Notes on Intuitionistic Fuzzy Sets
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Intuitionistic fuzzy evolution problem with nonlocal conditions

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Abstract: In this manuscript, we investigate the existence and uniqueness of solutions for the intuitionistic fuzzy evolution problem with non-local conditions, employing a generalized Caputo derivative ${}^{C}_{gH}D^{\gamma}_{0+}$ of order $0 < \gamma < 1$. Our methodology involves utilizing the intuitionistic fuzzy semigroup and the concept of contraction mapping.

Keywords: Generalized intuitionistic fuzzy Caputo derivative, Intuitionistic fuzzy fractional evolution problem, Intuitionistic fuzzy semi-group, Mean-square calculus.
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1 Introduction

The theory of fuzzy subsets, introduced by Lotfi Zadeh in 1965 [16], has become a foundational concept in fuzzy logic. It was not until 1986, however, that Atanassov expanded this notion by introducing the concept of intuitionistic fuzzy subsets [2]. This extension, known as the theory of intuitionistic fuzzy sets, has gained significant importance and has become a critical area for investigating various problems.

In [4], the authors examine the following nonlinear intuitionistic fuzzy fractional evolution problem:

$$\begin{cases} {}_{gH}^{C}D^{q}x(t) = Ax(t) + f(t, x(t)) \\ x(t_0) = x_0 \in IF^1, \end{cases} \tag{1}$$

Additionally, the authors in [11] demonstrated the existence and uniqueness of solutions for the following non-local intuitionistic fuzzy fractional differential equations:

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)) \\ x(0) = x_0 + g(t_1, t_2, \dots, t_p, x(.)), \end{cases} \tag{2}$$

In this paper, we investigate the existence and uniqueness of solutions for a specific intuitionistic fuzzy fractional evolution problem with non-local conditions:

$$\begin{cases} {}_{gH}^{C}D_{0+}^{\gamma}\mathfrak{u}(t) = \mathcal{A}\mathfrak{u}(t) + \mathcal{F}(t, \mathfrak{u}(t)), \quad t \in I = [0, T] \\ \mathfrak{u}(0) = \mathfrak{u}_0 + h(t_1, t_2, \dots, t_n, \mathfrak{u}(.)), \end{cases} \tag{3}$$

where $\mathcal{A} : IF^1 \to IF^1$ is an operator generating an intuitionistic fuzzy semigroup $(\mathcal{S}(t))_{t\geq 0}$ on IF^1, ${}_{gH}^{C}D_{0+}^{\gamma}$ is the generalized Caputo fractional derivative of order $\gamma \in (0, 1)$, $\mathfrak{u}_0 \in IF^1$, and $\mathcal{F} : I \times IF^1 \to IF^1$ and h are given functions.

To address this problem, we will first introduce the foundational concepts and necessary results from intuitionistic fuzzy set theory. We will then explore the generalized Caputo derivative for intuitionistic fuzzy sets and the embedding theorem. Additionally, we will investigate the intuitionistic fuzzy semigroup and introduce relevant notions for the L_2 space in Section 2.

In Section 3, we will present specific assumptions related to our problem, derive the solution formula for Problem 3, and prove both the existence and uniqueness of its solution.

Finally, Section 4 will provide a concise conclusion.

2 Preliminaires

In this part we will try to present all the basic concepts necessary in our study.

Definition 1. [9] *The set of intuitionistic fuzzy numbers is defined by:*

$$IF^1 = IF(\mathbb{R}) = \{\langle \mathfrak{u}, \mathfrak{v} \rangle : \mathbb{R} \longrightarrow [0, 1]^2 \ , \ 0 \leq \mathfrak{u} + \mathfrak{v} \leq 1\},$$

and it checks the following properties:

1) For all $\langle \mathfrak{u}, \mathfrak{v} \rangle \in IF^1$ *is normal, i.e.: There exist* $a, b \in \mathbb{R}$ *such that:* $\mathfrak{u}(a) = 1$ *and* $\mathfrak{v}(b) = 1$.

2) For all $\langle \mathfrak{u}, \mathfrak{v} \rangle \in IF^1$ *is intuitionistic convex, that is to say:*

- $\mathfrak{u}$ *is fuzzy convex:* $\mathfrak{u}(\lambda a + (1-\lambda)b) \geq \min\{\mathfrak{u}(a), \mathfrak{u}(b)\}$, $\forall a, b \in \mathbb{R}$, $\forall \lambda \in [0, 1]$.
- $\mathfrak{v}$ *is fuzzy concave:* $\mathfrak{v}(\lambda a + (1-\lambda)b) \leq \max\{\mathfrak{v}(a), \mathfrak{v}(b)\}$, $\forall a, b \in \mathbb{R}$, $\forall \lambda \in [0, 1]$.

3) For all $\langle \mathfrak{u}, \mathfrak{v} \rangle \in IF^1$, $\mathfrak{u}$ *is lower continuous and* $\mathfrak{v}$ *is upper continuous.*

4) $supp\langle \mathfrak{u}, \mathfrak{v} \rangle = \overline{\{a \in \mathbb{R} \ , \ \mathfrak{v}(a) < 1\}}$ *is bounded.*

And we define zero intuitionistic fuzzy set by:

$$\tilde{0}(a) = \begin{cases} (1, 0) \ ; \ a = 0, \\ (0, 1) \ ; \ a \neq 0 \end{cases}$$

Definition 2. [9] *For* $\alpha \in [0, 1]$, *we define the upper and lower* α*-cut as follows:*

$$[\langle \mathfrak{u}, \mathfrak{v} \rangle]_\alpha = \{a \in \mathbb{R} \ , \ \mathfrak{u}(a) \geq \alpha\}.$$
$$[\langle \mathfrak{u}, \mathfrak{v} \rangle]^\alpha = \{a \in \mathbb{R} \ , \ \mathfrak{v}(a) \leq 1 - \alpha\}.$$

And we can write:

$$[\langle \mathfrak{u}, \mathfrak{v} \rangle]_\alpha = \left[[\langle \mathfrak{u}, \mathfrak{v} \rangle]_l^+ (\alpha), [\langle \mathfrak{u}, \mathfrak{v} \rangle]_r^+ (\alpha) \right]$$

and

$$[\langle \mathfrak{u}, \mathfrak{v} \rangle]^\alpha = \left[[\langle \mathfrak{u}, \mathfrak{v} \rangle]_l^- (\alpha), [\langle \mathfrak{u}, \mathfrak{v} \rangle]_r^- (\alpha) \right] .$$

