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**Notes on Intuitionistic Fuzzy Sets**
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# Complex intuitionistic fuzzy evolution equations

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**Abstract:** In this work, we define a complex intuitionistic fuzzy strongly continuous semigroup and some properties. We use this theory to prove the existence and uniqueness of a mild solution to the Complex intuitionistic fuzzy evolution equations.

**Keywords:** Complex intuitionistic fuzzy sets, Complex intuitionistic fuzzy evolution equations, Mild solution.
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## 1 Introduction

The concept of intuitionistic fuzzy is introduced by K. Atanassov (1983) [1, 2, 3]. This concept is a generalization of fuzzy theory introduced by L. Zadeh [12]. Several works made in the study of the Cauchy problem with intuitionistic fuzzy initial condition [7].

The concept of complex fuzzy sets as sets with complex membership functions was first introduced by Ramot et al., who in [11] demonstrated the increased expressive power gained by endowing a set $S$ with a complex membership function $\mu_S(x) = r_S(x)e^{i\phi_S(x)}$, where $r_S(x)$ and $\phi_S(x)$ are real-valued functions with $r_S$ solely responsible for the fuzzy information and $\phi_S$ functioning as a phase term containing additional crisp information.

In [5], we discussed the existence and uniqueness for a solution of the intuitionistic fuzzy differential equation

$$\begin{cases} U'(t) = \mathcal{H}(t, U(t)), & t \in I \\ U(0) = U_0 \in \mathbb{F}, \end{cases} \tag{1}$$

In this paper, we consider the following problem:

$$\begin{cases} U'(t) = \mathcal{A}U(t) + \mathcal{H}(t, U(t)), & t \in I, \\ U(0) = U_0 \in \mathbb{F}, \end{cases} \tag{2}$$

where $\mathcal{A}$ is the generator of a strongly continuous semigroup $\{\mathcal{T}(t),\, t \geq 0\}$ on $\mathbb{F}$ and $\mathcal{H} : I \times \mathbb{F} \to \mathbb{F}$ which we take to be continuous in both arguments and satisfies some conditions. And it will be extensive as an initiation to study other concept, stability, etc., which is defined in the fuzzy case [6].

## 2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let us denote by $P_k(\mathbb{R})$ the set of all nonempty compact convex subsets of $\mathbb{R}$.

**Definition 1.** *We denote*

$$IF = \left\{(u, v) : \mathbb{R} \to [0,1]^2 \;|\forall x \in \mathbb{R} \,/ 0 \leq u(x) + v(x) \leq 1\right\}$$

*where*

1. $(u, v)$ *is normal i.e. there exists* $x_0, x_1 \in \mathbb{R}$ *such that* $u(x_0) = 1$ *and* $v(x_1) = 1$.
2. $u$ *is fuzzy convex and* $v$ *is fuzzy concave.*
3. $u$ *is upper semicontinuous and* $v$ *is lower semicontinuous*
4. $supp(u, v) = cl(\{x \in \mathbb{R} : v(x) < 1\})$ *is bounded.*

*For* $\alpha \in [0, 1]$ *and* $(u, v) \in IF$*, we define*

$$[(u, v)]^\alpha = \{x \in \mathbb{R} \mid v(x) \leq 1 - \alpha\}$$

*and*

$$[(u, v)]_\alpha = \{x \in \mathbb{R} \mid u(x) \geq \alpha\}.$$

**Remark 1.** *We can consider* $[(u, v)]_\alpha$ *as* $[u]^\alpha$ *and* $[(u, v)]^\alpha$ *as* $[1 - v]^\alpha$ *in the fuzzy case.*

**Definition 2.** *The intuitionistic fuzzy zero is intuitionistic fuzzy set defined by*

$$0_{(1,0)}(x) = \begin{cases} (1, 0), & x = 0 \\ (0, 1), & x \neq 0 \end{cases}.$$

**Definition 3.** *Let* $(u,v)$ ,$(u',v') \in IF$ *and* $\lambda \in \mathbb{R}$*, we define the addition by:*

$$((u,v) \oplus (u',v'))(z) = \left( \sup_{z=x+y} \min(u(x), u'(y)); \inf_{z=x+y} \max(v(x), v'(y)) \right)$$

$$\lambda(u,v) = \begin{cases} (\lambda u, \lambda v), & if\ \lambda \neq 0 \\ 0_{(0,1)}, & if\ \lambda = 0 \end{cases}.$$

According to Zadeh's extension principle, we have addition and scalar multiplication in intuitionistic fuzzy number space $IF$ as follows:

$$[(u,v) \oplus (z,w)]^\alpha = [(u,v)]^\alpha + [(z,w)]^\alpha, \tag{3}$$

$$[\lambda(u,v)]^\alpha = \lambda[(u,v)]^\alpha, \tag{4}$$

$$[(u,v) \oplus (z,w)]_\alpha = [(u,v)]_\alpha + [(z,w)]_\alpha, \tag{5}$$

$$[\lambda(u,v)]_\alpha = \lambda[(u,v)]_\alpha, \tag{6}$$

where $(u,v),(z,w) \in IF$ and $\lambda \in \mathbb{R}$.

