Intuitionistic fuzzy T-sets based solution technique for multiple objective linear programming problems under imprecise environment

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Abstract: Technique to find Pareto-optimal solutions to multiple objective linear programming problems under imprecise environment is discussed in this paper. In 1997, Angelov formulated an optimization technique under intuitionistic fuzzy environment. Several other researchers have worked on it in recent years. In optimization technique under imprecise environment, it is observed that the prime intention to maximize up-gradation of most misfortunate is better served if some constraints present in existing, well established techniques are removed. In classical intuitionistic fuzzy optimization techniques, it is also observed that membership functions and non-membership functions are not utilised in the way they are defined; and in some cases, constraints in those existing techniques may make the problem infeasible. Hence in this paper, new functions: $T^{(+)}$-characteristic functions and $T^{(-)}$-characteristic functions, are introduced to supersede membership functions and non-membership functions respectively; and subsequently new set: Intuitionistic fuzzy T-set is introduced to supersede intuitionistic fuzzy set to represent impreciseness. Moreover in this paper, one general algorithm has been developed to find Pareto optimal solutions to multiple objective linear programming problems under imprecise environment. A real life industrial application model further illustrates the limitations of existing technique as well as advantages of using proposed technique. Finally conclusions are drawn.

Keywords: Intuitionistic fuzzy sets, Intuitionistic fuzzy optimization, $T^{(+)}$-characteristic functions, $T^{(-)}$-characteristic functions, Intuitionistic fuzzy T-sets.

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1 Introduction

Multiple objective linear programming is the process of optimizing systematically and simultaneously a collection of objective functions. By assuming that the decision maker (DM) has imprecise aspiration level for each objective function, mathematicians have proposed several methods in literature for characterizing Pareto optimal solutions to multiple objective linear programming problems (MOLPP) [9–11]. In one such approach, fuzzy set theory is used (Bellman and Zadeh, 1970) [3]. As pointed out by Zimmermann in 1976, and later in 1978, various kinds of uncertainties can be categorized as fuzziness [13]. In fact, Zimmermann and successive researchers converted MOLPP into single objective optimization problem to find Pareto optimal solutions by applying fuzziness of the DM's aspiration with respect to goals of imprecise objective functions (and constraints till they are symmetric) [13, 14].

Recently Jimenez and Bilbao (2009) showed that fuzzy efficient solutions may not be Pareto optimal in case that one of the fuzzy goals is fully achieved [8]. Their procedure extended two phase approach of Guu, Wu (1997, 1999) [6, 7] and approach of Dubois, Fortemps (1999) [4] to attain Pareto optimal solutions. But according to Wu et al (2015), proposed approach by Jimenez and Bilbao (2009) cannot guarantee to be one general procedure to attain Pareto optimal solutions to MOLPP under impreciseness [8]. In their proposed approach, Wu et al modified definition of membership functions of fuzzy sets and introduced redefined membership functions [12].

But we observe that those definitions of redefined membership functions and their usage in mathematical models as well as in numerical examples in the article by Wu et al are not analogous to one another. It is clear from their mathematical model or given numerical examples that only one part of redefined membership functions, where values of objective functions lie between goals and goals plus tolerances is used. But in existing techniques, these realities are not reflected in form of constraints during formulation of the model. Thus in problem formulation, redefined membership functions are strictly monotone over entire domain (values of objective functions) where as they are not so in definition. We find that classical fuzzy optimization technique, developed by Zimmermann and later used by many researchers, also has similar drawbacks!

We note that Wu et al (2015) suggested removal of upper bound (at unity) of membership functions of fuzzy sets [12]. But corresponding lower bound still remains at zero. In those numerical examples by Wu et al (2015), those additional constraints (arising from definition of redefined membership functions that objective values cannot exceed sums of goals and tolerances) could make no impact for minimization type of objective functions (similar conclusions can be drawn for objective functions of maximization type). But in many cases, if the lower bound of membership functions remain intact at zero, optimization models become infeasible under fuzzy environment. Point to be noted.

Our deep analysis suggests the cause to lie in definition of membership functions of fuzzy sets. Membership functions of fuzzy sets provide satisficing results when extreme ends of imprecise information can be quantified within the boundary of zero and one. But it may not be always logically/mathematically correct to quantify or measure impreciseness within some bounded subset of real line.

On the other hand, fuzzy set theory has been widely developed and several modifications and generalizations have appeared. One of them is concept of intuitionistic fuzzy (IF) sets that was introduced by K. T. Atanassov in 1986 [2]. IF sets consider not only membership values but also non-membership values such that sum of these values does not exceed unity for any IF objective function [2]. The advantages of using IF sets to represent impreciseness in optimization models are manifold and well known. Plamen P. Angelov (1997) first introduced the solution technique of optimization model under IF environment [1]. In subsequent years, researchers have extended this technique to MOLPP under IF environment. Analogously, this technique has many limitations. In this paper, on one side, limitations of existing optimization technique under IF environment are discussed and on another side, new set is introduced to represent impreciseness; as well as an algorithm is developed applying the new set to solve MOLPP under impreciseness.

The rest of the paper is organized as follows: Section 2 is occupied with limitations of existing IF optimization technique. Section 3 introduces definition of new set: Intuitionistic fuzzy T-set to supersede IF set to measure impreciseness. Some related definitions are also given in Section 3. Section 4 develops the theory of optimization technique involving proposed intuitionistic fuzzy T-sets. Section 5 contains proposed algorithm to find Pareto optimal solutions to MOLPP under impreciseness. In section 6, numerical example as well as one model on real life industrial application is taken not only to highlight the limitations of existing intuitionistic fuzzy optimization technique but also to discuss advantages of using proposed technique. Section 7 concludes our article.

2 Limitations of existing intuitionistic fuzzy optimization technique

Consider MOLPP with k objective functions $z_i(x), i = 1 \ldots k$, each of minimizing type, as follows:

$$Min \ (z_1(x), z_2(x) \ldots z_k(x))^T$$
$$subject \ to \ x \in X \tag{1}$$

where $X = \{x \in \mathbb{R}^n : Ax \le b, x \ge 0\}, b = (b_1, b_2 \ldots b_m) \in \mathbb{R}^m$ and A is $m \times n$ matrix. Assuming that the DM has IF aspiration levels for each of the objective functions $z_i(x), i = 1 \ldots k$ of model (1), several methods have been proposed in literature for characterizing Pareto optimal solutions to it. As per existing literature, it is known as MOLPP under IF environment. Usually the linear membership function and linear non-membership function of the ith IF objective function (of minimization type) $z_i(x), i = 1 \ldots k$ of model (1) are denoted by $\mu_i(z_i(x)), \nu_i(z_i(x)), i = 1 \ldots k$ respectively and are defined as follows:

$$\mu_i(z_i(x)) = \begin{cases} 1 & , z_i(x) < L_i \\ \dfrac{U_i - z_i(x)}{U_i - L_i} & , z_i(x) \in [L_i, U_i], \\ 0 & , z_i(x) > U_i \end{cases} \quad \forall i = 1 \ldots k \quad \text{and} \quad \nu_i(z_i(x)) = \begin{cases} 0 & , z_i(x) < L_i \\ \dfrac{z_i(x) - L_i}{W_i - L_i} & , z_i(x) \in [L_i, W_i], \\ 1 & , z_i(x) > W_i \end{cases} \quad \forall i = 1 \ldots k$$

where L_i is the goal of the ith IF objective function $z_i(x), i = 1...k$ and U_i and $W_i(U_i \le W_i)$ are the goal plus tolerance values of membership function $\mu_i(z_i(x)), i = 1...k$ and non-membership function $v_i(z_i(x)), i = 1...k$ of the ith IF objective function $z_i(x), i = 1...k$, respectively. Following the IF optimization technique by Angelov (1997) and other researchers, if $\alpha = \min_{i=1...k}\{\mu_i(z_i(x))\}$ and $\beta = \max_{i=1...k}\{v_i(z_i(x))\}$, i.e. if α denotes the minimal level of acceptance and β denotes the maximal level of rejection of IF objectives and constraints, the single objective linear programming problem (LPP) may be obtained as follows: [1]

$$\max \alpha - \beta$$
$$\text{subject to } \mu_i(z_i(x)) \ge \alpha, \forall i = 1...k, v_i(z_i(x)) \le \beta, \forall i = 1...k, \alpha \ge \beta, \ \alpha + \beta \le 1, \ \beta \ge 0, x \in X. \tag{2}$$