With,

$$[\langle \mathfrak{u}, \mathfrak{v} \rangle]_l^+ (\alpha) = \inf\{x \in \mathbb{R}^n \ ; \ \mathfrak{u}(x) \geq \alpha\},$$
$$[\langle \mathfrak{u}, \mathfrak{v} \rangle]_r^+ (\alpha) = \sup\{x \in \mathbb{R}^n \ ; \ \mathfrak{u}(x) \geq \alpha\},$$
$$[\langle \mathfrak{u}, \mathfrak{v} \rangle]_l^- (\alpha) = \inf\{x \in \mathbb{R}^n \ ; \ \mathfrak{v}(x) \leq 1 - \alpha\},$$
$$[\langle \mathfrak{u}, \mathfrak{v} \rangle]_r^- (\alpha) = \sup\{x \in \mathbb{R}^n \ ; \ \mathfrak{v}(x) \leq 1 - \alpha\}.$$

Proposition 1. *[9] Let* $\langle \mathfrak{u}_1, \mathfrak{v}_1 \rangle, \langle \mathfrak{u}_2, \mathfrak{v}_2 \rangle \in IF^1$, *we have :*

1) $\langle \mathfrak{u}_1, \mathfrak{v}_1 \rangle = \langle \mathfrak{u}_2, \mathfrak{v}_2 \rangle \Leftrightarrow [\langle \mathfrak{u}_1, \mathfrak{v}_1 \rangle]_\alpha = [\langle \mathfrak{u}_2, \mathfrak{v}_2 \rangle]_\alpha$, $[\langle \mathfrak{u}_1, \mathfrak{v}_1 \rangle]^\alpha = [\langle \mathfrak{u}_2, \mathfrak{v}_2 \rangle]^\alpha$, $\forall \alpha \in [0, 1]$.

2) $(\langle \mathfrak{u}_1, \mathfrak{v}_1 \rangle \oplus \langle \mathfrak{u}_2, \mathfrak{v}_2 \rangle)(x) = \left(\sup_{z=x+y} \min(\mathfrak{u}_1(x), \mathfrak{u}_2(y)), \inf_{z=x+y} \max(\mathfrak{v}_1(x), \mathfrak{v}_2(y)) \right),$

and according to the extension of Zadeh, we have:

$$[\langle \mathfrak{u}_1, \mathfrak{v}_1 \rangle \oplus \langle \mathfrak{u}_2, \mathfrak{v}_2 \rangle]_\alpha = [\langle \mathfrak{u}_1, \mathfrak{v}_1 \rangle]_\alpha + [\langle \mathfrak{u}_2, \mathfrak{v}_2 \rangle]_\alpha ,$$
$$[\langle \mathfrak{u}_1, \mathfrak{v}_1 \rangle \oplus \langle \mathfrak{u}_2, \mathfrak{v}_2 \rangle]^\alpha = [\langle \mathfrak{u}_1, \mathfrak{v}_1 \rangle]^\alpha + [\langle \mathfrak{u}_2, \mathfrak{v}_2 \rangle]^\alpha .$$

3) $\lambda\langle \mathfrak{u}_1, \mathfrak{v}_1\rangle = \langle \lambda\mathfrak{u}_1, \lambda\mathfrak{v}_1\rangle$, $\forall\lambda \in \mathbb{R}$,

and according to the extension of Zadeh, we have:

$$[\lambda\langle \mathfrak{u}_1, \mathfrak{v}_1\rangle]_\alpha = \lambda\,[\langle \mathfrak{u}_1, \mathfrak{v}_1\rangle]_\alpha\,,$$
$$[\lambda\langle \mathfrak{u}_1, \mathfrak{v}_1\rangle]^\alpha = \lambda\,[\langle \mathfrak{u}_1, \mathfrak{v}_1\rangle]^\alpha\,.$$

If $\lambda = 0$*, then* $\lambda\langle \mathfrak{u}_1, \mathfrak{v}_1\rangle = \tilde{0}$.

Theorem 2. [9] *Let* $\mathcal{M} = \{\mathcal{M}_\alpha, \mathcal{M}^\alpha, \alpha \in [0,1]\}$ *be the family of subsets of* $\mathbb{R}$*, for which the following properties are verified:*

1) $\mathcal{M}_\alpha \subset \mathcal{M}^\alpha$ *for all* $\alpha \in [0,1]$.

2) $\alpha \leq \beta \Rightarrow \mathcal{M}_\beta \subset \mathcal{M}_\alpha$ *and* $\mathcal{M}^\beta \subset \mathcal{M}^\alpha$ *for all* $\alpha, \beta \in [0,1]$.

3) $\mathcal{M}_\alpha$ *and* $\mathcal{M}^\alpha$ *are two non-empty compact convex subsets in* $\mathbb{R}$ *for all* $\alpha \in [0,1]$.

4) For all non-decreansing sequences $\alpha_i \to \alpha$ *on* $[0,1]$*, we have:*

$$\mathcal{M}_\alpha = \cap_i \mathcal{M}_{\alpha_i}\ ,\ \mathcal{M}^\alpha = \cap_i \mathcal{M}^{\alpha_i}.$$

Then, we define $\mathfrak{u}$ *and* $\mathfrak{v}$ *by:*

$$\mathfrak{u}(a) = \begin{cases} 0 & \text{if } a \notin \mathcal{M}_0, \\ \sup\{\alpha \in [0,1] : a \in \mathcal{M}_\alpha\} & \text{if } a \in \mathcal{M}_0, \end{cases}$$
$$\mathfrak{v}(a) = \begin{cases} 1 & \text{if } a \notin \mathcal{M}^0, \\ 1 - \sup\{\alpha \in [0,1] : a \in \mathcal{M}^\alpha\} & \text{if } a \in \mathcal{M}^0. \end{cases}$$

Therefore, $\langle \mathfrak{u}, \mathfrak{v}\rangle \in IF^1$, $\mathcal{M}_\alpha = [\langle \mathfrak{u}, \mathfrak{v}\rangle]_\alpha$ *and* $\mathcal{M}^\alpha = [\langle \mathfrak{u}, \mathfrak{v}\rangle]^\alpha$.

Remark 1. *i) The family* $\{[\langle \mathfrak{u}, \mathfrak{v}\rangle]_\alpha, [\langle \mathfrak{u}, \mathfrak{v}\rangle]^\alpha\ ,\ \alpha \in [0,1]\}$ *satisfies the previous properties of Theorem 2.*

ii) For all $\alpha \in [0,1]$ *we have:*

$$[\langle \mathfrak{u}, \mathfrak{v}\rangle]_\alpha \subset [\langle \mathfrak{u}, \mathfrak{v}\rangle]^\alpha\,.$$