We denote

$$[(u,v)]_l^+(\alpha) = \inf\{x \in \mathbb{R} \mid u(x) \geq \alpha\},$$
$$[(u,v)]_r^+(\alpha) = \sup\{x \in \mathbb{R} \mid u(x) \geq \alpha\},$$
$$[(u,v)]_l^-(\alpha) = \inf\{x \in \mathbb{R} \mid v(x) \leq 1-\alpha\},$$
$$[(u,v)]_r^-(\alpha) = \sup\{x \in \mathbb{R} \mid v(x) \leq 1-\alpha\}.$$

**Remark 2.**

$$\left[(u,v)\right]_\alpha = \left[[(u,v)]_l^+(\alpha), [(u,v)]_r^+(\alpha)\right],$$
$$\left[(u,v)\right]^\alpha = \left[[(u,v)]_l^-(\alpha), [(u,v)]_r^-(\alpha)\right].$$

**Theorem 1.** *let* $\mathcal{M} = \{M_\alpha,\ M^\alpha\ :\ \alpha \in [0,1]\}$ *be a family of subsets in* $\mathbb{R}$ *satisfying Conditions* $(i)-(iv)$

**i)** $\alpha \leq \beta \Rightarrow M_\beta \subset M_\alpha$ *and* $M^\beta \subset M^\alpha$

**ii)** $M_\alpha$ *and* $M^\alpha$ *are nonempty compact convex sets in* $\mathbb{R}$ *for each* $\alpha \in [0,1]$.

**iii)** *for any nondecreasing sequence* $\alpha_i \to \alpha$ *on* $[0,1]$*, we have* $M_\alpha = \bigcap_i M_{\alpha_i}$ *and* $M^\alpha = \bigcap_i M^{\alpha_i}$.

**iv)** *For each* $\alpha \in [0,1]$, $M_\alpha \subset M^\alpha$ *and define* $u$ *and* $v$*, by*

$$u(x) = \begin{cases} 0 & if\ x \notin M_0 \\ \sup\{\alpha \in [0,1]\ :\ x \in M_\alpha\} & if x \in M_0 \end{cases},$$

$$v(x) = \begin{cases} 1 & if\ x \notin M^0 \\ 1 - \sup\{\alpha \in [0,1]\ :\ x \in M^\alpha\} & if x \in M^0 \end{cases}.$$

*Then* $(u,v) \in IF$.

*Proof.* See [10]. □

The space $IF$ is metrizable by the distance of the following form:

$$d_\infty((u,v),(z,w)) = \frac{1}{4}\sup_{0<\alpha\le 1}|[(u,v)]_r^+(\alpha) - [(z,w)]_r^+(\alpha)|$$

$$+\frac{1}{4}\sup_{0<\alpha\le 1}|[(u,v)]_l^+(\alpha) - [(z,w)]_l^+(\alpha)|$$

$$+\frac{1}{4}\sup_{0<\alpha\le 1}|[(u,v)]_r^-(\alpha) - [(z,w)]_r^-(\alpha)|$$

$$+\frac{1}{4}\sup_{0<\alpha\le 1}|[(u,v)]_l^-(\alpha) - [(z,w)]_l^-(\alpha)|,$$

where $|.|$ denotes the usual Euclidean norm in $\mathbb{R}$.

**Theorem 2.** *$(IF, d_\infty)$ is a complete metric space.*

*Proof.* See [10]. □

We recall the definition of a complex fuzzy set:

**Definition 4.** *A complex fuzzy set $A$, defined on a universe of discourse $X$, is characterized by a membership function $\mu_A(x)$ that assigns any element $x \in X$ a complex-valued grade of membership in $A$. By definition $\mu_A(x)$ a value in the unit circle in the complex plane in the polar case. And a value in the unit square in $\mathbb{C}$ in the Cartesian case.*

## 2.1 Complex intuitionistic fuzzy set

In this section, we recall some basic notion on complex intuitionistic fuzzy sets defined in [4]. As the definition of complex fuzzy set, we give here a definition of complex intuitionistic fuzzy set:

**Definition 5.** *A complex intuitionistic fuzzy set $A$, defined on a universe of discourse $X$, is characterized by a membership function $\mu_A(x)$ and non-membership function $\nu_A(x)$ that assigns any element $x \in X$ a complex-valued grade of membership and non-membership in $A$. By definition, the values $\mu_A(x)$, $\nu_A(x)$ and $\mu_A(x) + \nu_A(x)$ may receive all lie within the unit circle in the complex plane in the polar case. And $\mu_A$, $\nu_A$ and $\mu_A(x) + \nu_A(x)$ a value in the unit square in $\mathbb{C}$ in the Cartesian case.*

## 2.2 Cartesian representation of complex grades of membership and non-membership

The complex membership function $\mu$, is defined as

$$\mu(V, z) = \mu_R(V) + i\mu_I(z),$$

likewise, we can define the complex non-membership function as

$$\nu(V, z) = \nu_R(V) + i\nu_I(z),$$

where $V$ is to be interpreted as a set in a intuitionistic fuzzy set of sets and $z$ as an element of $V$. This definition can be easily extended to $\mathbb{R}$, for $x \in \mathbb{R}$, let

$$f_1(x) = u(x) + iv(x) \qquad \text{and} \qquad f_2(x) = u'(x) + iv'(x),$$

where $f = (u, u') : \mathbb{R} \to [0,1]^2$ and $g = (v, v') : \mathbb{R} \to [0,1]^2$. For ease of notation, denote $\mathcal{F}$ by $(f, g)$. Thus, $f_1$, $f_2$ assigns to each $x \in \mathbb{R}$ a value in the unit square in $\mathbb{C}$, representing a complex grade of membership and non-membership. Note that $u$, $v$, $u'$ and $v'$ considered individually define non-complex fuzzy sets in $\mathbb{R}$.

Now, for $f = (u, u')$, $g = (v, v') : \mathbb{R} \to [0,1]^2$, $\alpha$-level sets are classically defined as follows:

$$[f]^\alpha = [(u, u')]^\alpha = \{x \in \mathbb{R} \,|\, u'(x) \leq 1 - \alpha\}\,; \qquad [f]_\alpha = [(u, u')]_\alpha = \{x \in \mathbb{R} \,|\, u(x) \geq \alpha\}$$

and

$$[f]^0 = [(u, u')]^0 = \overline{\{x \in \mathbb{R} \,|\, u'(x) < 1\}}; \qquad [f]_0 = [(u, u')]_0 = \overline{\{x \in \mathbb{R} \,|\, u(x) > 0\}}.$$