It may be observed that during formulation of constraints $\mu_i(z_i(x)) \ge \alpha, i = 1...k$ and $v_i(z_i(x)) \le \beta, i = 1...k$ in model (2), conventionally, researchers (Angelov and others) have employed $\frac{U_i - z_i(x)}{U_i - L_i}, i = 1...k$ as membership functions $\mu_i(z_i(x)), i = 1...k$ and $\frac{z_i(x) - L_i}{W_i - L_i}, i = 1...k$ as non-membership functions $v_i(z_i(x)), i = 1...k$ respectively [1]. But $\forall i = 1...k$, $\frac{U_i - z_i(x)}{U_i - L_i}$ is strictly monotonic (decreasing for minimizing type of objectives) function $z_i(x)$, $\forall i = 1...k$, whereas as per definition, $\mu_i(z_i(x)) \ \forall i = 1...k$ is strictly monotonic (decreasing for minimizing type of objectives) for $z_i(x) \in [L_i, U_i] \ \forall i = 1...k$ only; and also $\forall i = 1...k$, $\frac{z_i(x) - L_i}{W_i - L_i}$ is strictly monotonic (increasing for minimizing type of objectives) function $\forall z_i(x)$, whereas $v_i(z_i(x)) \ \forall i = 1...k$ is strictly monotonic for $z_i(x) \in [L_i, W_i] \ \forall i = 1...k$ only.

To handle this effectively and efficiently, constraints $\alpha \ge \beta$, $\alpha + \beta \le 1$, $\beta \ge 0$ were added to model (2) by the researchers! In this paper, next, we plan to show that presence of these constraints $\alpha \ge \beta$, $\alpha + \beta \le 1$, $\beta \ge 0$ (and hence $\alpha \ge 0$) may make model (2) (and hence model (1)) *infeasible* or may hinder from obtaining *most preferable* Pareto optimal solutions to MOLPP (1) under IF environment in some cases.

It may be further recalled that usually in literature, IF objectives and/or constraints are represented by triangular IF numbers. And $\forall i = 1...k$, triangular IF goal of the ith IF objective function of minimization type is denoted by $(a_i; b_i, b_i')$ with $\mu_i(z_i(a_i)) = 1$, $v_i(z_i(a_i)) = 0$, $\mu_i(z_i(b_i)) = 0$ and $v_i(z_i(b_i')) = 1$, and $b_i < b_i'$ (equality occurs in case of fuzzy set), as shown in Fig. 1.
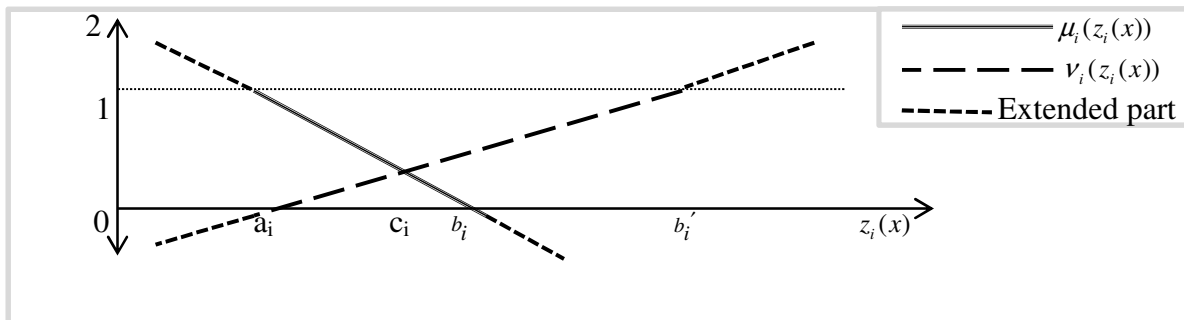

Figure 1. Strictly monotonic functions (not in scale) extended in both directions

Now, let us put our concentration on following result:

Result 1. *Let* $(a_i; b_i, b_i')$ *be triangular IF goal to* i^{th} *objective function* $z_i(x), \forall i = 1\ldots k$ *of minimization type to MOLPP (1). Then for any* $a_i'' < a_i$, *deployment of strictly monotonic part of definition of membership and non-membership function during computation yield that sum of membership value and non-membership value exceeds unity always i.e.* $\mu_i(z_i(a_i'')) + \nu_i(z_i(a_i'')) > 1, \forall a_i'' < a_i$ *with* $a_i'', a_i \in X$.

Proof: Suppose that $(a_i; b_i, b_i')$ is the triangular IF goal of ith objective function $z_i(x), i = 1\ldots k$ of minimization type to MOLPP (1) under IF environment. Then as per definition $\forall i = 1\ldots k$, we have: $\mu_i(z_i(a_i)) = 1$ and $\nu_i(z_i(a_i)) = 0 = \mu_i(z_i(b_i))$, $\nu_i(z_i(b_i')) = 1$ with $b_i < b_i'$. For any $i = 1\ldots k$, we may select any $a_i'' < a_i$ such that $a_i'' = a_i'' = z_i(x')$ for some $x' \in X$. Then, strictly

monotonic part of membership function yields $\mu_i(z_i(a_i'')) = \dfrac{b_i - a_i''}{b_i - a_i}$ and strictly monotonic part

of non-membership function yields $\nu_i(z_i(a_i'')) = -\dfrac{a_i - a_i''}{b_i' - a_i}$. Consequently, $\forall i = 1\ldots k$, we get

$\mu_i(z_i(a_i'')) + \nu_i(z_i(a_i'')) = \dfrac{b_i - a_i''}{b_i - a_i} - \dfrac{a_i - a_i''}{b_i' - a_i} = 1 + \dfrac{(b_i' - b_i)(a_i - a_i'')}{(b_i - a_i)(b_i' - a_i)} > 1$. Hence the result. $\qquad\square$

Now, constraint $\alpha + \beta \le 1$ in model (2) implies that sum of minimal value of acceptance and maximal value of rejection cannot exceed unity. But from Fig. 1 as well as from result 1, $\forall i = 1\ldots k$, it is clear that if strictly monotonic parts are deployed, sum of membership and non-membership values of objective functions $z_i(x), i = 1\ldots k$ always exceeds unity in extended part (as depicted graphically in Fig. 1 by using dotted line on left side of a_i). Hence if constraint $\alpha + \beta \le 1$ is present, objective functions $z_i(x), i = 1\ldots k$ may not attain any values lower than a_i. But since objective functions $z_i(x), i = 1\ldots k$ are of minimization type, any values lower than a_i to objective functions $z_i(x), i = 1\ldots k$ are more preferable to DM. Further, numerical example 6.1.1 in section 6 shows that presence of constraint $\alpha + \beta \le 1$ may result in less preferable optimal values for each and every objective functions $z_i(x), i = 1\ldots k$ to MOLPP under imprecise environment. Hence we may propose removal of constraint $\alpha + \beta \le 1$ during formulation of model (2).