Definition 3. [9] *Let* $\langle \mathfrak{u}_1, \mathfrak{v}_1\rangle, \langle \mathfrak{u}_2, \mathfrak{v}_2\rangle \in IF^1$*, we define the following two distances on* IF^1*:*

$$\begin{aligned} d_\infty\left(\langle \mathfrak{u}_1, \mathfrak{v}_1\rangle, \langle \mathfrak{u}_2, \mathfrak{v}_2\rangle\right) = &\frac{1}{4}\sup_{\alpha\in(0,1]} \mid [\langle \mathfrak{u}_1, \mathfrak{v}_1\rangle]_r^+(\alpha) - [\langle \mathfrak{u}_2, \mathfrak{v}_2\rangle]_r^+(\alpha) \mid \\ &+ \frac{1}{4}\sup_{\alpha\in(0,1]} \mid [\langle \mathfrak{u}_1, \mathfrak{v}_1\rangle]_l^+(\alpha) - [\langle \mathfrak{u}_2, \mathfrak{v}_2\rangle]_l^+(\alpha) \mid \\ &+ \frac{1}{4}\sup_{\alpha\in(0,1]} \mid [\langle \mathfrak{u}_1, \mathfrak{v}_1\rangle]_r^-(\alpha) - [\langle \mathfrak{u}_2, \mathfrak{v}_2\rangle]_r^-(\alpha) \mid \\ &+ \frac{1}{4}\sup_{\alpha\in(0,1]} \mid [\langle \mathfrak{u}_1, \mathfrak{v}_1\rangle]_l^-(\alpha) - [\langle \mathfrak{u}_2, \mathfrak{v}_2\rangle]_l^-(\alpha) \mid, \end{aligned}$$

and, for $p \in [0; +\infty[$ *we have:*

$$
\begin{aligned}
d_p\left(\langle \mathfrak{u}_1,\mathfrak{v}_1\rangle,\langle \mathfrak{u}_2,\mathfrak{v}_2\rangle\right) = (\ & \frac{1}{4}\int_0^1 \mid [\langle \mathfrak{u}_1,\mathfrak{v}_1\rangle]_r^+(\alpha) - [\langle \mathfrak{u}_2,\mathfrak{v}_2\rangle]_r^+(\alpha) \mid^p d\alpha \\
& + \frac{1}{4}\int_0^1 \mid [\langle \mathfrak{u}_1,\mathfrak{v}_1\rangle]_l^+(\alpha) - [\langle \mathfrak{u}_2,\mathfrak{v}_2\rangle]_l^+(\alpha) \mid^p d\alpha \\
& + \frac{1}{4}\int_0^1 \mid [\langle \mathfrak{u}_1,\mathfrak{v}_1\rangle]_r^-(\alpha) - [\langle \mathfrak{u}_2,\mathfrak{v}_2\rangle]_r^-(\alpha) \mid^p d\alpha \\
& + \frac{1}{4}\int_0^1 \mid [\langle \mathfrak{u}_1,\mathfrak{v}_1\rangle]_l^-(\alpha) - [\langle \mathfrak{u}_2,\mathfrak{v}_2\rangle]_l^-(\alpha) \mid^p d\alpha)^{\frac{1}{p}}\ ),
\end{aligned}
$$

then, (IF^1, d_p) *is a complete metric space.*

Now, we move on to define the Hukuhara difference between two intuitionistic fuzzy numbers.

Definition 4. [10] *Let* $\langle \mathfrak{u}_1,\mathfrak{v}_1\rangle, \langle \mathfrak{u}_2,\mathfrak{v}_2\rangle \in IF^1$*, then the generalized difference of Hukuhara between two numbers of* IF^1 *is defined by:*

$$\langle \mathfrak{u}_1,\mathfrak{v}_1\rangle \ominus_{gH} \langle \mathfrak{u}_2,\mathfrak{v}_2\rangle = \langle \mathfrak{u}_3,\mathfrak{v}_3\rangle \ \Leftrightarrow\ \langle \mathfrak{u}_1,\mathfrak{v}_1\rangle = \langle \mathfrak{u}_2,\mathfrak{v}_2\rangle \oplus \langle \mathfrak{u}_3,\mathfrak{v}_3\rangle.$$

For what follows, let $\mathfrak{g} : [0,T] \longrightarrow IF^1$ be a function with an intuitionistic fuzzy value, then its representation $\alpha-$level is:

$$
\begin{aligned}
[\mathfrak{g}]_\alpha &= [\mathfrak{g}_{\alpha,l}, \mathfrak{g}_{\alpha,r}], \\
[\mathfrak{g}]^\alpha &= \left[\mathfrak{g}^{\alpha,l}, \mathfrak{g}^{\alpha,r}\right],
\end{aligned}
$$

with, $\mathfrak{g}_{\alpha,l}$, $\mathfrak{g}_{\alpha,r}$, $\mathfrak{g}^{\alpha,l}$ and $\mathfrak{g}^{\alpha,r}$ are the bounds (left and right) of the function at the $\alpha-$level.

Definition 5. [10] *Let* $\mathfrak{g} : [0,T] \longrightarrow IF^1$*, then the generalized Hukuhara derivative of* $\mathfrak{g}$ *at* t_0 *is defined by:*

$$\mathfrak{g}^{'}_{gH}(t_0) = \lim_{t\to t_0} \frac{\mathfrak{g}(t) \ominus_{gH} \mathfrak{g}(t_0)}{t - t_0},$$

if $\mathfrak{g}^{'}_{gH}(t_0) \in IF^1$*, and we say that* $\mathfrak{g}$ *is generalized Hukuhara differentiable* ($gH-$*differentiable*) *at* t_0*.*

Therefore, if $\mathfrak{g}_{\alpha,l}$, $\mathfrak{g}_{\alpha,r}$, $\mathfrak{g}^{\alpha,l}$ and $\mathfrak{g}^{\alpha,r}$ should be continuous functions at t_0 and should be differentiable at t_0, and their derivatives should exist, we can separate two types of $gH-$differentiability for a function with value in IF^1. We say that $\mathfrak{g}$ is $[(i) - gH]-$differentiable at t_0 if:

$$
\begin{aligned}
\left[\mathfrak{g}^{'}_{gH}\right]_\alpha &= \left[(\mathfrak{g}_{\alpha,l})^{'}, (\mathfrak{g}_{\alpha,r})^{'}\right], \\
\left[\mathfrak{g}^{'}_{gH}\right]^\alpha &= \left[\left(\mathfrak{g}^{\alpha,l}\right)^{'}, \left(\mathfrak{g}^{\alpha,r}\right)^{'}\right].
\end{aligned}
$$

We say that $\mathfrak{g}$ is $[(ii) - gH]-$differentiable at t_0 if:

$$
\begin{aligned}
\left[\mathfrak{g}^{'}_{gH}\right]_\alpha &= \left[(\mathfrak{g}_{\alpha,r})^{'}, (\mathfrak{g}_{\alpha,l})^{'}\right], \\
\left[\mathfrak{g}^{'}_{gH}\right]^\alpha &= \left[\left(\mathfrak{g}^{\alpha,r}\right)^{'}, \left(\mathfrak{g}^{\alpha,l}\right)^{'}\right].
\end{aligned}
$$

Remark 2. *We can define the generalized derivative of higher order by:*

$$\begin{cases} \mathfrak{g}^0 = \mathfrak{g}, \\ \mathfrak{g}_{gH}^{(n)} = (\mathfrak{g}_{gH}^{(n-1)})'_{gH}. \end{cases}$$