We use the above to define $(\alpha, \beta)$-level sets for $\mathcal{F} = (f, g)$, $0 < \alpha,\ \beta \leq 1$:

$$[\mathcal{F}]^{(\alpha,\beta)} = [(f, g)]^{(\alpha,\beta)} = [f]^\alpha \cap [g]^\beta, \tag{7}$$

and

$$[\mathcal{F}]_{(\alpha,\beta)} = [(f, g)]_{(\alpha,\beta)} = [f]_\alpha \cap [g]_\beta. \tag{8}$$

Consider the following set of conditions as an alternative definition of $[\mathcal{F}]^{(\alpha,\beta)}$ and $[\mathcal{F}]_{(\alpha,\beta)}$:

$$[\mathcal{F}]^{(\alpha,\beta)} = \{x \in \mathbb{R} \,|\, u'(x) \leq 1 - \alpha,\ v'(x) \leq 1 - \beta\}\,, \tag{9}$$

$$[\mathcal{F}]^{(\alpha,0)} = \overline{\{x \in \mathbb{R} \,|\, u'(x) \leq 1 - \alpha,\ v'(x) < 1\}}, \tag{10}$$

$$[\mathcal{F}]^{(0,\beta)} = \overline{\{x \in \mathbb{R} \,|\, u'(x) < 1,\ v'(x) \leq 1 - \beta\}}, \tag{11}$$

$$[\mathcal{F}]^{(0,0)} = \overline{\{x \in \mathbb{R} \,|\, u'(x) < 1,\ v'(x) < 1\}}, \tag{12}$$

and

$$[\mathcal{F}]_{(\alpha,\beta)} = \{x \in \mathbb{R} \,|\, u(x) \geq \alpha,\ v(x) \geq \beta\}\,, \tag{13}$$

$$[\mathcal{F}]_{(\alpha,0)} = \overline{\{x \in \mathbb{R} \,|\, u(x) \geq \alpha,\ v(x) > 0\}}, \tag{14}$$

$$[\mathcal{F}]_{(0,\beta)} = \overline{\{x \in \mathbb{R} \,|\, u(x) > 0,\ v(x) \geq \beta\}}, \tag{15}$$

$$[\mathcal{F}]_{(0,0)} = \overline{\{x \in \mathbb{R} \,|\, u(x) > 0,\ v(x) > 0\}}. \tag{16}$$

Note that (9) and (12) are equivalent to definition (7), likewise (13) and (16) are equivalent to definition (8) for the corresponding $\alpha$, $\beta$, but (10), (11) and (14), (15) are not: (7) and (8) may not yield closed sets in the case when exactly one of $\alpha$, $\beta$ is equal to 0, but (10), (11) and (14), (15) would yield the respective closures of those sets.

For $f, g \in IF$, we have $[f]^\alpha \cap [g]^\beta$, $[f]_\alpha \cap [g]_\beta$ are always compact and $[f]^1 \cap [g]^1 \subset [f]^\alpha \cap [g]^\beta \subset [f]^0 \cap [g]^0$ and $[f]_1 \cap [g]_1 \subset [f]_\alpha \cap [g]_\beta \subset [f]_0 \cap [g]_0$ are nonempty as in order to ensure this, it is sufficient that $[f]^1 \cap [g]^1$ and $[f]_1 \cap [g]_1$ be nonempty, meaning that there should exist some $x_0,\, x_1 \in \mathbb{R}$ such that $x_0 \in [f]^1$, i.e., $u'(x_0) = 0$, $x_0 \in [g]^1$, i.e., $v'(x_0) = 0$ and $x_1 \in [f]_1$, i.e.,$u(x_1) = 1$, $x_1 \in [g]_1$, i.e., $v(x_1) = 1$. With that in mind, we define the following set:

$$\hat{IF}^2 = \left\{ \Big( (u, u'),\, (v, v') \Big) \in IF \times IF \,|\, \exists x_0,\, x_1 \in \mathbb{R},\, s.t\, u(x_1) = v(x_1) = 1,\, u'(x_0) = v'(x_0) = 0 \right\}. \tag{17}$$

Then for $(f, g) \in \hat{IF}^2$, $[\mathcal{F}]^{(\alpha,\beta)} = [f]^\alpha \cap [g]^\beta$, $[\mathcal{F}]_{(\alpha,\beta)} = [f]_\alpha \cap [g]_\beta \in \mathcal{P}_k(\mathbb{R})$ for all $\alpha,\, \beta \in [0, 1]$. And the compactness of the $[\mathcal{F}]^{(\alpha,\beta)}$ sets guarantees the complete equivalence of definition (7) and the set of definitions (9)–(12), and the complete equivalence of definition (8) and the set of definitions (13)–(16).

We recall that $IF$ is closed under addition and scalar multiplication, to establish a similar result for $\hat{IF}^2$. For functions $f = (u, u')$, $g = (v, v') \in IF$, addition and scalar multiplication can be defined via level sets as (3)-(6).

For $\mathcal{F} = (f, g) = \big((u, u'), (v, v')\big)$, $\mathcal{G} = (f', g') = \big((x, x'), (y, y')\big) \in \hat{IF}^2$ and $\lambda$ is a scalar, let

$$\mathcal{F} + \mathcal{G} = (f, g) + (f', g') = (f + f', g + g'), \tag{18}$$

$$\lambda\mathcal{F} = \lambda(f, g) = (\lambda f, \lambda g). \tag{19}$$

**Theorem 3** ([4]). *$\hat{IF}^2$ is closed under addition and scalar multiplication.*

Consider the product metric on $IF^2 = IF \times IF$, $\hat{d}_\infty :\, IF^2 \times IF^2 \to \mathbb{R}^+$ by:

$$\hat{d}_\infty(\mathcal{F}, \mathcal{G}) = \max\{d_\infty(f, f')\,,\, d_\infty(g, g')\}, \quad \mathcal{F} = (f, g),\, \mathcal{G} = (f', g') \in \hat{IF}^2. \tag{20}$$

Since $\hat{IF}^2 \subset IF^2$, $\hat{d}_\infty$ is also a metric for $\hat{IF}^2$. Hence, $(\hat{IF}^2, \hat{d}_\infty)$ is a complete metric space.