On the other hand, classical membership functions of IF objective functions $z_i(x), i = 1\ldots k$ have upper bound at unity. And constraints $\alpha + \beta \le 1, \beta \ge 0$ automatically imply another constraint viz. $\alpha \le 1$. But this resultant constraint $\alpha \le 1$ may not allow objective functions $z_i(x), i = 1\ldots k$ to attain any values lower than a_i in model (2). In fact presence of this constraint may contract the feasible space! Here IF goals and/or tolerances are imprecise in nature and objective functions $z_i(x), i = 1\ldots k$, are of minimization type. Consequently DM get more satisfaction when optimal values of $z_i(x), i = 1\ldots k$ become as minimum as possible (DM wishes for any value not exceeding a_i!). So, in some cases, resultant constraint $\alpha \le 1$ may result in less preferable optimal solutions to IF objective functions $z_i(x), i = 1\ldots k$, as shown by

numerical example 6.1.2. in section 6 of this paper. Consequently, it may be justified not to impose any upper bound for membership functions as well as any lower bound for non-membership functions of IF objective functions $z_i(x), i = 1 \ldots k$.

On the other hand, the constraint $\alpha \geq \beta$ in model (2) implies that minimal level of acceptance cannot be less than maximum level of rejection. This constraint may result in IF objective functions $z_i(x), i = 1 \ldots k$ not to attain any value higher than c_i. (In Fig. 1, c_i denotes abscissa of the point of intersection of membership function and non-membership function of IF objective function $z_i(x), i = 1 \ldots k$). Here goals and tolerances of IF objective functions $z_i(x), i = 1 \ldots k$ are imprecise in nature. And in many cases, optimal solutions of model (2) exist only if objective functions $z_i(x), i = 1 \ldots k$ may attain values higher than c_i. In such cases, constraint $\alpha \geq \beta$ may contract the feasible space and model (2) may become infeasible. We have shown the case by numerical example 6.2.1 in Sect. 6. Hence it may be justified to remove constraint $\alpha \geq \beta$ during formulation of model (2).

Again constraints $\alpha \geq \beta$, $\beta \geq 0$ in model (2) automatically imply $\alpha \geq 0$. It means minimal optimal value of acceptance cannot be negative. Further the constraint $\alpha \geq 0$ implies that IF objective functions $z_i(x), i = 1 \ldots k$ cannot attain any value higher than b_i, $i = 1, \ldots, k$. (In Fig. 1, $b_i, i = 1 \ldots k$ denotes the abscissa of the point of intersection between membership function and IF objective function $z_i(x)$, $i = 1, \ldots, k$). Consequently constraint $\alpha \geq 0$ may contract the feasible space. In many cases, optimal solutions of model (2) exist only if objective functions $z_i(x), i = 1 \ldots k$ can attain values higher than b_i. Therefore the constraint $\alpha \geq 0$, although arising indirectly, may make model (2) (and hence MOLPP (1)) infeasible, as shown by numerical example 6.2.2 in section 6 in this paper. Hence constraint $\alpha \geq 0$ needs to be avoided during formulation of model (2). Consequently, it may be proposed not to keep any restriction on lower bound of membership functions of IF objective functions $z_i(x), i = 1 \ldots k$. Therefore classical membership functions may be transformed into strictly monotonic functions, keeping all essential characteristics intact, which seems more useful in optimization technique under IF environment. Analogously, removal of constraint $\beta \geq 0$ in model (2) may be proposed.

Again, definition of non-membership functions of IF objective functions $z_i(x), i = 1 \ldots k$ restrict the upper bound at unity. Corresponding constraint $\beta \leq 1$ in model (2) implies that IF objective functions $z_i(x), i = 1 \ldots k$ cannot attain any value higher than b'_i (in Fig. 1). Consequently, it may contract the feasible space and model (2) may become infeasible. We show it through numerical example 6.2.3 in section 6. Hence constraint $\beta \leq 1$ has to be avoided during formulation of model (2). Consequently, it may be suggested not to keep any upper bound of non-membership functions of IF objective functions $z_i(x), i = 1 \ldots k$. Finally, classical non-membership functions may be transformed into strictly monotonic functions, keeping all essential characteristics intact, which seems more useful in optimization technique under IF environment.

Hence conventional usages of membership functions and non-membership functions of IF objective functions may *not* to be consistent with how these are defined. We may term it as limitation of existing IF optimization technique and hence existing IF optimization technique

may not be one general technique to determine Pareto optimal solutions to MOLPP (1) under imprecise environment.

3   Definitions

The characteristic functions of crisp sets assign values of either 1 or 0 to each element in the universal set thereby discriminating between members and non-members of crisp set under consideration. On the other hand, membership functions of fuzzy sets generalize characteristic functions such that values assigned to the element of universal set fall within a specified range and indicate the membership values of these elements to the set in question. But we have observed that fuzzy membership functions fail to discriminate between 'yes' and 'certainly yes' by assigning membership value 1 in both cases as well as to discriminate between 'no', and 'certainly no' by assigning membership value 0 in both cases. And analogously, in IF set, membership functions and non-membership functions fail to discriminate between 'yes' and 'certainly yes' as well as between 'no', and 'certainly no' by assigning same values to both of them.

Consider one example: Suppose we plan to purchase an *Apple iPhone 6 Plus 64GB Gold* from online market place. Our budget is \$800 and we cannot wait for more than two days. Suppose A is the IF sub set of retailers based on these criterions. Then one retailer x_1 promising to deliver the phone in **one day** and **at \$799** may be assigned membership value unity and non-membership value zero in IF set A. But suppose we search more and find another retailer x_2, who promises to deliver the phone in **one day** and **at \$749.** Then retailer x_2 have to be assigned membership value unity as well as non-membership value zero in IF set A. Since initially we had no information/knowledge about retailer x_2, we assigned membership value 1 and non-membership value zero to x_1. Therefore either we have to treat both retailers x_1 and x_2 at par or in order to reflect reality properly, we have to alter pre assigned membership and non-membership values of IF set A. It may be termed as drawback of definitions of membership and non-membership functions of intuitionistic fuzzy set A.

Hence under imprecise environment, membership and non-membership functions of IF sets fail to explain the case satisfactorily. An element of an IF set may lie partly or never lie or must lie in that set. The concepts of lying, partly lying or not lying are well measured by membership and non-membership functions of IF set. But it does not suit the case of either must lying or never lying.

On the other hand, in optimization technique under impreciseness, one interesting, and useful property of membership (non-membership) functions of IF sets is that higher (lower) values of membership (non-membership) functions imply preferable solutions to IF objective functions. These ideas can be further integrated, and new functions that not only handle underlying issues well but also preserve characteristics of membership and non-membership functions may be defined. Consequently an IF set may be superseded by a new set to represent impreciseness. Suppose IR denotes the set of real numbers. Then the following definitions may be proposed:

Definition 1. Let S denotes universal set and A is any subset of S. The $T^{(+)}$-*characteristic function* of A is denoted by $T_A^{(+)}$ and is defined as $T_A^{(+)} : S \rightarrow IR$ such that it assigns one real

number $T_A^{(+)}(x)$ to each element $x \in S$. Higher the value of $T_A^{(+)}(x)$, larger the value of membership of $x \in S$ in A.

And $T^{(-)}$-*characteristic function* of A is denoted by $T_A^{(-)}$ and is defined as $T_A^{(-)} : S \to IR$ such that it assigns one real number $T_A^{(-)}(x)$ to each element $x \in S$. Lower the value of $T_A^{(-)}(x)$, larger the value of non-membership of $x \in S$ in A.

Definition 2. Let S denotes universal set. *Intuitionistic fuzzy T-subset A* of universal set S is defined as the ordered triplet $A = \{(a, T^{(+)}(a), T^{(-)}(a)) : \forall a \in S\}$, where $T^{(+)}$-characteristic function $T_A^{(+)} : S \to IR$ assigns real number $T_A^{(+)}(x)$ as membership value of each $x \in S$ and $T^{(-)}$-characteristic function $T_A^{(-)} : S \to IR$ assigns real number $T_A^{(-)}(x)$ as non-membership value of each $x \in S$.