Definition 6. [10] *Let* $\mathfrak{g} : [0,T] \longrightarrow IF^1$*, we say that* $\mathfrak{g}$ *is of class* $\mathcal{C}^m$*,* $m \in \mathbb{N}$ *if* $\mathfrak{g}_{gH}^{(m)}$ *exists and continues, with respect to metric* d_∞*.*

If $\mathfrak{g}_{\alpha,l}, \mathfrak{g}_{\alpha,r}, \mathfrak{g}^{\alpha,l}$and $\mathfrak{g}^{\alpha,r}$ are Riemann integrable on $[0,T]$, then,

$$\int_{[0,T]} \mathfrak{g} = \left\{ \left[\int_{[0,T]} \mathfrak{g}_{\alpha,l}, \int_{[0,T]} \mathfrak{g}_{\alpha,r} \right], \left[\int_{[0,T]} \mathfrak{g}^{\alpha,l}, \int_{[0,T]} \mathfrak{g}^{\alpha,r} \right] \right\}.$$

Definition 7. [10] *Let* $\mathfrak{g} : [0,T] \longrightarrow IF^1$*, we say that* $\mathfrak{g}$ *is integrable on* $[0,T]$*, if* $\mathfrak{g}_{\alpha,l}, \mathfrak{g}_{\alpha,r}, \mathfrak{g}^{\alpha,l}$ *and* $\mathfrak{g}^{\alpha,r}$ *are integrable on* $[0,T]$*.*

2.1 Intuitionistic fuzzy generalized Caputo derivative

Let $\mathfrak{g} : [0,T] \longrightarrow IF^1$ be an intuitionistic fuzzy-valued integrable function on $[0,T]$, and let $\gamma \in (n-1, n]$ with $n \in \mathbb{N}^*$. Then its $\alpha-$levels are defined by:

$$[\mathfrak{g}]_\alpha = [\mathfrak{g}_{\alpha,l}, \mathfrak{g}_{\alpha,r}],$$
$$[\mathfrak{g}]^\alpha = \left[\mathfrak{g}^{\alpha,l}, \mathfrak{g}^{\alpha,r}\right],$$

where, $\mathfrak{g}_{\alpha,l}, \mathfrak{g}_{\alpha,r}, \mathfrak{g}^{\alpha,l}, \mathfrak{g}^{\alpha,r} \in \mathcal{C}^n([O,T])$.
Then,

$$\mathcal{N}_\alpha = \left[\frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-n-1} (\mathfrak{g}_{\alpha,l})^{(n)}(s), \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-n-1} (\mathfrak{g}_{\alpha,r})^{(n)}(s) \right]$$

and

$$\mathcal{N}^\alpha = \left[\frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-n-1} (\mathfrak{g}^{\alpha,l})^{(n)}(s), \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-n-1} (\mathfrak{g}^{\alpha,r})^{(n)}(s) \right].$$

Proposition 3. *The family* $\{\mathcal{N}_\alpha, \mathcal{N}^\alpha, \alpha \in (0,1)\}$ *defines an intuitionistic fuzzy element.*

Definition 8. *The intuitionistic fuzzy preceding item is called the generalized Caputo derivative of* $\mathfrak{g}$*, we denote it by* ${}^C D_{0+}^\gamma \mathfrak{g}$*. We say that* $\mathfrak{g}$ *is* ${}^{cf}[(i)-gH]-$*differentiable at* t_0 *if:*

$$\left[{}_{gH}^{C} D_{0+}^\gamma \mathfrak{g}\right]_\alpha = \left[{}^C D_{0+}^\gamma \mathfrak{g}_{\alpha,l}, {}^C D_{0+}^\gamma \mathfrak{g}_{\alpha,r}\right],$$

$$\left[{}_{gH}^{C} D_{0+}^\gamma \mathfrak{g}\right]^\alpha = \left[{}^C D_{0+}^\gamma \mathfrak{g}^{\alpha,l}, {}^C D_{0+}^\gamma \mathfrak{g}^{\alpha,r}\right],$$

and that $\mathfrak{g}$ *is* ${}^{cf}[(ii)-gH]-$*differentiable at* t_0 *if:*

$$\left[{}_{gH}^{C} D_{0+}^\gamma \mathfrak{g}\right]_\alpha = \left[{}^C D_{0+}^\gamma \mathfrak{g}_{\alpha,r}, {}^C D_{0+}^\gamma \mathfrak{g}_{\alpha,l}\right],$$

$$\left[{}_{gH}^{C} D_{0+}^\gamma \mathfrak{g}\right]^\alpha = \left[{}^C D_{0+}^\gamma \mathfrak{g}^{\alpha,r}, {}^C D_{0+}^\gamma \mathfrak{g}^{\alpha,l}\right].$$

As in the previuos definition, we will give the difinition of intuitionistic fuzzy fractional Riemann–Liouville integral. If the $\alpha-$levels of $\mathfrak{g} : [0, T] \longrightarrow IF^1$ are given by:

$$[\mathfrak{g}]_\alpha = [\mathfrak{g}_{\alpha,l}, \mathfrak{g}_{\alpha,r}]\,,$$
$$[\mathfrak{g}]^\alpha = \left[\mathfrak{g}^{\alpha,l}, \mathfrak{g}^{\alpha,r}\right].$$

and $\mathfrak{g}_{\alpha,l}, \mathfrak{g}_{\alpha,r}, \mathfrak{g}^{\alpha,l}, \mathfrak{g}^{\alpha,r}$ are Riemann integrable on $(0, T]$. Since the family:

$$\left\{[\mathfrak{g}]_\alpha = [\mathfrak{g}_{\alpha,l}, \mathfrak{g}_{\alpha,r}]\,, [\mathfrak{g}]^\alpha = \left[\mathfrak{g}^{\alpha,l}, \mathfrak{g}^{\alpha,r}\right]\right\}$$

builds an intuitionistic element and the integral preserves the monotony, then the family $\{\mathcal{A}_\alpha, \mathcal{A}^\alpha, \alpha \in (0, 1)\}$, where:

$$\mathcal{A}_\alpha = \left[\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\mathfrak{g}_{\alpha,l}(s), \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\mathfrak{g}_{\alpha,r}(s)\right],$$

and

$$\mathcal{A}^\alpha = \left[\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\mathfrak{g}^{\alpha,l}(s), \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\mathfrak{g}^{\alpha,r}(s)\right],$$

defines an intuitionistic fuzzy element, which is the Riemann–Liouville fractional integral of $\mathfrak{g}$ on $(0, T)$, which we denote: $\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\mathfrak{g}(s)$.

Definition 9. *The Riemann–Liouville fractional integral of* $\mathfrak{g}$ *on* $(0, T)$ *is defined as:*

$${}_{gH}I^{\gamma}_{0^+}\mathfrak{g}(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\mathfrak{g}(s)ds$$

with, $\gamma \in (n-1, n)$.