It will also prove useful to define a zero element in $\hat{IF}^2$. Recall that on $IF$ we define zero element $0_{(1,0)} \in IF$ by

$$0_{(1,0)}(x) = \begin{cases} (1, 0) & , x = 0 \\ (0, 1) & , x \neq 0 \end{cases}.$$

The zero element on $\hat{IF}^2$ then reads

$$\hat{0} = \big(0_{(1,0)}, 0_{(1,0)}\big) \in IF^2.$$

We have $\hat{0}(0) = \big((1, 0), (1, 0)\big)$, verifying that $\hat{0} \in \hat{IF}^2$.

**Theorem 4** ([4]). *$\hat{IF}^2 \subset IF \times IF$ is embeddable into a Banach space.*

**Remark 3.** *In the same manner can be defined $\hat{IF}^n$, $n \geq 3$ and it is shown that is embeddable into a Banach space.*

## 2.3 Polar representation of complex grades of membership and non-membership

The polar representation of the membership function $\mu$, is defined as

$$\mu(V, z) = r(V)e^{i\sigma\phi(z)},$$

likewise, we can define the polar representation of complex non-membership function as

$$\nu(V, z) = r'(V)e^{i\sigma\phi'(z)},$$

where $\sigma$ is a scaling factor, does not translate directly to and from the respective Cartesian representation. Therefore, the two representations of the corresponding extension to $\mathbb{R}$ are not equivalent as defined, which will be seen below. Thus, depending on the application, one may be more appropriate to use than the other.

For $x \in \mathbb{R}$, the polar form of $f_1$ and $f_2$ is defined as follows:

$$f_1(x) = r(x)e^{2\pi\phi(x)i}, \qquad f_2(x) = r'(x)e^{2\pi\phi'(x)i},$$

where $f = (r, r'), g = (\phi, \phi') : \mathbb{R} \to [0, 1]^2$.

We denote $f_1 = (r, \phi)$ and $f_2 = (r', \phi')$. The scaling factor is taken to be $2\pi$ , allowing the range of $f_1$ and $f_2$ to be the entire unit circle. Because $e^{2\pi i\phi}$ is periodic, we take the value of $\phi$ giving the maximum distance from $e^0$, $\phi = 0.5$, to be the "maximum" membership value.

$$[f]^\alpha = [(r, r')]^\alpha = \{x \in \mathbb{R} \,|\, r'(x) \leq 1 - \alpha\},$$

and

$$[f]_\alpha = [(r, r')]_\alpha = \{x \in \mathbb{R} \,|\, r(x) \geq \alpha\}.$$

And we define the level sets for $g = (\phi, \phi')$, denoted $[g]^{\langle\alpha\rangle}$ and $[g]_{\langle\alpha\rangle}$, must be defined differently to account for the periodicity:

$$[g]^{\langle\alpha\rangle} = \{x \in \mathbb{R} \,|\, \phi'(x) \in [\alpha, 1 - \alpha],\ \alpha \in (0, 0.5]\},\tag{21}$$

$$[g]_{\langle\alpha\rangle} = \{x \in \mathbb{R} \,|\, \phi(x) \in [\alpha, 1 - \alpha],\ \alpha \in (0, 0.5]\},\tag{22}$$

$$[g]^{\langle 0\rangle} = \overline{\{x \in \mathbb{R} \,|\, 0 < \phi'(x) < 1\}},\tag{23}$$

$$[g]_{\langle 0\rangle} = \overline{\{x \in \mathbb{R} \,|\, 0 < \phi(x) < 1\}},\tag{24}$$

$$[g]^{\langle\alpha\rangle} = [(\phi, \phi')]^{\langle 1-\alpha\rangle}, \quad [g]_{\langle\alpha\rangle} = [(\phi, \phi')]_{\langle 1-\alpha\rangle}, \text{ for all } \alpha \in [0, 1].\tag{25}$$

For $\mathcal{F} = (f, g)$, We can then define the level sets $[\mathcal{F}]^{\langle\alpha,\beta\rangle}$ and $[\mathcal{F}]_{\langle\alpha,\beta\rangle}$ as

$$[\mathcal{F}]^{\langle\alpha,\beta\rangle} = [(f, g)]^{\langle\alpha,\beta\rangle} = [f]^{\langle\alpha\rangle} \cap [g]^{\langle\beta\rangle}, \text{ and } [\mathcal{F}]_{\langle\alpha,\beta\rangle} = [(f, g)]_{\langle\alpha,\beta\rangle} = [f]_{\langle\alpha\rangle} \cap [g]_{\langle\beta\rangle},\tag{26}$$

or by the relations:

$$[\mathcal{F}]^{\langle\alpha,\beta\rangle} = \{x \in \mathbb{R} \,|\, r'(x) \leq 1-\alpha,\, \phi'(x) \in [\beta, 1-\beta]\}\,, \tag{27}$$

$$[\mathcal{F}]_{\langle\alpha,\beta\rangle} = \{x \in \mathbb{R} \,|\, r(x) \geq \alpha,\, \phi(x) \in [\beta, 1-\beta]\}\,, \tag{28}$$

$$[\mathcal{F}]^{\langle\alpha,0\rangle} = \overline{\{x \in \mathbb{R} \,|\, r'(x) \leq 1-\alpha,\, 0 < \phi'(x) < 1\}}, \tag{29}$$

$$[\mathcal{F}]_{\langle\alpha,0\rangle} = \overline{\{x \in \mathbb{R} \,|\, r(x) \geq \alpha,\, 0 < \phi(x) < 1\}}, \tag{30}$$

$$[\mathcal{F}]^{\langle 0,\beta\rangle} = \overline{\{x \in \mathbb{R} \,|\, r'(x) < 1,\, \phi'(x) \in [\beta, 1-\beta]\}}, \tag{31}$$