E.g. $T^{(+)}$-characteristic function $T_A^{(+)}$ of intuitionistic fuzzy T-subset A of universal set S may be defined as $T_A^{(+)}(x) = 0$ or < 0 according as $x \notin A$ or x is certainly not in A; and $T_A^{(+)}(x) = 1$ or > 1 according as $x \in A$ or x is certainly in A, always keeping monotonicity of $T_A^{(+)}$ intact such that higher the value of $T^{(+)}$-characteristic function, larger the value of membership of $x \in S$. Similarly, $T^{(-)}$-characteristic function $T_A^{(-)}$ of intuitionistic fuzzy T-subset A of universal set S may be defined.

Hence IF membership function of IF set may be one special case of $T^{(+)}$-characteristic function of proposed intuitionistic fuzzy T-set and IF non-membership function of IF set may be one special case of $T^{(-)}$-characteristic function of proposed intuitionistic fuzzy T-set. Consequently IF set may be one special case of intuitionistic fuzzy T-set for representing impreciseness. In this paper, we propose intuitionistic fuzzy T-sets to supersede intuitionistic fuzzy sets to represent impreciseness to solve MOLPP under imprecise environment.

Definition 3. Let S be the universal set. Then *intuitionistic fuzzy T-subset B* of intuitionistic fuzzy T-set A is denoted by $B \subseteq A$ and is defined as the triplet $B = \{(b, T_B^{(+)}(b), T_B^{(-)}(b)) : T_B^{(+)}(b) \leq T_A^{(+)}(b), T_B^{(-)}(b) \geq T_A^{(-)}(b), \forall b \in S\}$.

E.g. suppose $S = \{a, b, c, d\}$ is the universal set and two intuitionistic fuzzy T-sets A and B are $A = \{(a, 0.3, 0.2), (b, 0.6, -0.2), (c, -0.3, 1), (d, 1.2, 0)\}$, $B = \{(a, 0.2, 0.3), (b, -0.3, 1), (c, -1, 1), (d, 0.6, 1)\}$. Then $B \subseteq A$.

Definition 4. Let S be the universal set and A and B are two intuitionistic fuzzy T-subsets of S. We say that A and B are *equal* and denote it by $A = \underline{B}$ iff $A \subseteq B$ and $B \subseteq A$.
E.g. suppose $S = \{a, b, c, d\}$ is the universal set and two intuitionistic fuzzy T-subsets A and B are $A = \{(a, 0.3, 0.2), (b, 0.6, -0.2), (c, -0.3, 1), (d, 1.2, 0)\}$ and $B = \{(a, 0.3, 0.2), (b, 0.6, -0.2), (c, -0.3, 1), (d, 1.2, 0)\}$. Then $A = B$.

Definition 5. The *union* or *disjunction* of two intuitionistic fuzzy T-subsets A and B of universal set S, denoted by $A \cup B$, is defined as the ordered triplet $A \cup B = \{(x, T_{A \cup B}^{(+)}(x), T_{A \cup B}^{(-)}(x)) : x \in S\}$, where $T^{(+)}$-characteristic function of intuitionistic fuzzy T-subset $A \cup B$ is defined as $T_{A \cup B}^{(+)}(x) = \max\{T_A^{(+)}(x), T_B^{(+)}(x)\}, \forall x \in S$ and $T^{(-)}$-characteristic function of intuitionistic fuzzy T-subset $A \cup B$ is defined as $T_{A \cup B}^{(-)}(x) = \min\{T_A^{(-)}(x), T_B^{(-)}(x)\}, \forall x \in S$.

E.g. suppose $S = \{a,b,c,d\}$ is the universal set and two intuitionistic fuzzy T-subsets of S are $A = \{(a,0.3,0.2),(b,0.6,-0.2),(c,-0.3,1),(d,1.2,0)\}$ and $B = \{(a,0.6,0.3),(b,-0.3,1),(c,1,0),(d,0.6,-0.1)\}$. Then intuitionistic fuzzy T-subset $A \cup B$ is given by $A \cup B = \{(a,0.6,0.2),(b,0.6,-0.2),(c,1,0),(d,1.2,-0.1)\}$.

Definition 6. The *intersection* or *conjunction* of two intuitionistic fuzzy T-subsets A and B of universal set S, denoted by $A \cap B$, is defined as the ordered triplet $A \cap B = \{(x,T_{A \cap B}^{(+)}(x),T_{A \cap B}^{(-)}(x)) : x \in S\}$, where $T^{(+)}$-characteristic function of intuitionistic fuzzy T-subset $A \cap B$ is defined as $T_{A \cap B}^{(+)}(x) = \min\{T_A^{(+)}(x),T_B^{(+)}(x)\}, \forall x \in S$ and $T^{(-)}$-characteristic function of intuitionistic fuzzy T-subset $A \cap B$ is defined as $T_{A \cap B}^{(-)}(x) = \max\{T_A^{(-)}(x),T_B^{(-)}(x)\}, \forall x \in S$.

E.g. suppose that $S = \{a,b,c,d\}$ is the universal set and intuitionistic fuzzy T-subsets of S are $A = \{(a,0.3,0.2),(b,0.6,-0.2),(c,-0.3,1),(d,1.2,0)\}$ and $B = \{(a,0.6,0.3),(b,-0.3,1),(c,1,0),(d,0.6,-0.1)\}$. Then intuitionistic fuzzy T-subset $A \cap B$ is given by $A \cap B = \{(a,0.3,0.3),(b,-0.3,1),(c,-0.3,1),(d,0.6,0)\}$.

Definition 7. The *complement* of intuitionistic fuzzy T-subset A of universal set S, denoted by $\overline{A}$ or A^c, is defined as the ordered triplet $\overline{A} = \{(x,T_A^{(-)}(x),T_A^{(+)}(x)) : x \in S\}$.

E.g. suppose that $S = \{a,b,c,d\}$ is the universal set and intuitionistic fuzzy T-subset of S is $A = \{(a,1.2,-0.2),(b,-0.3,0.8),(c,1,-0.2),(d,0,1.5)\}$. Then intuitionistic fuzzy T-subset $\overline{A}$ or A^c is given by $\overline{A} = \{(a,-0.2,1.2),(b,0.8,-0.3),(c,-0.2,1),(d,1.5,0)\}$.

When intuitionistic fuzzy T-sets represent impreciseness, we may refer the uncertain or imprecise environment as *T-intuitionistic* environment. We define *intuitionistic fuzzy T-efficient solutions* or *T-Pareto-optimal solutions* to MOLPP (1) for suitable $T^{(+)}$-characteristic functions and $T^{(-)}$-characteristic functions under T-intuitionistic environment.

Definition 8. A decision plan $x_0 \in X$ is said to be an *intuitionistic fuzzy T-efficient solution* or *T-Pareto-optimal solution* to the MOLPP (1) under T-intuitionistic environment if there does not exist another $y \in X$ such that $T^+(z(x_0)) \leq T^+(z(y)), T^-(z(y)) \leq T^-(z(x_0)), \forall i, i \neq j$; and $T^+(z(x_0)) < T^+(z(y)), T^-(z(y)) < T^-(z(x_0))$ for at least one j.

It may be observed that the concepts of intuitionistic fuzzy T-Pareto optimal solutions defined in terms of intuitionistic fuzzy T-sets are natural extension of IF efficient solutions or M-N-Pareto optimal solutions defined in terms of membership and non- membership functions of IF sets, which are another natural extension of M-Pareto optimal solutions defined in terms of membership functions of fuzzy sets, which are another natural extension of Pareto optimal solutions defined in terms of crisp sets.