2.2 Embedding theorem and intuitionistic fuzzy $\alpha-$semigroup

Since the elements of IF^1 are closed (Hausdorff topology) and convex, so we can apply the result of [15].

Theorem 4. *We can extend* IF^1 *in a normed space.*

Proof. See [4]. □

Theorem 5. *There exists a Banach space* X *such that* IF^1 *can be embedded as a convex cone* $\mathcal{C}$ *with vertex* 0 *in* X*. Furthermore, the following conditions hold true:*

1. *The embedding* j *is isometric,*
2. *The addition in* X *induces the addition in* IF^1*,*
3. *The multiplication by a non-negative real number in* X *induces the corresponding operation in* IF^1*,*
4. $\mathcal{C} - \mathcal{C} = \{a - b, a, b \in \mathcal{C}\}$ *is dense in* X*,*
5. $\mathcal{C}$ *is closed.*

Definition 10. *A continuous one-parameter intuitionistic fuzzy* $\alpha-$*semigroup* $\{\mathcal{T}_\alpha(t), t \geq 0\}$ *of operators on* IF^1 *is defined by the following conditions:*

1. *For any fixed* $t \geq 0$, $\mathcal{T}_\alpha(t)$ *is a continuous operator defined on* IF^1 *into* IF^1.
2. *For any* $\langle \mathfrak{u}, \mathfrak{v} \rangle$, $\mathcal{T}_\alpha(t) \langle \mathfrak{u}, \mathfrak{v} \rangle$ *is strongly continuous in* t*, with the metric* d_1.
3. $\mathcal{T}_\alpha \left((t+s)^{\frac{1}{\alpha}} \right) = \mathcal{T}_\alpha \left((t)^{\frac{1}{\alpha}} \right) \mathcal{T}_\alpha \left((s)^{\frac{1}{\alpha}} \right)$.
4. *For all* $\langle \mathfrak{u}, \mathfrak{v} \rangle, \langle \mathfrak{x}, \mathfrak{y} \rangle \in IF^1$ *we have:*
$$d_1 \left(\mathcal{T}_\alpha(t) \langle \mathfrak{u}, \mathfrak{v} \rangle, \mathcal{T}_\alpha(t) \langle \mathfrak{x}, \mathfrak{y} \rangle \right) \leq M e^{wt^\alpha} \ , \ \forall t \geq 0 \ , \ M > 0.$$

We call such a family $\mathcal{T}_\alpha(t)$ simply intuitionistic fuzzy $\alpha-$semigroup of type w. The strict $\alpha-$infinitesimal generator A_α of an intuitionistic fuzzy $\alpha-$semigroup $\mathcal{T}_\alpha(t)$ is defined by:
$$A_\alpha x = \lim_{t \to 0} \mathcal{T}_\alpha^{(\alpha)}(t) \langle \mathfrak{u}, \mathfrak{v} \rangle , \langle \mathfrak{u}, \mathfrak{v} \rangle \in IF^1.$$
The right side exists in IF^1.
We define the domain of A_α, by:
$$D(A_\alpha) = \left\{ \langle \mathfrak{u}, \mathfrak{v} \rangle \in IF^1, \lim_{t \to 0} \mathcal{T}_\alpha^{(\alpha)}(t) \langle \mathfrak{u}, \mathfrak{v} \rangle \text{ exist} \right\}.$$

Lemma 1. *If the family* $\{\mathcal{T}_\alpha(t), t \geq 0\}$ *is an intuitionistic fuzzy* $\alpha-$*semigroup of type* w*, then* $j\, \mathcal{T}_\alpha(t)\, j^{-1}$ *is a nonlinear* $\alpha-$*semigroup of type* w *on* $\mathcal{C}$.

Lemma 2. *If* A_α *is an intuitionistic fuzzy infinitesimal generator of an intuitionistic fuzzy* $\alpha-$*semigroup of type* $w\{\mathcal{T}_\alpha(t)\}_{t \geq 0}$. *Then* $j\, A_\alpha\, j^{-1}$ *is the infinitesimal generator of* $j\, \mathcal{T}_\alpha(t)\, j^{-1}$.

Proof. See [4]. □

2.3 The L_2-space

Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a complete probability space.

Definition 11. (cf. [14, 17]) *An intuitionistic fuzzy random variable* * *(IFRV for short) is a Borel measurable function* $X : (\Omega, \mathcal{A}) \longrightarrow (IF^1, d_\infty)$.

The norm $\| \ \|$ of an intuitionistic fuzzy number $(\mathfrak{u}, \mathfrak{v}) \in IF^1$ is defined by:
$$\| (\mathfrak{u}, \mathfrak{v}) \| = d_\infty \left((\mathfrak{u}, \mathfrak{v}), 0_{(1,0)} \right) = \| [(\mathfrak{u}, \mathfrak{v})]_0 \| = \frac{1}{2} \sup_{a \in [(\mathfrak{u}, \mathfrak{v})]_0} | a | + \frac{1}{2} \inf_{b \in [(\mathfrak{u}, \mathfrak{v})]^0} | b | .$$
If $E \| X \| < \infty$, then the expected value EX exists. X is called a second-order IFRV, provided $E \| X \|^2 < \infty$. Let,
$$L(\Omega, \mathcal{A}, \mathcal{P}) = \left\{ X \ | \ X \text{ is an IFRV with } \int_\Omega d_\infty(X, 0_{(1,0)})^2 d\mathcal{P}(\omega) < \infty \right\}.$$

* The concept of "intuitionistic fuzzy random variable" has already been researched, though in a completely different sense, by Parvathi and Radhika in [14], and another approach to the concept is also discussed by Zainali, Akbari and Noughabi in [17].

The family of all second-order IFRVs is denoted by $L_2(IF^1)$ (L_2 for short). Any two IFRVs X and Y are called equivalent if $\mathcal{P}(X \neq Y) = 0$. All of the equivalent elements in L_2 are identified. Define:

$$\varphi(X,Y) = \left(\int_\Omega d_\infty(X,Y)^2 d\mathcal{P}\right)^{\frac{1}{2}}, \; X,Y \in L_2.$$

The norm $\| X \|_2$ of an element $X \in L_2$ is defined by:

$$\| X \|_2 = \varphi(X, 0_{(1,0)}) = \left(\int_\Omega d_\infty(X, 0_{(1,0)})^2 d\mathcal{P}\right)^{\frac{1}{2}}.$$

Proposition 6. *(L_2, φ) is a complete metric space.*

In addition φ satisfies that, for any $X, Y, Z \in L_2$, $\lambda, k \in \mathbb{R}$:

1) $\varphi(X+Z, Y+Z) = \varphi(X,Y)$.

2) $\varphi(\lambda X, \lambda Y) = | \lambda | \varphi(X,Y)$.

3) $\varphi(\lambda X, kX) \leq | \lambda - k | \| X \|_2$.