$$[\mathcal{F}]_{\langle 0,\beta\rangle} = \overline{\{x \in \mathbb{R} \,|\, r(x) > 0,\, \phi(x) \in [\beta, 1-\beta]\}}, \tag{32}$$

$$[\mathcal{F}]^{\langle 0,0\rangle} = \overline{\{x \in \mathbb{R} \,|\, r'(x) < 1,\, 0 < \phi'(x) < 1\}}, \tag{33}$$

$$[\mathcal{F}]_{\langle 0,0\rangle} = \overline{\{x \in \mathbb{R} \,|\, r(x) > 0,\, 0 < \phi(x) < 1\}}, \tag{34}$$

together with

$$[\mathcal{F}]^{\langle\alpha,\beta\rangle} = [\mathcal{F}]^{\langle\alpha,1-\beta\rangle}, \quad \text{and} \quad [\mathcal{F}]_{\langle\alpha,\beta\rangle} = [\mathcal{F}]_{\langle\alpha,1-\beta\rangle}, \quad \text{for all}\, \alpha, \beta \in [0,1]. \tag{35}$$

It is clear that, for $g = (\phi, \phi') \in IF$, $[g]^{\langle\alpha\rangle} \subset [g]^{\alpha}$ and $[g]_{\langle\alpha\rangle} \subset [g]_{\alpha}$ for all $\alpha \in [0, 0.5]$. However, $[g]^{\langle\alpha\rangle}$, $[g]_{\langle\alpha\rangle}$ need not be compact or convex. In order to address this issue, we define

$$\hat{\mathcal{G}} = \left\{(u,v) : \mathbb{R} \to [0,1]^2 \text{ satisfying all of the following conditions}\right\},$$

1. There exists $x_0, x_1 \in \mathbb{R}$ such that $u(x_0) = v(x_1) = 0.5$.

2. $u$ and $v$ are monotone.

3. $u$ is upper semi-continuous on $K_1$ and lower semi-continuous on $K_2$, with

$$K_1 = \{x \in \mathbb{R} \,|\, 0 < u(x) \leq 0.5\}\,, \quad \text{and} \quad K_2 = \{x \in \mathbb{R} \,|\, 0.5 \leq u(x) < 1\}\,.$$

4. $v$ is lower semi-continuous on $K_1'$ and upper semi-continuous on $K_2'$, with

$$K_1' = \{x \in \mathbb{R} \,|\, 0 < v(x) \leq 0.5\}\,, \quad \text{and} \quad K_2' = \{x \in \mathbb{R} \,|\, 0.5 \leq v(x) < 1\}\,.$$

5. $\overline{K_1 \cup K_2}$ and $\overline{K_1' \cup K_2'}$ are compact.

**Theorem 5** ([4]). *There exists an embedding $l : \hat{\mathcal{G}} \to IF \times IF$.*

Now, we define

$$\hat{IF}^2_* = \left\{ \Big( (r, r'), (\phi, \phi') \Big) \in IF \times \hat{\mathcal{G}} \,|\, \exists\, x_0, x_1 \in \mathbb{R} \,\text{s.t}\; r(x_0) = 1,\; r'(x_1) = 0,\; \phi(x_0) = \phi'(x_1) = 0.5 \right\}.$$

Note that, for $\mathcal{F} \in \hat{IF}^2_*$, definition (26) is equivalent to the set of definitions (27)–(35).

**Theorem 6** ([4]). *$\hat{IF}^2_*$ is embeddable into a Banach space.*

The following results, therefore, apply equally to the space $\hat{IF}^2$ in the Cartesian case and to the space $\hat{IF}^2_*$ in the polar case.

For brevity, we shall let $\mathbb{F} = \hat{IF}^2$ when dealing with the Cartesian complex form, and $\mathbb{F} = \hat{IF}^2_*$ when dealing with the polar complex form.

We define differentiability as in terms of the Hukuhara difference. For $\mathcal{F}, \mathcal{G} \in \mathbb{F}$, if there exists $\mathcal{K} \in \mathbb{F}$ such that $\mathcal{G} + \mathcal{K} = \mathcal{F}$, we write $\mathcal{F} - \mathcal{G} = \mathcal{K}$ and call $\mathcal{K}$ the difference of $\mathbb{F}$ and $\mathcal{G}$.

**Definition 6.** *We call a mapping $\mathcal{R} : I = [0, a] \to \mathbb{F}$ is differentiable at $t_0 \in I$ if there exists some $\mathcal{R}'(t_0) \in \mathbb{F}$ such that the following limits exist and are equal to $\mathcal{R}'(t_0) \in \mathbb{F}$:*

$$\lim_{h \to 0^+} \frac{\mathcal{R}(t_0 + h) - \mathcal{R}(t_0)}{h} \qquad \text{and} \lim_{h \to 0^+} \frac{\mathcal{R}(t_0) - \mathcal{R}(t_0 - h)}{h}.$$

Let $\mathcal{R} : I \to \mathbb{F}$ be a continuous mapping. We define $\mathcal{S} : [0, a] \to \mathbb{F}$ by

$$\mathcal{S}(t) = \int_0^t \mathcal{R}(s)ds, \quad t \in I.$$

Note that

$$\frac{d}{dt}\mathcal{S}(t) = \mathcal{S}'(t) = \mathcal{R}(t), \quad t \in I.$$

# 3 Main results

## 3.1 Complex intuitionistic fuzzy semigroups

We give here a definition of semigroup on $\mathbb{F}$ similar to that defined in the fuzzy case in [6, 8, 9].