4 Optimization technique using proposed intuitionistic fuzzy T-sets

Within the scope of multiple objective decision making theory, Pareto optimality of solutions is necessary condition in order to guarantee the rationality of decision. In a minimization problem, an uncertain goal of DM may be to achieve "substantially less" than some value £. This type of statements may be quantified by eliciting corresponding suitable intuitionistic fuzzy T-sets. Also intuitionistic fuzzy T-sets can be constructed for uncertain objective functions when no such information is available. In this section, solution technique using intuitionistic fuzzy T-sets for MOLPP under T-intuitionistic environment (i.e. under imprecise environment) is discussed.

Analogous to IF optimization technique by Angelov and several other researchers, if $T^{(+)}$-characteristic functions and $T^{(-)}$-characteristic functions of objective functions $z_i(x), i = 1...k$ under T-intuitionistic environment are denoted by $T_i^{(+)}(z_i(x)), i = 1...k$ and $T_i^{(-)}(z_i(x)), i = 1...k$ respectively, MOLPP model (1) is equivalent to the following problem:

$$\max \ (T_1^{(+)}(z_1(x)), T_2^{(+)}(z_2(x))...T_k^{(+)}(z_k(x)))^T, \ \min \ (T_1^{(-)}(z_1(x)), T_2^{(-)}(z_2(x))...T_k^{(-)}(z_k(x)))^T \quad (3)$$

Subject to $x \in X$. Using min-max operator by Bowman (1976), and if

$$v = \max_{i=1...k}\{-T_i^{(+)}(z_i(x)) : x \in X\} = -\alpha, \text{ and } w = \min_{i=1...k}\{-T_i^{(-)}(z_i(x)) : x \in X\} = -\beta,$$

model (3) may be converted into single objective optimization problem as follows:

$$\max \ \alpha - \beta \text{ subject to } T_i^{(+)}(z_i(x)) \geq \alpha, i = 1...k, T_i^{(-)}(z_i(x)) \leq \beta, i = 1...k, x \in X. \quad (4)$$

Suppose the optimal value of objective function $\alpha - \beta$ of model (4) is $\alpha* - \beta*$ (here * denotes optimality). The relationships between T-Pareto optimal solutions of model (4) and Pareto optimal solutions to the MOLPP (1) under impreciseness/T-intuitionistic can be characterized by following theorems one by one:

Theorem 1. *If $x* \in X$ is unique optimal solution to model (4), then $x*$ is Pareto-optimal solution to the MOLPP (1) under T-intuitionistic environment.*

Theorem 2. *If $x* \in X$ is a Pareto-optimal solution of the MOLPP (1) under T-intuitionistic environment, then $x*$ is an intuitionistic fuzzy T-Pareto-optimal solution to model (4) for some $< T_i^{(+)}, T_i^{(-)} >$, i=1...k.*

From theorem 1 and theorem 2, it is clear that if the uniqueness of the optimal solution $x*$ to model (4) is not guaranteed, it feels necessary to perform Pareto-optimality test for $x*$. And Pareto-optimality test for $x*$ can be performed by solving optimization problem with decision variables $x = (x_1, x_2,..., x_n)^T$, $\Omega = (\Omega_1, \Omega_2,...,\Omega_k)^T$ and $\mho = (\mho_1, \mho_2,...,\mho_k)^T$ by solving the following model:

$$\max \sum_{i=1}^{k} \Omega_i + \sum_{i=1}^{k} \mho_i$$

$$\text{subject to } T_i^{(+)}(z_i(x)) - \Omega_i \geq T_i^{(+)}(z_i(x^*)), \tag{5}$$

$$i = 1 \ldots k, T_i^{(-)}(z_i(x)) + \mho_i \leq T_i^{(-)}(z_i(x^*)), i = 1 \ldots k, x \in X, \Omega_i, \mho_i \geq 0, i = 1 \ldots k.$$

Theorem 3 *Let* $\overline{x}, \overline{\Omega}$ *and* $\overline{\mho}$ *are optimal solutions to model (5). Then*

(1) *If* $\overline{\Omega}_i = 0 = \overline{\mho}_i, \forall i = 1 \ldots k$, *then x* is Pareto optimal solution of the MOLPP (1) under* T-intuitionistic *environment.*

(2) *If at least one* $\overline{\Omega}_i > 0$ *or* $\overline{\mho}_i > 0$, *then x* is not Pareto optimal solution to MOLPP under* T-intuitionistic *environment (1). Instead of x*,* $\overline{x}$ *is Pareto optimal solution to the MOLPP under* T-intuitionistic *environment (1).*

(Proofs of analogous theorems are recorded in literature).

5 Proposed general algorithm to solve MOLPP under imprecise environment

The above ideas can be further integrated into a general framework, and an algorithm can be developed to find Pareto optimal solutions to MOLPP (1) under imprecise environment, which have the additional property of being intuitionistic fuzzy T-Pareto optimal solutions for chosen intuitionistic fuzzy T-sets. The steps of the proposed algorithm may be synthesized as follows:

Step 1. Convert imprecise linguistic information to objective functions and/or constraints and construct optimization problem, as model (1). Request the DM to specify goals and tolerances for these imprecise objective functions and constraints.

Step 2. Elicit suitable $T^{(+)}$-characteristic functions for imprecise objectives $z_i(x), i = 1 \ldots k$ in such a way that higher values of $T^{(+)}$-characteristic functions yield preferable values to objectives. Also elicit suitable $T^{(-)}$-characteristic functions $T_i^{(-)}(z_i(x))$ for imprecise objectives in such a way that lower values of $T^{(-)}$-characteristic functions yield preferable values for objectives. Well-defined $T^{(+)}$-characteristic functions and $T^{(-)}$-characteristic functions preserve the amazing characteristic and their values are always finite.

Step 3. Construct single objective LPP as follows:

$$\max \ \alpha - \beta \text{ subject to } T_i^+(z_i(x)) \geq \alpha, i = 1, \ldots, k,$$

$$T_i^-(z_i(x)) \leq \beta, i = 1, \ldots, k, \alpha \geq \beta, \beta \geq 0, x \in X. \tag{A}$$

Step 4. Solve model (A). If intuitionistic fuzzy T-Pareto optimal solutions exist, go to step 8. Otherwise go to step 5.

Step 5. If model (A) is infeasible, remove constraint $\alpha \geq \beta$ and add constraint $\alpha \geq 0$ to it. The modified model becomes as follows:

$$\max \ \alpha - \beta \ \text{subject to} \ T_i^+(z_i(x)) \geq \alpha, \text{i} = 1, \ldots, \text{k},$$

$$T_i^-(z_i(x)) \leq \beta, \text{i} = 1, \ldots, \text{k}, \alpha \geq 0, \beta \geq 0, x \in X. \tag{B}$$

Solve this modified model (B). If it has intuitionistic fuzzy T-Pareto optimal solutions, go to step 8. Otherwise, go to step 6.

Step 6. Remove the newly added constraint $\alpha \geq 0$ and add the constraint $\beta \leq 1$ in last modified model. The modified model becomes

$$\max \ \alpha - \beta \ \text{subject to} \ T_i^+(z_i(x)) \geq \alpha, \text{i} = 1 \ldots \text{k},$$

$$T_i^-(z_i(x)) \leq \beta, \text{i} = 1 \ldots \text{k}, \beta \leq 1, \alpha \ \text{unrestricted in sign}, x \in X. \tag{C}$$

Solve model (C). If it has intuitionistic fuzzy T-Pareto optimal solutions, go to step 8. Otherwise, remove constraint $\beta \leq 1$ from it and solve. If it has intuitionistic fuzzy T-Pareto optimal solutions, go to step 8. Otherwise *the model is infeasible*, even under IF environment; go to step 7.

Step 7. Ask decision maker to modify goals and tolerances for IF objectives and constraints. Go to step 2.

It may be noted that goals and/or tolerances of IF objective functions should better lie within individual maximum and minimum values.

