2 + 0.1 = 0.3,$$

$$e_A^\alpha(x_2) = |1-0.7| + |0-0.2| = 0.3 + 0.2 = 0.5,$$

$$e_A^\alpha(x_3) = |1-0.6| + |0-0.3| = 0.4 + 0.3 = 0.7.$$

Thus, the α-intuitionistic fuzzy entropy is $E_\alpha(A) = \frac{1}{3}(0.3+0.5+0.7) = \frac{1.5}{3} = 0.5$. Now, for $A^c = \{\langle x_1, 0.1, 0.8\rangle, \langle x_2, 0.2, 0.7\rangle, \langle x_3, 0.3, 0.6\rangle\}$ and $B = \{\langle x_1, 0, 1\rangle, \langle x_2, 0, 1\rangle, \langle x_3, 0, 1\rangle\}$, we have $I_I(\langle 0.1, 0.8\rangle, \langle 0, 1\rangle) = \langle \max\{0.8, 0\}, \min\{0.1, 1\}\rangle = \langle 0.8, 0.1\rangle$.

Similarly, $I_I(\langle 0.2, 0.7\rangle, \langle 0, 1\rangle) = \langle 0.7, 0.2\rangle$ and $I_I(\langle 0.3, 0.6\rangle, \langle 0, 1\rangle) = \langle 0.6, 0.3\rangle$. Therefore, using Definition 3.2, we get $I_s(A^c, B) = \left\langle \frac{1}{3}(0.8+0.7+0.6),\ \frac{1}{3}(0.1+0.2+0.3)\right\rangle = \langle 0.7,\ 0.2\rangle$. Hence, using Definition 3.3, we find $I_D(A^c,B) = \frac{|1-0.7|+|0-0.2|}{2} = \frac{0.3+0.2}{2} = \frac{0.5}{2}$. Consequently, we obtain $E_\alpha(A) = 2I_D(A^c,B)$.

6 Conclusion

In this paper, we introduced the implication deviation, formulated using an implication operator and a universal quantifier within the framework of intuitionistic fuzzy sets. This measure plays a pivotal role in defining fundamental concepts such as inclusion and similarity measures, highlighting its foundational importance in the analysis of intuitionistic fuzzy sets. Furthermore, we proposed the α-intuitionistic fuzzy entropy and the β-intuitionistic fuzzy entropy to capture the underlying uncertainty of intuitionistic fuzzy sets by incorporating both the degree of hesitation and the behavior of the membership and non-membership functions, particularly in situations where these two values are equal. We also established a relationship between the proposed implication deviation and the entropy of an intuitionistic fuzzy set. The present work is formulated using the intuitionistic fuzzy quantifier $\forall_{139,3}$. Although this quantifier possesses several desirable mathematical properties, the proposed framework is not inherently restricted to it. An interesting direction for future research is to investigate implication deviation using other intuitionistic fuzzy quantifiers or more general aggregation operators, which may provide greater flexibility and extending the scope of the proposed theory. Another promising direction is to investigate the behavior of implication deviation under different classes of intuitionistic fuzzy implication operators, such as Goguen-type [**?**], Zadeh-type [**?**], modal, and weak [**?**] implications. A systematic comparative study may reveal whether these operators induce genuinely different notions of deviation, similarity, and inclusion. Further, implication deviation could serve as a useful framework for comparing and classifying intuitionistic fuzzy implications, thereby strengthening its connection with the broader theory of intuitionistic fuzzy implications. Moreover, we have used a particular type of negation operator in this paper as its behaviour aligns with that of the standard fuzzy negation operator when the degree of hesitancy is zero. However, other intuitionistic fuzzy negation operators [**?**, **?**] available in the literature can also be incorporated, potentially leading to different properties and insights. On the other hand, the piecewise structure of the proposed entropy measures arises from the underlying principle of quantifying deviation from distinct extremal uncertainty states, each of which contributes uniquely to the characterization of uncertainty. While the current formulation effectively captures the underlying intuition and enables the desired theoretical properties to be established, it would be worthwhile to investigate whether an equivalent unified non-piecewise formulation can be developed that preserves the influence of these extremal points while yielding more compact analytical expressions. We believe that this paper will open several new directions in intuitionistic fuzzy set and intuitionistic fuzzy logic in future.

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