**Definition 7.** *A family $\{\mathcal{T}(t), t \geq 0\}$ of operators from $\mathbb{F}$ into itself is a complex intuitionistic fuzzy strongly continuous semigroup on $\mathbb{F}$ if*

*(i) $\mathcal{T}(t)(0) = i$ , the identity mapping on $\mathbb{F}$,*

*(ii) $\mathcal{T}(t + s) = \mathcal{T}(t)\mathcal{T}(s)$ for all $t, s \geq 0$,*

*(iii) the function $\mathcal{H} : [0, \infty[\to \mathbb{F}$, defined by $\mathcal{H}(t) = \mathcal{T}(t)U$ is continuous at $t = 0$ for all $U \in \mathbb{F}$ i.e.,*

$$\lim_{t \to 0^+} \mathcal{T}(t)U = U.$$

*(iv) There exist two constants $M > 0$ and $\omega$ such that*

$$\hat{d}_\infty\Big(\mathcal{T}(t)U, \mathcal{T}(t)V\Big) \leq M\, e^{\omega t}\, \hat{d}_\infty(U, V), \quad for\, t \geq 0, U, V \in \mathbb{F}.$$

*In particular, if $M = 1$ and $\omega = 0$, we say that $\{\mathcal{T}(t), t \geq 0\}$ is a contraction semigroup on $\mathbb{F}$.*

**Remark 4.** *The condition (iii) implies that the function $t \to \mathcal{T}(t)U$ is continuous on $[0, \infty[$ for all $U \in \mathbb{F}$.*

**Definition 8.** *Let $\{\mathcal{T}(t), t \geq 0\}$ be a strongly continuous semigroup on $\mathbb{F}$ and $U \in \mathbb{F}$. If for $h > 0$ very small, the Hukuhara difference $\mathcal{T}(h)U - U$ exits, we define*

$$\mathcal{A}U = \lim_{h \to 0^+} \frac{\mathcal{T}(h)U - U}{h},$$

*whenever this limit exists in the metric space $(\mathbb{F}, \hat{d}_\infty)$. Then the operator $\mathcal{A}$ defined on*

$$D(\mathcal{A}) = \left\{ U \in \mathbb{F} \,|\, \lim_{h \to 0^+} \frac{\mathcal{T}(h)U - U}{h} \textit{ exists} \right\} \subset \mathbb{F}$$

*is called the infinitesimal generator of the semigroup $\{\mathcal{T}(t), t \geq 0\}$.*

**Lemma 1.** *Let $\mathcal{A}$ be the generator of a semigroup $\{\mathcal{T}(t), t \geq 0\}$ on $\mathbb{F}$, then for all $U \in \mathbb{F}$ such that $\mathcal{T}(t)U \in D(\mathcal{A})$ for all $t \geq 0$, the mapping $\mathcal{F} : t \to \mathcal{T}(t)U$ is differentiable and*

$$\mathcal{F}'(t) = \frac{d}{dt}\Big(\mathcal{T}(t)U\Big) = \mathcal{A}\,\mathcal{T}(t)U, \quad \forall t \geq 0.$$

*Proof.* For $U \in \mathbb{F}$, $t \geq 0$ and $h$ very small, since $\mathcal{T}(t)U \in D(\mathcal{A})$, so

$$\begin{aligned}
\lim_{h \to 0^+} \frac{\mathcal{F}(t+h) - \mathcal{F}(t)}{h} &= \lim_{h \to 0^+} \frac{\mathcal{T}(t+h)U - \mathcal{T}(t)U}{h} \\
&= \lim_{h \to 0^+} \frac{\mathcal{T}(h)\mathcal{T}(t)U - \mathcal{T}(t)U}{h} \\
&= \mathcal{A}\mathcal{T}(t)U.
\end{aligned}$$

This completes the proof. □

**Example 1.** *Define on $\mathbb{F}$ the family of linear operators $\mathcal{T}(t) : \mathbb{F} \to \mathbb{F}$ by*

$$\mathcal{T}(t)U = \mathcal{T}(t)(f, g) = (e^{nt} f\,,\, e^{mt} g), \quad U = (f, g) \in \mathbb{F},\, n, m \in \mathbb{N},\, t \geq 0.$$

*1. $\{\mathcal{T}(t)\,, t \geq 0\}$ is a semigroup on $\mathbb{F}$. It is easy to see that $\pi(t)$ is well-defined.*

*(i) for $U \in \mathbb{E}$, we have*

$$\mathcal{T}(0)U = U.$$

*(ii)* *For* $U = (f,g) \in \mathbb{F}$, $t, s \geq 0$, *we have*

$$\begin{aligned} \mathcal{T}(t+s)U &= \mathcal{T}(t)(f,g) = (e^{n(t+s)}f, e^{m(t+s)}g) \\ &= (e^{nt}e^{ns}f, e^{mt}e^{ms}g) \\ &= \mathcal{T}(t)(e^{ns}f, e^{ms}g) \\ &= \mathcal{T}(t)\mathcal{T}(s)(f,g) \\ &= \mathcal{T}(t)\mathcal{T}(s)U. \end{aligned}$$

*(iii)* *For* $U = (f,g) \in \mathbb{F}$, $t \geq 0$ *and* $l = \max\{n, m\}$, *we have*

$$\begin{aligned} &\hat{d}_\infty(\mathcal{T}(t)U, U) = \hat{d}_\infty(\mathcal{T}(t)(f,g), (f,g)) \\ &= \hat{d}_\infty\left((e^{nt}f, e^{mt}g), (f,g)\right) \\ &= \max\left\{d_\infty\left(e^{nt}f, f\right), d_\infty\left(e^{mt}g, g\right)\right\} \\ &= \max\{d_\infty\left((e^{nt}-1)f, 0_{(1,0)}\right), d_\infty\left((e^{mt}-1)g, 0_{(1,0)}\right)\} \\ &= \max\left\{(e^{nt}-1)d_\infty\left(f, 0_{(1,0)}\right), (e^{mt}-1)d_\infty\left(g, 0_{(1,0)}\right)\right\} \\ &\leq (e^{lt}-1)\max\left\{d_\infty\left(f, 0_{(1,0)}\right), d_\infty\left(g, 0_{(1,0)}\right)\right\} \\ &= (e^{lt}-1)\hat{d}_\infty\left((f,g), (0_{(1,0)}, 0_{(1,0)})\right) \\ &= (e^{lt}-1)\hat{d}_\infty\left(U, \tilde{0}\right) \to 0 \, as \, t \to 0. \end{aligned}$$

*Then,* $\lim_{t\to 0} \mathcal{T}(t)U = U$.