Step 8. These solutions are intuitionistic fuzzy T-Pareto optimal solutions under imprecise environment with specified goals and tolerances. To test whether these are also Pareto optimal solutions to model (1), solve model (5). Let $\bar{x}, \bar{\Omega}$ and $\bar{\bar{\mho}}$ are optimal solutions of model (5) in step 4. Then following two cases may arise:

   a. If $\bar{\Omega}_i = 0 = \bar{\bar{\mho}}_i \forall i = 1 \ldots k$, x^* are Pareto-optimal solutions to MOLPP (1) under T-intuitionistic environment.

   b. If $\bar{\Omega}_i > 0$ or $\bar{\bar{\mho}}_i > 0$ for at least one i, then x^* are not Pareto optimal solutions to MOLPP (1) under T-intuitionistic environment. Instead of x^*, $\bar{x}$ are Pareto optimal solutions to MOLPP (1) under T-intuitionistic environment.

These solutions are thus Pareto-optimal to MOLPP (1) under T-intuitionistic environment. Supply these optimal solutions to the DM. If he/she is satisfied with these solutions, stop. Otherwise go to Step 7.

6 Numerical examples

6.1 Numerical Example 1

Consider MOLPP under imprecise environment as follows:

$$\textit{imprecise} \ \max \ z_1(x) = 5x_1 + 5x_2, \textit{imprecise} \ \max \ z_2(x) = 3x_1 - 8.2x_2$$

$$\textit{subject to} \ 5x_1 + 7x_2 \leq 12, 9x_1 + x_2 \leq 10, -5x_1 + 3x_2 \leq 3, x_1, x_2 \geq 0. \tag{6}$$

To construct classical membership and non-membership functions for IF objective functions $z_i(x)$, $i = 1, 2$. Individual maximum and minimum of IF objective functions $z_i(x)$, $i = 1, 2$ are computed and are given in Table 1.

Objective functions	Individual maximum values	Individual minimum values
$z_1(x)$	10	0
$z_2(x)$	3.33	-14.06

Table 1. Individual maximum and minimum of objective functions

Objective functions	Goals	Tolerances for	
		membership functions	non-membership functions
$z_1(x)$	8	1.5	2
$z_2(x)$	-2	2	2.5

Table 2. Goals and tolerances of objective functions

Objective functions	Goals	Tolerances for	
		membership functions	non-membership functions
$z_1(x)$	8	1.5	2
$z_2(x)$	-2	2	2.5

Table 3. Goals and tolerances of objective functions

6.1.1. *Better result on removal of constraint* $\alpha + \beta \leq 1$

Suppose DM specifies goals, tolerances of IF objective functions as given in Table 2. Based on the information, we compute optimal solutions in existing IF optimization technique (by using Lingo 15.0.32). Next, we remove the constraint $\alpha + \beta \leq 1$ and solve the same problem with same goals, tolerances of IF objective functions in proposed algorithm (by using Lingo 15.0.32). The intuitionistic fuzzy T-Pareto optimal solutions and finally Pareto optimal solutions are obtained as given in Table 4.

Here it may be observed that proposed algorithm generates more preferable Pareto optimal solutions with both objective functions holding more preferable optimal values along with higher levels of acceptances as well as lower levels of rejections than existing IF optimization technique.

6.1.2. *Better result without adding constraint* $\alpha \leq 1$ (already constraint $\alpha + \beta \leq 1$ is not present)

Suppose DM specifies goals, tolerances of IF objective functions as given in Table 3. Based on the information, we compute optimal solutions in existing IF optimization technique (by using Lingo 15.0.32). Next, we remove the constraint $\alpha \leq 1$ and solve the same problem with same goals, tolerances of IF objective functions in proposed algorithm (by using Lingo 15.0.32). The intuitionistic fuzzy T-Pareto optimal solutions and finally Pareto optimal solutions are obtained as given in Table 4.

Here it may be observed that proposed algorithm may generate more preferable Pareto optimal solutions to MOLPP (6) with both objective functions holding more preferable optimal values along with higher levels of acceptances as well as lower levels of rejections than existing IF optimization technique.

No.	Optimal solutions by existing IF optimization technique (* denotes optimality)	Pareto Optimal solutions by proposed algorithm (⁻ denotes Pareto optimality):
6.1.1	$\alpha^* = 1$, $\beta^* = 0$, $x_1^* = 0.993$, $x_2^* = 0.607$, $z_1(x^*) = 8$, $z_2(x^*) = -2$, $\mu_1(z_1(x^*)) = \mu_2(z_2(x^*)) = 1$, $\nu_1(z_1(x^*)) = \nu_2(z_2(x^*)) = 0$.	$\bar{\Omega}_1 = \bar{\Omega}_2 = 0 = \bar{\upsilon}_1 = \bar{\upsilon}_2$, $\bar{x}_1 = 1.045$, $\bar{x}_2 = 0.594$, $z_1(\bar{x}) = 8.197$, $z_2(\bar{x}) = -1.738$, $T_1^{(+)}(z_1(\bar{x})) = T_2^{(+)}(z_2(\bar{x})) = 1.131$, $T_1^{(-)}(z_1(\bar{x})) = -0.098$, $T_2^{(-)}(z_2(\bar{x})) = -0.105$.
6.1.2	$\alpha^* = 1$, $\beta^* = 0$, $x_1^* = 0.993$, $x_2^* = 0.607$, $z_1(x^*) = 8$, $z_2(x^*) = -2$, $\mu_1(z_1(x^*)) = \mu_2(z_2(x^*)) = 1$, $\nu_1(z_1(x^*)) = \nu_2(z_2(x^*)) = 0$.	$\bar{\Omega}_1 = \bar{\Omega}_2 = 0 = \bar{\upsilon}_1 = \bar{\upsilon}_2$, $\bar{x}_1 = 1.045$, $\bar{x}_2 = 0.594$, $z_1(\bar{x}) = 8.197$, $z_2(\bar{x}) = -1.738$, $T_1^{(+)}(z_1(\bar{x})) = 1.131 = T_2^{(+)}(z_2(\bar{x})) = 1.131$, $T_1^{(-)}(z_1(\bar{x})) = -0.098$, $T_2^{(-)}(z_2(\bar{x})) = -0.105$.

Table 4. Results of model (6) in different cases

Objective functions	Individual maximum values	Individual minimum values
$z_1(x) = 5x_1 + 5x_2$	10	0
$z_2(x) = 5x_1 + x_2$	6	0
$z_3(x) = 3x_1 - 8.2x_2$	3.33	-14.06

Table 5. Individual maximum and minimum of objective functions

Objective functions	Goals	Tolerances for	
		membership functions	non-membership functions
$z_1(x)$	7	1.5	2
$z_2(x)$	2	2	2.5
$z_3(x)$	−2	2	2.5

Table 6. Goals and tolerances of objective functions

6.2 Numerical Example 2

Consider MOLPP under IF environment as follows:

$$imprecise \ \max 5x_1 + 5x_2, imprecise \ \min \ 5x_1 + x_2, imprecise \ \max \ 3x_1 - 8.2x_2$$
$$subject \ to \ 5x_1 + 7x_2 \leq 12, 9x_1 + x_2 \leq 10, -5x_1 + 3x_2 \leq 3, x_1, x_2 \geq 0. \tag{7}$$

To construct classical membership and non-membership functions for each IF objective function $z_i(x), i = 1, 2, 3$, individual maximum and minimum of each objective function may be obtained as in Table 5.

6.2.1. *Solution found after removal of constraint $\alpha \geq \beta$*

Suppose that DM specifies goals, tolerances of imprecise objective functions as given in Table 6. Based on the information, we compute optimal solutions by using existing IF optimization technique. Solving the problem by using Lingo 15.0.32, it is found that the *problem has no feasible solution*. Next, we remove the constraint $\alpha > \beta$ and solve the same

problem with same goals, tolerances of IF objective functions in proposed algorithm (by using Lingo 15.0.32). The intuitionistic fuzzy T-Pareto optimal solutions and finally Pareto optimal solutions are obtained as given in Table 9.