Definition 12. *Let $(X_n)_{n\geq 1}$ be a sequence in L_2, we call that X_n converges in mean square, or ms-converges, to X as $n \to \infty$, if $\varphi(X_n, X) \to 0$, and we write $X_n \to^{m.s} X$ or $\lim_{n\to\infty} X_n = X$.*

Definition 13. *Let T be a finite or an infinite interval in $\mathbb{R}$. A mapping $X : T \longrightarrow L_2$ is called a second-order intuitionistic fuzzy stochastic process (IFSP for short). If X is continuous at a $t \in T$ with respect to the metric φ, then we call X continuous in mean square or ms-continuous at t. If X is ms-continuous at every $t \in T$, then we call X ms-continuous.*

3 Nonlocal intuitionistic fuzzy evolution problem

In this part, we are interested in studying the existence and uniqueness of the solution to the following problem:

$$\begin{cases} {}^C_{gH}D^\gamma_{0^+}\mathfrak{u}(t) = \mathcal{A}\mathfrak{u}(t) + \mathcal{F}(t, \mathfrak{u}(t)), & t \in I = [0,T] \\ \mathfrak{u}(0) = \mathfrak{u}_0 + h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.)), \end{cases} \tag{4}$$

with $0 < t_1 < t_2 < \cdots < t_n < T$.
We assume the following hypotheses:

(H_1) $\mathcal{A} : D(\mathcal{A}) \subset IF^1 \longrightarrow IF^1$ is an operator that generates a strongly continuous semigroup $(\mathcal{S}(t))_{t\geq 0}$. There are two constants $\mathcal{M}$ and $\omega \in \mathbb{R}^*_+$ such that:

$$\varphi(\mathcal{S}(t)\mathfrak{u}, \mathcal{S}(t)\mathfrak{v}) \leq \mathcal{M}e^{\omega t}\varphi(\mathfrak{u}, \mathfrak{v}), \; \forall t \geq 0, \; \mathfrak{u}, \mathfrak{v} \in L_2 \cap D(\mathcal{A}).$$

(H_2) $\mathcal{F} : I \times L_2 \longrightarrow L_2$ is an ms-continuous intuitionistic fuzzy mapping with respect to t, which satisfies a generalized Lipschitz condition, i.e., there exists a constant $\mathcal{M}_F$ such that:

$$\varphi(\mathcal{F}(t, \mathfrak{u}), \mathcal{F}(t, \mathfrak{v})) \leq \mathcal{M}_F\varphi(\mathfrak{u}, \mathfrak{v}), \; \forall t \geq 0, \; \mathfrak{u}, \mathfrak{v} \in L_2.$$

(H_3) $h : I^n \times L_2 \longrightarrow L_2$ satisfies a generalized Lipschitz condition, i.e., there exists a constant $\mathcal{M}_h$ such that:

$$\varphi(h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.)), h(t_1, t_2, \ldots, t_n, \mathfrak{v}(.))) \leq \mathcal{M}_h \varphi(\mathfrak{u}, \mathfrak{v}) \ , \ \forall t \in I \ , \ \mathfrak{u}, \mathfrak{v} \in L_2.$$

Lemma 3. *Let $\mathcal{F} : [a, b] \longrightarrow IF^1$ be a fuzzy-valued function such that $\mathcal{F}'_{gH} \in C^{IF^1}([a, b]) \cap L^{IF^1}([a, b])$, then:*

$$_{gH}I^{\gamma}_{0+} \left({}^{C}_{gH}D^{\gamma}_{0+}\mathcal{F} \right)(t) = \mathcal{F}(t) \ominus_{gH} \mathcal{F}(a).$$

We denote:

$$sgn(\mathfrak{u}) = \begin{cases} +, & \text{if } \mathfrak{u} \text{ is } {}^{cf}[(i) - gH]\text{−differentiable,} \\ \ominus(-1), & \text{if } \mathfrak{u} \text{ is } {}^{cf}[(ii) - gH]\text{−differentiable.} \end{cases}$$

Definition 14. *Let $\mathfrak{u} : I \longrightarrow L_2$.*

1. *If $\mathfrak{u}$ is ${}^{cf}[(i)-gH]$−differentiable, then the problem (4) is equivalent to the integral equation:*

$$\mathfrak{u}(t) = \mathcal{S}(t) \left[\mathfrak{u}_0 + h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.)) \right] + \frac{1}{\Gamma(\gamma)} \int_0^t (t - s)^{\gamma - 1} \mathcal{F}(s, \mathfrak{u}(s)) ds.$$

2. *If $\mathfrak{u}$ is ${}^{cf}[(ii) - gH]$−differentiable, then the problem (4) is equivalent to the integral equation:*

$$\mathfrak{u}(t) = \mathcal{S}(t) \left[\mathfrak{u}_0 + h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.)) \right] \ominus \frac{-1}{\Gamma(\gamma)} \int_0^t (t - s)^{\gamma - 1} \mathcal{F}(s, \mathfrak{u}(s)) ds.$$

Theorem 7. [6] *Consider $\mathfrak{V} : T \longrightarrow X$ to be a set of continuous functions. Then $\mathfrak{V}$ is a relative compact set if and only if $\mathfrak{V}$ is equicontinuous and for any $t \in T$, $\mathfrak{V}(t)$ is a relative compact set in X.*

Theorem 8. [6] *Let $\mathfrak{V}$ be a closed convex subset of a Banach space X. If $A : \mathfrak{V} \longrightarrow \mathfrak{V}$ is continuous and $\mathfrak{V} = A(\mathfrak{V})$ is compact, then A has a fixed point in $\mathfrak{V}$.*

Theorem 9. *Suppose that the hypotheses (H_1)–(H_3) are verified, then problem (4) admits a unique solution in $[0, \theta]$ with:*

$$\theta = \min \left\{ T, \frac{1}{\omega} \log \left(\frac{\sigma - \beta}{\mathcal{M} K_h} + \frac{K_F}{\Gamma(\gamma + 1)} \right), \frac{1}{\omega} \log \left(\frac{1}{\mathcal{M}\mathcal{M}_h} + \frac{\mathcal{M}_F}{\Gamma(\gamma + 1)} \right) \right\}.$$

With,

$$\varphi(\mathcal{F}(t,), 0_{(1,0)}) \leq K_F,$$

$$\varphi(h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.)), 0_{(1,0)}) \leq K_h.$$

Proof. Just before commencing the demonstration, we give:

$$\mathcal{H}(\mathfrak{u}, \mathfrak{v}) = \sup_{0 \leq t \leq \theta} \varphi(\mathfrak{u}(t), \mathfrak{v}(t)).$$