*(iv)* *For* $U = (f,g)\, V = (f', g') \in \mathbb{F}$ *and* $t \geq 0$, *we have*

$$\begin{aligned} &\hat{d}_\infty\left(\mathcal{T}(t)U\,,\, \mathcal{T}(t)V\right) \\ &= \hat{D}\left(\left(e^{nt}f, e^{mt}g\right)\,,\, \left(e^{nt}f', e^{mt}g'\right)\right) \\ &= \max\left\{d_\infty\left(e^{nt}f, e^{nt}f'\right)\,,\, d_\infty\left(e^{mt}g, e^{mt}g'\right)\right\} \\ &= \max\left\{e^{nt}d_\infty\left(f, f'\right)\,,\, e^{mt}d_\infty\left(g, g'\right)\right\} \\ &\leq e^{lt}\max\left\{d_\infty\left(f, f'\right)\,,\, d_\infty\left(g, g'\right)\right\} \\ &\leq e^{lt}\hat{d}_\infty\left((f,g)\,,\,(f',g')\right) \\ &= e^{lt}\hat{d}_\infty(U, V). \end{aligned}$$

2. *The linear operator* $\mathcal{A}:\ U = (f,g) \to \mathcal{A}U = (nf, mg)$ *is the infinitesimal generator of the semigroup* $\{\mathcal{T}(t), t \geq 0\}$. *Indeed, for* $U \in \mathbb{F}$ *and* $h \geq 0$ *very small, we have*

$$\begin{aligned} \left(\left(e^{nh}-1\right)f\,,\, \left(e^{mh}-1\right)g\right) + U &= \left(\left(e^{nh}-1\right)f + f\,,\, \left(e^{mh}-1\right)g + g\right) \\ &= \left(e^{nh}f\,,\, e^{mh}g\right) \\ &= \mathcal{T}(h)U. \end{aligned}$$

*then the difference* $\mathcal{T}(h)U - U$ *exists and equal* $\left(\left(e^{nh}-1\right)f\,,\, \left(e^{mh}-1\right)g\right)$.
*Therefore, we have*

$$\begin{aligned} \frac{\mathcal{T}(t)U - U}{h} &= \frac{1}{h}\left(\left(e^{nh}-1\right)f\,,\, \left(e^{mh}-1\right)g\right) \\ &= \left(\left(\frac{e^{nh}-1}{h}\right)f\,,\, \left(\frac{e^{mh}-1}{h}\right)g\right). \end{aligned}$$

*Hence,*

$$
\begin{aligned}
&\hat{d}_\infty\left(\frac{\mathcal{T}(t)U-U}{h}\,,\,(nf,mg)\right)\\
&=\hat{d}_\infty\left(\left(\left(\frac{e^{nh}-1}{h}\right)f\,,\,\left(\frac{e^{mh}-1}{h}\right)g\right)\,,\,(nf,mg)\right)\\
&=\max\left\{d_\infty\left(\left(\frac{e^{nh}-1}{h}\right)f\,,\,nf\right)\,,\,d_\infty\left(\left(\frac{e^{mh}-1}{h}\right)g\,,\,mg\right)\right\}\\
&=\max\left\{d_\infty\left(nf+\left(\sum_{k=2}^{\infty}\frac{(nh)^k}{k!}\right)f\,,\,nf\right)\,,\,d_\infty\left(mg+\left(\sum_{k=2}^{\infty}\frac{(mh)^k}{k!}\right)g\,,\,mg\right)\right\}\\
&=\max\left\{d_\infty\left(\left(\sum_{k=2}^{\infty}\frac{(nh)^k}{hk!}\right)f\,,\,0_{(1,0)}\right)\,,\,d_\infty\left(\left(\sum_{k=2}^{\infty}\frac{(mh)^k}{hk!}\right)g\,,\,0_{(1,0)}\right)\right\}\\
&=\max\left\{\left(\sum_{k=2}^{\infty}\frac{(nh)^k}{hk!}\right)d_\infty\left(f\,,\,0_{(1,0)}\right)\,,\,\left(\sum_{k=2}^{\infty}\frac{(mh)^k}{hk!}\right)d_\infty\left(g\,,\,0_{(1,0)}\right)\right\}\\
&\le\left(\sum_{k=2}^{\infty}\frac{(lh)^k}{hk!}\right)\max\left\{d_\infty\left(f\,,\,0_{(1,0)}\right)\,,\,d_\infty\left(g\,,\,0_{(1,0)}\right)\right\}\\
&=\left(\sum_{k=2}^{\infty}\frac{(lh)^k}{hk!}\right)\hat{d}_\infty\left((f\,,\,g)\,,\,\left(0_{(1,0)}\,,\,0_{(1,0)}\right)\right)\\
&=\frac{e^{lh}-1-lh}{h}\hat{d}_\infty\left(U\,,\,\hat{0}\right)\to 0\,\textit{as}\,t\to 0.
\end{aligned}
$$

*Then*

$$\mathcal{A}U=\lim_{h\to 0^+}\frac{\mathcal{T}(h)U-U}{h}=(nf,mg).$$

## 3.2 Complex intuitionistic fuzzy evolution equations

Let $\mathcal{C}(I,\mathbb{F})$ denote the set of all continuous maps from $I$ to $\mathbb{F}$ and let $\hat{d}_\infty^{\mathcal{C}}$ denote a metric on $\mathcal{C}(I,\mathbb{F})$ defined as

$$\hat{d}_\infty^{\mathcal{C}}(\mathcal{F},\mathcal{G})=\sup_{t\in I}\hat{d}_\infty(\mathcal{F}(t),\mathcal{G}(t)),\quad \mathcal{F},\mathcal{G}\in\mathcal{C}(I,\mathbb{F}).$$

It follows that $(\mathcal{C}(I,\mathbb{F}),\hat{d}_\infty^{\mathcal{C}})$ is a complete metric space.