Here it may be observed that proposed algorithm may generate Pareto optimal solutions to MOLPP (7) under imprecise environment whereas existing IF optimization technique may make the MOLPP infeasible.

Objective functions	Goals	Tolerances for	
		membership functions	non-membership functions
$z_1(x)$	8	1.5	2
$z_2(x)$	1.5	2	2.5
$z_3(x)$	−2	2	2.5

Table 7. Goals and tolerances of objective functions

Objective functions	Goals	Tolerances for	
		membership functions	non-membership functions
$z_1(x)$	9.5	1.5	2
$z_2(x)$	1.5	2	2.5
$z_3(x)$	−2	2	2.5

Table 8. Goals and tolerances of objective functions

6.2.2. *Solution found after removal of constraint $\alpha \geq 0$* (so constraint $\alpha \geq \beta$ is not present)

Suppose that DM specifies goals, tolerances of imprecise objective functions as given in Table 7. Based on the information, we compute optimal solutions by using existing IF optimization technique. Solving the problem by using Lingo 15.0.32, it is found that the *problem has no feasible solution*. Next, we remove the constraint $\alpha \geq 0$ and solve the same problem with same goals, tolerances of IF objective functions in proposed algorithm (by using Lingo 15.0.32). The intuitionistic fuzzy T-Pareto optimal solutions and finally Pareto optimal solutions are obtained as given in Table 9.

Here it may be observed that proposed algorithm may generate Pareto optimal solutions to MOLPP (7) under imprecise environment whereas existing IF optimization technique may make the MOLPP infeasible.

6.2.3. *Solution found without adding constraint $\beta \leq 1$*
    (so constraints $\alpha \geq \beta$ and $\alpha \geq 0$ are not present)

Suppose that DM specifies goals, tolerances of imprecise objective functions as given in Table 8. Based on the information, we compute optimal solutions by using existing IF optimization technique. Solving the problem by using Lingo 15.0.32, it is found that the *problem has no feasible solution*. Next, we remove the constraint $\beta \leq 1$ and solve the same problem with same goals, tolerances of IF objective functions in proposed algorithm (by using

Lingo 15.0.32). The intuitionistic fuzzy T-Pareto optimal solutions and finally Pareto optimal solutions are obtained as given in Table 9.

Here it may be observed that proposed algorithm may generate Pareto optimal solutions to MOLPP (7) under imprecise environment whereas existing IF optimization technique may make the MOLPP infeasible.

No.	Optimal solutions by existing IF optimization technique	Pareto Optimal solutions by proposed algorithm ($^-$ denotes Pareto optimality):
6.2.1	No feasible solution found	$\bar{\Omega}_1 = \bar{\Omega}_2 = \bar{\Omega}_3 = 0 = \bar{\mho}_1 = \bar{\mho}_2 = \bar{\mho}_3, \bar{x}_1 = 0.561, \bar{x}_2 = 0.624,$ $z_1(\bar{x}) = 5.927,\ z_2(\bar{x}) = 3.431,\ z_3(\bar{x}) = -3.431,$ $T_1^{(+)}(z_1(\bar{x})) = T_2^{(+)}(z_2(\bar{x})) = T_3^{(+)}(z_3(\bar{x})) = 0.284,$ $T_1^{(-)}(z_1(\bar{x})) = 0.537, T_2^{(-)}(z_2(\bar{x})) = T_3^{(-)}(z_3(\bar{x})) = 0.572.$
6.2.2	No feasible solution found	$\bar{\Omega}_1 = \bar{\Omega}_2 = \bar{\Omega}_3 = 0 = \bar{\mho}_1 = \bar{\mho}_2 = \bar{\mho}_3, \bar{x}_1 = 0.576, \bar{x}_2 = 0.710,$ $z_1(\bar{x}) = 6.43, z_2(\bar{x}) = 3.59,\ z_3(\bar{x}) = -4.094,$ $T_1^{(+)}(z_1(\bar{x})) = T_2^{(+)}(z_2(\bar{x})) = T_3^{(+)}(z_3(\bar{x})) = -0.046,$ $T_1^{(-)}(z_1(\bar{x})) = 0.785, T_2^{(-)}(z_2(\bar{x})) = T_3^{(-)}(z_3(\bar{x})) = 0.837.$
6.2.3.	No feasible solution found	$\bar{\Omega}_1 = \bar{\Omega}_2 = \bar{\Omega}_3 = 0 = \bar{\mho}_1 = \bar{\mho}_2 = \bar{\mho}_3, \bar{x}_1 = 0.675, \bar{x}_2 = 0.820,$ $z_1(\bar{x}) = 7.477,\ z_2(\bar{x}) = 4.197,\ z_3(\bar{x}) = -4.697,$ $T_1^{(+)}(z_1(\bar{x})) = T_2^{(+)}(z_2(\bar{x})) = T_3^{(+)}(z_3(\bar{x})) = -0.349,$ $T_1^{(-)}(z_1(\bar{x})) = 1.011,\ T_2^{(-)}(z_2(\bar{x})) = T_3^{(-)}(z_3(\bar{x})) = 1.079.$

Table 9. Results of model (7) in different cases

6.3   Application of proposed algorithm in large scale steel plant

6.3.1. *Usage of T-sets in optimization technique in steel-iron industry problem*

This example is based on paper titled *A multiple objective model for purchasing of bulk raw materials of a large-scale integrated steel plant* by Zhen Gao, Lixin Tang (2003) published in *International Journal of Production Economics* [5]. Instead of assigning crisp goals of the objective functions, IF goals may be assigned to imprecise objective functions. Moreover, in place of membership and non-membership functions, $T^{(+)}$-characteristic functions and $T^{(-)}$-characteristic functions may be used in this MOLPP with imprecise goals. Here the problem is to purchase raw materials of a large scale steel plant. In steel-iron industry, selection of appropriate items is the key to reduce production cost. Decision on quantities of raw materials, using strong professional knowledge on steel iron metallurgy, is another key objective. Selection of vendors to keep stability and quality of supply of raw materials is also key objective in steel-iron industry. Using two dimensional vectors x_{ij} denoting order quantity of jth item of raw materials from i^{th} vendor, and assuming the model as single time phase model, the constraints under consideration may be taken as follows: purchasing budget constraint, production demand constraint, inventory capacity constraint, technological requirement constraint, vendor resource constraint. Suppose the problem is to solve MOLPP with seven vendors and thirteen items that belong to four large kinds of bulk raw materials –

ore, lump ore, pellet, coal [5]. Therefore analogous to Gao (2003), the MOLPP with imprecise objective functions may be taken as follows:

$$\text{Imprecise min } z_1(x) = 0.112x_{11} + 0.127x_{31} + 0.122x_{41} + 0.115x_{51} + 0.119x_{71} + 0.0654x_{12} + 0.0621x_{22}$$
$$+ 0.0586x_{32} + 0.0602x_{62} + 0.195x_{33} + 0.185x_{53} + 0.09521x_{14} + 0.0975x_{34}$$

$$\text{Imprecise min } z_2(x) = 0.1x_{11} + 0.155x_{31} + 0.17x_{41} + 0.12x_{51} + 0.2x_{71} + 0.1x_{12} + 0.25x_{22} + 0.15x_{32} + 0.3x_{62}$$
$$+ 0.15x_{33} + 0.12x_{53} + 0.1x_{14} + 0.15x_{34}$$

$$\text{Imprecise min } z_3(x) = 0.2x_{11} + 0.1x_{31} + 0.15x_{41} + 0.17x_{51} + 0.13x_{71} + 0.2x_{12} + 0.1x_{22} + 0.15x_{32} + 0.22x_{62}$$
$$+ 0.15x_{33} + 0.17x_{53} + 0.2x_{14} + 0.15x_{34}$$

$$\text{subject to } z_1(x) \leq 16.373, 1.2x_{11} + 0.9x_{31} + x_{41} + 1.1x_{51} + 0.95x_{71} \geq 60,$$
$$1.25x_{12} + 0.95x_{22} + 1.15x_{32} + 1.05x_{62} \geq 30, 1.3x_{33} + 1.1x_{53} \geq 10, \ 1.12x_{14} + 1.24x_{34} \geq 70,$$
$$2x_{12} + 2x_{22} + 2x_{32} + 2x_{62} + 3x_{33} + 3x_{53} = x_{11} + x_{31} + x_{41} + x_{51} + x_{71} + x_{14} + x_{34}, x_{ij} \geq 0,$$
$$\forall i = 1...7, j = 1...4. \tag{8}$$

Proposed algorithm may be used to find Pareto optimal solutions to this MOLPP under T-intuitionistic environment.