Let $\mathcal{B} = \{\mathfrak{u} \in L_2, \mathcal{H}(\mathfrak{u}, \mathfrak{u}_0) \leq \sigma\}$ be the space of mean-square-continuous fuzzy intuitionistic applications.
Let $\mathfrak{R} : \mathcal{B} \longrightarrow \mathcal{B}$ be an operator defined by:

$$\mathfrak{R}\mathfrak{u}(t) = \begin{cases} \mathcal{S}(t)\left[\mathfrak{u}_0 + h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.))\right] + \dfrac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \mathcal{F}(s, \mathfrak{u}(s))ds, \\ \mathcal{S}(t)\left[v_0 + h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.))\right] \ominus \dfrac{-1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \mathcal{F}(s, \mathfrak{u}(s))ds \end{cases}$$

First, showing that the operator $\mathfrak{R}$ is ms-continuous and $\mathcal{H}(\mathfrak{R}\mathfrak{u}, \mathfrak{u}_0) \leq \sigma$, it follows that $\mathcal{F}$ is ms-continuous. Suppose that $\mathfrak{u}$ is ${}^{cf}[(i) - gH]-$differentiable, then we have:

$$\begin{aligned}
&\varphi\left(\mathfrak{R}\mathfrak{u}(t+\xi), \mathfrak{R}\mathfrak{u}(t)\right) \\
&= \varphi(\mathcal{S}(t+\xi)\left[\mathfrak{u}_0 + h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.))\right] + \frac{1}{\Gamma(\gamma)} \int_0^{t+\xi} (t+\xi-s)^{\gamma-1} \mathcal{F}(s, \mathfrak{u}(s))ds, \\
&\mathcal{S}(t)\left[\mathfrak{u}_0 + h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.))\right] + \frac{1}{\Gamma(\gamma)} \int_0^{t} (t-s)^{\gamma-1} \mathcal{F}(s, \mathfrak{u}(s))ds) \\
&\leq \varphi(\mathcal{S}(t+\xi)\mathfrak{u}_0, \mathcal{S}(t)\mathfrak{u}_0) + \varphi(\mathcal{S}(t+\xi)h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.)), \mathcal{S}(t)h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.))) \\
&\quad + \varphi\left(\frac{1}{\Gamma(\gamma)} \int_0^{t+\xi} (t+\xi-s)^{\gamma-1} \mathcal{F}(s, \mathfrak{u}(s))ds, \frac{1}{\Gamma(\gamma)} \int_0^{t} (t-s)^{\gamma-1} \mathcal{F}(s, \mathfrak{u}(s))ds\right) \\
&\leq \varphi(\mathcal{S}(t+\xi)\mathfrak{u}_0, \mathcal{S}(t)\mathfrak{u}_0) + \varphi(\mathcal{S}(t+\xi)h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.)), \mathcal{S}(t)h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.))) \\
&\quad + \varphi\left(\frac{1}{\Gamma(\gamma)} \int_0^{\xi} (t+\xi-s)^{\gamma-1} \mathcal{F}(s, \mathfrak{u}(s))ds, 0_{(1,0)}\right) \\
&\quad + \varphi\left(\frac{1}{\Gamma(\gamma)} \int_\xi^{t+\xi} (t+\xi-s)^{\gamma-1} \mathcal{F}(s, \mathfrak{u}(s))ds, \frac{1}{\Gamma(\gamma)} \int_0^{t} (t-s)^{\gamma-1} \mathcal{F}(s, \mathfrak{u}(s))ds\right) \\
&\leq \mathcal{M}e^{\omega t}\left(\varphi(\mathcal{S}(\xi)\mathfrak{u}_0, \mathfrak{u}_0) + \varphi(\mathcal{S}(\xi)h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.)), h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.)))\right) \\
&\quad + \varphi\left(\frac{1}{\Gamma(\gamma)} \int_0^{\xi} (t+\xi-s)^{\gamma-1} \mathcal{F}(s, \mathfrak{u}(s))ds, 0_{(1,0)}\right) \\
&\quad + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \varphi\left(\mathcal{F}(s+\xi, \mathfrak{u}(s+\xi)), \mathcal{F}(s, \mathfrak{u}(s))\right) ds.
\end{aligned}$$

Now if $\mathfrak{u}$ is ${}^{cf}[(ii) - gH]-$differentiable, inside the identical way we have got:

$$\begin{aligned}
&\varphi\left(\mathfrak{R}\mathfrak{u}(t+\xi), \mathfrak{R}\mathfrak{u}(t)\right) \\
&= \varphi(\mathcal{S}(t+\xi)\left[\mathfrak{u}_0 + h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.))\right] \ominus \frac{-1}{\Gamma(\gamma)} \int_0^{t+\xi} (t+\xi-s)^{\gamma-1} \mathcal{F}(s, \mathfrak{u}(s))ds, \\
&\mathcal{S}(t)\left[\mathfrak{u}_0 + h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.))\right] \ominus \frac{-1}{\Gamma(\gamma)} \int_0^{t} (t-s)^{\gamma-1} \mathcal{F}(s, \mathfrak{u}(s))ds) \\
&\leq \mathcal{M}e^{\omega t}\left(\varphi(\mathcal{S}(\xi)\mathfrak{u}_0, \mathfrak{u}_0) + \varphi(\mathcal{S}(\xi)h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.)), h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.)))\right) \\
&\quad + \varphi\left(\frac{1}{\Gamma(\gamma)} \int_0^{\xi} (t+\xi-s)^{\gamma-1} \mathcal{F}(s, \mathfrak{u}(s))ds, 0_{(1,0)}\right) \\
&\quad + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \varphi\left(\mathcal{F}(s+\xi, \mathfrak{u}(s+\xi)), \mathcal{F}(s, \mathfrak{u}(s))\right) ds.
\end{aligned}$$

It is evident that, when $\xi \mapsto 0$:

$$\varphi(\mathcal{S}(\xi)\mathfrak{u}_0, \mathfrak{u}_0) \to 0,$$

$$\varphi(\mathcal{S}(\xi)h(t_1, t_2, \dots, t_n, \mathfrak{u}(.)), h(t_1, t_2, \dots, t_n, \mathfrak{u}(.))) \to 0,$$

$$\varphi\left(\frac{1}{\Gamma(\gamma)}\int_0^{\xi}(t+\xi-s)^{\gamma-1}\mathcal{F}(s,\mathfrak{u}(s))ds, 0_{(1,0)}\right) \to 0.$$

Thus, consistently with the dominated convergence theorem, we find:

$$\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\varphi\left(\mathcal{F}(s+\xi, \mathfrak{u}(s+\xi)), \mathcal{F}(s, \mathfrak{u}(s))\right)ds \to 0.$$