**Definition 9.** *We say that $U:I\to\mathbb{F}$ is a mild solution to the problem (2) if and only if $U\in\mathcal{C}(I,\mathbb{F})$, and for all $t\ge 0$ and $U$ satisfies the integral equation*

$$U(t)=U_0+\int_0^t\mathcal{T}(t-s)\mathcal{H}(s,U(s))ds,\quad t\in I.$$

**Definition 10.** *A mapping $\mathcal{H}:\mathbb{F}\to\mathbb{F}$ is Holder continuous if there exists a constant $L>0$ and a constant $0<\alpha\le 1$ such that*

$$\hat{d}_\infty(\mathcal{H}(X),\mathcal{H}(Y))\le L(\hat{d}_\infty(X,Y))^\alpha,\quad \forall X,Y\in\mathbb{F}.$$

**Definition 11.** *A mapping $\mathcal{H} : I \times \mathbb{F} \to \mathbb{F}$ is Lipschitzian with respect to the second argument if there exists a constant $M > 0$ such that*

$$\hat{d}_\infty(\mathcal{H}(t, X), \mathcal{H}(t, Y)) \leq M\hat{d}_\infty(X, Y), \quad \forall X, Y \in \mathbb{F}, \quad t \geq 0.$$

**Theorem 7.** *Let $\mathcal{H} : I \times \mathbb{F} \to \mathbb{F}$ be Lipschitzian with respect to the second argument with constant $N$, $\mathcal{A}$ is the generator of a strongly continuous semigroup $\{\mathcal{T}(t), t \geq 0\}$ and for each $U_0 \in \mathbb{F}$ such that $\mathcal{T}(t)U_0 \in \mathbb{F}$ for all $t \geq 0$. Then, there exists a unique mild solution to the problem (2) on $I$.*

*Proof.* Define the operator $\mathcal{O}$ on $\mathcal{C}(I, \mathbb{F})$ by

$$\mathcal{O}U(t) = \mathcal{T}(t)U_0 + \int_0^t \mathcal{T}(t-s)\mathcal{H}(s, U(s))ds, \quad t \in I. \tag{36}$$

It is easy to see that $\mathcal{O}$ is well-defined.
For $U, V \in \mathcal{C}(I, \mathbb{F})$ and $t \in I$, we have

$$\begin{aligned}
&\hat{d}_\infty(\mathcal{O}U(t)\,,\, \mathcal{O}V(t)) \\
&= \hat{d}_\infty\left(\mathcal{T}(t)U_0 + \int_0^t \mathcal{T}(t-s)\mathcal{H}(s, U(s))ds\,,\, \mathcal{T}(t)U_0 + \int_0^t \mathcal{T}(t-s)\mathcal{H}(s, V(s)))ds\right) \\
&\leq \hat{d}_\infty\left(\int_0^t \mathcal{T}(t-s)\mathcal{H}(s, U(s))ds\,,\, \int_0^t \mathcal{T}(t-s)\mathcal{H}(s, V(s)))ds\right) \\
&\leq \int_0^t \hat{d}_\infty\left(\mathcal{T}(t-s)\mathcal{H}(s, U(s))\,,\, \mathcal{T}(t-s)\mathcal{H}(s, V(s)))\right) ds \\
&\leq M\int_0^t e^{\omega(t-s)}\hat{d}_\infty\left(\mathcal{H}(s, U(s))\,,\, \mathcal{H}(s, V(s)))\right) ds \\
&\leq MNe^{\omega a}\int_0^t \hat{d}_\infty\left(U(s)\,,\, V(s))\right) ds \\
&\leq tMNe^{\omega a}\hat{d}^{\mathcal{C}}_\infty\left(U\,,\, V)\right).
\end{aligned}$$

By the some way we have

$$\begin{aligned}
\hat{d}_\infty(\mathcal{O}^2U(t)\,,\, \mathcal{O}^2V(t)) &\leq MNe^{\omega a}\int_0^t \hat{d}_\infty\left(\mathcal{O}U(s),\, \mathcal{O}V(s))\right) ds \\
&\leq M^2N^2e^{2\omega a}\hat{d}^{\mathcal{C}}_\infty\left(U\,,\, V\right)\int_0^t sds \\
&= \frac{M^2N^2e^{2\omega a}t^2}{2}\hat{d}^{\mathcal{C}}_\infty\left(U\,,\, V\right).
\end{aligned}$$

Then, we can prove that for all $p \in \mathbb{N}^*$, we have

$$\hat{d}^{\mathcal{C}}_\infty(\mathcal{O}^pU\,,\, \mathcal{O}^pV) \leq \hat{d}_\infty(\mathcal{O}^pU(t)\,,\, \mathcal{O}^pV(t)) \leq \frac{M^pN^pe^{p\omega a}t^2}{p!}\hat{d}^{\mathcal{C}}_\infty\left(U\,,\, V\right).$$

Therefore, there exists $q > 0$ such that $\dfrac{M^qN^qe^{q\omega a}t^2}{q!} < 1$, since

$$\lim_{p \to +\infty} \frac{M^pN^pe^{p\omega a}t^2}{p!} = 0.$$

Then, $\mathcal{O}^q$ is a contraction and, by the Banach fixed point theorem, the operator $\mathcal{O}^q$ has a unique fixed point $X$ such that $\mathcal{O}^q U = U$. Thus by the uniqueness of $U$, $U$ is the unique mild solution of (2) (since $\mathcal{O}^q \mathcal{O} U = \mathcal{O}\mathcal{O}^q U = \mathcal{O}U$). □

**Corollary 1.** *Let $\mathcal{H} : I \times \mathbb{F} \to \mathbb{F}$ Holder continuous with constant $L$, then there exists a unique mild solution to the problem (2) on $I$.*

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