Different cases may be considered based on different goals and tolerances (these are supplied by DM or presumed values) and the results are tabulated in Table 10. There, in cases I, II and III, it is shown that there exists no optimal solution if classical IF membership and non-membership functions are used along with existing optimization technique.

Cases		I			II			III		
Objective functions		$z_1(x)$	$z_2(x)$	$z_3(x)$	$z_1(x)$	$z_2(x)$	$z_3(x)$	$z_1(x)$	$z_2(x)$	$z_3(x)$
Goals		16	18.5	27.5	15.5	22.5	23.8	15.1	20.4	24.4
Tolerances of	$T_i^{(+)}{}_{(z_i(x))}$	0.3	0.3	0.3	0.5	0.5	0.5	0.4	0.4	0.4
	$T_i^{(-)}{}_{(z_i(x))}$	0.5	0.5	0.5	0.7	0.7	0.7	0.6	0.6	0.6
Solutions by existing technique		*No feasible solution found*			*No feasible solution found*			*No feasible solution found*		
Intuitionistic fuzzy T-Pareto optimal solutions		15.833	18.333	27.333	15.829	22.829	24.129	15.734	21.034	25.034
Pareto optimal solutions		15.833	18.333	27.333	15.829	22.829	24.129	15.734	21.034	25.034
Optimal grades of	$T_i^{(+)}{}_{(z_i(x))}$	1.56	1.56	1.56	0.34	0.34	0.34	-0.58	-0.58	-0.58
	$T_i^{(-)}{}_{(z_i(x))}$	-0.33	-0.33	-0.33	0.47	0.47	0.47	1.06	1.06	1.06
Remarks		1. Removal of $\alpha + \beta \leq 1$, $\beta \geq 0$ (and not adding $\alpha \leq 1$ gives optimal solution. 2. *More acceptable optimal values* than crisp method to each objective function (Gao et al, 2003)			1. Removal of $\alpha \geq \beta$ gives optimal solution. 2. *More acceptable optimal values* than crisp method to each objective function (Gao et al, 2003)			1. Removal of $\alpha \leq 0$ (and not adding $\beta \geq 1$) gives optimal solution. 2. *More acceptable optimal values* than crisp method to each objective function (Gao et al, 2003)		

Table 10. Purchase of bulk raw materials of a large-scale integrated steel plant with IF goals

But if strictly monotonic $T^{(+)}$-characteristic functions and $T^{(-)}$-characteristic functions, as proposed in this paper, are used, proposed algorithm may generate intuitionistic fuzzy T-Pareto optimal solutions and hence Pareto optimal solutions. Moreover the Pareto optimal solutions in case I, case II and case III in Table 8 are more preferable for all objective functions to the DM than crisp solution as given by Gao (2003). In proposed algorithm, the optimal values of $z_1(x), z_2(x), z_3(x)$ in case I, case II and case III are Pareto optimal. And result in case I is better than solution number 2, case II is better than solution number 3 and solution number 4, case III is better than solution number 1 in crisp optimization by Gao (2003).

6.3.2. *T-sets to solve MOLPP under impreciseness when little information is available*

It is discussed in this paper that traditional membership function may not be the only function whose higher value gives preferable value of objective function. Similar statement can be made for non-membership function as well.

Now in numerical example 6.3.1, $T^{(+)}$-characteristic functions and $T^{(-)}$-characteristic functions are constructed analogous to traditional membership and non-membership functions. But for minimizing type of objective functions $z_i(x)$, one example of $T^{(+)}$-characteristic function may be $-z_i(x)$ and of $T^{(-)}$-characteristic function may be $z_i(x)$ itself. These may be useful specially when goals and/or tolerances cannot be determined by DM or even a DM may not be available at all; so that traditional membership and/or non-membership functions cannot be defined for IF objectives. Using $-z_i(x)$ as $T^{(+)}$-characteristic function and $z_i(x)$ as $T^{(-)}$-characteristic function for each IF objective function $z_i(x) \forall i = 1, 2, 3$ and applying proposed algorithm on MOLPP (8) under imprecise environment, single objective LPP, analogous to model (4), may be obtained and by using Lingo 15.0.32, intuitionistic fuzzy T-Pareto optimal solutions may be obtained as follows:

$$\alpha^* = -23.704, \ \beta^* = 23.704, \ z_1(x^*) = 16.373, \ z_2(x^*) = 23.704, z_3(x^*) = 23.704.$$

To test the Pareto optimality of these intuitionistic fuzzy T-Pareto optimal solutions, one problem, analogous to model (5), may be solved and by using Lingo 15.0.32, Pareto optimal solutions $\bar{x}$, $\bar{\Omega}$ and $\bar{\mho}$ may be obtained as follows

$$\bar{\Omega}_1 = \bar{\Omega}_2 = \bar{\Omega}_3 = 0 = \bar{\mho}_1 = \bar{\mho}_2 = \bar{\mho}_3, z_1(\bar{x}) = 16.373, \ z_2(\bar{x}) = 23.704, z_3(\bar{x}) = 23.704.$$

This is another set of Pareto optimal solutions. It may be mentioned that $T^{(+)}$-characteristic functions and $T^{(-)}$-characteristic functions may be defined in many other suitable ways.

7 Conclusions

Technique to find Pareto optimal solutions to MOLPP under imprecise environment is discussed in this paper. IF technique is one of the richest apparatus for formulation of optimization problems under imprecise environment, thereby generating more satisfying result than crisp optimization technique. In this paper, limitations of existing IF optimization technique are discussed. It is identified that existing IF optimization technique may not only fail to correctly identify *the best solution among the good but also it may fail to find bad*

solution among the worst! Hence it may be concluded that existing IF optimization technique may not be a general procedure to obtain Pareto optimal solutions to MOLPP under impreciseness.

To overcome these limitations, new set viz. intuitionistic fuzzy T-set is proposed to supersede IF set for representing impreciseness. An algorithm is proposed involving intuitionistic fuzzy T-sets that generates most preferable Pareto optimal solutions in all cases whenever they exist and may be termed as a general procedure to attain Pareto optimality to MOLPP under impreciseness.

Moreover in real life situations, the DM may not be able to identify the goals or tolerances or both of imprecise objective functions; and in some cases, DM may not exist at all. In these scenarios, it is difficult to construct IF membership and non-membership functions of imprecise objective functions. But $T^{(+)}$-characteristic functions and $T^{(-)}$-characteristic functions of T-sets may be utilized to measure impreciseness and hence may be applied to yield Pareto optimal solutions to MOLPP under impreciseness. And in fact, usage of well-defined $T^{(+)}$-characteristic functions and $T^{(-)}$-characteristic functions of T-sets may remove the necessity to fetch information from DM and saves precious time of both.

Under these circumstances, it may be further concluded that issues of getting no solution as well as Pareto optimality having optimal membership value at unity by using existing IF optimization technique yielded from several published methods seems worthwhile or even necessary to reconsider. The same is true for nonlinear problems of this type as well.

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