Then, $\mathfrak{R}$ is ms-continuous in I.
As a result, we have that if $\mathfrak{u}$ is ${}^{cf}[(i)-gH]-$differentiable,

$$\begin{aligned}
\varphi\left(\mathfrak{R}\mathfrak{u}(t), \mathfrak{u}_0\right) &= \varphi\left(\mathcal{S}(t)\left[\mathfrak{u}_0 + h(t_1, t_2, \dots, t_n, \mathfrak{u}(.))\right] + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\mathcal{F}(s,\mathfrak{u}(s))ds, \mathfrak{u}_0\right) \\
&\le \varphi(\mathcal{S}(t)\mathfrak{u}_0, \mathfrak{u}_0) + \varphi(\mathcal{S}(t)h(t_1, t_2, \dots, t_n, \mathfrak{u}(.)), 0_{(1,0)}) \\
&\quad + \varphi\left(\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\mathcal{F}(s,\mathfrak{u}(s))ds, 0_{(1,0)})\right) \\
&\le \beta + \mathcal{M}e^{\omega t}\varphi(h(t_1, t_2, \dots, t_n, \mathfrak{u}(.)), 0_{(1,0)}) \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\varphi(\mathcal{F}(s,\mathfrak{u}(s)), 0_{(1,0)})ds \\
&\le \beta + \mathcal{M}e^{\omega T}K_h + \frac{T^{\gamma}K_F}{\Gamma(\gamma+1)}
\end{aligned}$$

Similarly, if $\mathfrak{u}$ is ${}^{cf}[(ii)-gH]-$differentiable, we obtain:

$$\begin{aligned}
\varphi\left(\mathfrak{R}\mathfrak{u}(t), \mathfrak{u}_0\right) &= \varphi\left(\mathcal{S}(t)\left[\mathfrak{u}_0 + h(t_1, t_2, \dots, t_n, \mathfrak{u}(.))\right] \ominus \frac{-1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\mathcal{F}(s,\mathfrak{u}(s))ds, \mathfrak{u}_0\right) \\
&\le \beta + \mathcal{M}e^{\omega T}K_h + \frac{T^{\gamma}K_F}{\Gamma(\gamma+1)}
\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{H}\left(\mathfrak{R}\mathfrak{u}(t), \mathfrak{u}_0\right) &= \sup_{0\le t\le\theta}\varphi\left(\mathfrak{R}\mathfrak{u}(t), \mathfrak{u}_0\right) \\
&\le \beta + \mathcal{M}e^{\omega\theta}K_h + \frac{\theta^{\gamma}K_F}{\Gamma(\gamma+1)} \\
&\le \sigma.
\end{aligned}$$

Hence, (L_2, φ) is a complete metric space.

Let $C([0,\theta], L_2) = \{u : [0,\theta] \longrightarrow L_2 \mid \mathfrak{u}(t) \text{ is ms-continuous}\}$. Let us show that this space is complete.
Let us show that $\mathcal{B}$ is a closed subset of $C([0,\theta], L_2)$.
Let $\{\mathfrak{u}_m\}$ be a sequence in $\mathcal{B}$ such that $\mathfrak{u}_m \to \mathfrak{u} \in C([0,\theta], L_2$. Then,

$$\varphi(\mathfrak{u}(t), \mathfrak{u}_0) \le \varphi(\mathfrak{u}(t), \mathfrak{u}_m(t)) + \varphi(\mathfrak{u}_m(t), \mathfrak{u}_0),$$

$$
\begin{aligned}
\mathcal{H}(\mathfrak{u}, \mathfrak{u}_0) &= \sup_{0 \le t \le \theta} \varphi(\mathfrak{u}(t), \mathfrak{u}_0) \\
&\le \mathcal{H}(\mathfrak{u}, \mathfrak{u}_m) + \mathcal{H}(\mathfrak{u}_m, \mathfrak{u}_0) \\
&\le \sigma + \beta.
\end{aligned}
$$

So $\mathfrak{u} \in \mathcal{B}$, which implies that $\mathcal{B}$ is a closed subset of $C([0, \theta], L_2)$. As a result, $\mathcal{B}$ is a complete metric space.

Now let us show that the operator $\mathfrak{R}$ is a contraction. Let $\mathfrak{u}, \mathfrak{v} \in \mathcal{B}$, we have:

$$
\begin{aligned}
\varphi\left(\mathcal{R}\mathfrak{u}(t), \mathcal{R}\mathfrak{v}(t)\right) &\le \varphi\left(\mathcal{S}(t)h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.)), \mathcal{S}(t)h(t_1, t_2, \ldots, t_n, \mathfrak{v}(.))\right) \\
&\quad + \varphi\left(\frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \mathcal{F}(s, \mathfrak{u}(s))ds, \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \mathcal{F}(s, \mathfrak{v}(s))ds\right) \\
&\le \mathcal{M}_h \mathcal{M} e^{\omega t} \varphi(\mathfrak{u}(t), \mathfrak{v}(t)) + \frac{\mathcal{M}_F T^\gamma}{\Gamma(\gamma+1)} \varphi(\mathfrak{u}(t), \mathfrak{v}(t)).
\end{aligned}
$$

We achieve,

$$
\begin{aligned}
\mathcal{H}(\mathfrak{R}\mathfrak{u}, \mathfrak{R}\mathfrak{v}) &\le \sup_{0 \le t \le \theta} \{\mathcal{M}_h \mathcal{M} e^{\omega t} \varphi(\mathfrak{u}t), \mathfrak{v}(t)) + \frac{\mathcal{M}_F}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \varphi(\mathfrak{u}(s), \mathfrak{v}(s))ds\} \\
&\le \left(\mathcal{M}_h \mathcal{M} e^{\omega\theta} + \frac{\mathcal{M}_F \theta^\gamma}{\Gamma(\gamma+1)}\right) \mathcal{H}(\mathfrak{u}, \mathfrak{v})
\end{aligned}
$$

Since, $\left(\mathcal{M}_h \mathcal{M} e^{\omega\theta} + \dfrac{\mathcal{M}_F \theta^\gamma}{\Gamma(\gamma+1)}\right) < 1$, then $\mathfrak{R}$ is a contracting operator.
Hence, $\mathfrak{R}$ admits a fixed point $\mathfrak{R}\mathfrak{u} = \mathfrak{u} \in C([0, \theta], L_2)$ is:

$$
\mathfrak{u}(t) = \begin{cases}
\mathcal{S}(t)\left[\mathfrak{u}_0 + h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.))\right] \\
\quad + \dfrac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \mathcal{F}(s, \mathfrak{u}(s))ds, & \text{if } \mathfrak{u} \text{ is } {}^{cf}[(i) - gH]\text{-differentiable,} \\
\mathcal{S}(t)\left[\mathfrak{u}_0 + h(t_1, t_2, \ldots, t_n, \mathfrak{u}(.))\right] \\
\quad \ominus \dfrac{-1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \mathcal{F}(s, \mathfrak{u}(s))ds, & \text{if } \mathfrak{u} \text{ is } {}^{cf}[(ii) - gH]\text{-differentiable.}
\end{cases}
$$

□

4 Conclusion

In conclusion, this paper provides important insights into the existence and uniqueness of solutions for intuitionistic fuzzy evolution problems with nonlocal conditions, employing generalized Caputo derivatives. By integrating concepts from semigroup theory, mean-square calculus techniques, and contraction mapping principles, our study establishes a robust mathematical framework that is crucial for effectively addressing these complex problems.

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