1st Int. Workshop on IFSs, Mersin, 14 Nov. 2014
Notes on Intuitionistic Fuzzy Sets
ISSN 1310–4926
Vol. 20, 2014, No. 5, 36–49

Resolution of a system of the max-min product intuitionistic fuzzy relation equations using LU-factorization

S. Melliani*, M. Elomari, L. S. Chadli and R. Ettoussi

LMACS – Laboratoire de Mathématiques Appliquées & Calcul Scientifique
Sultan Moulay Slimane University
PO Box 523, 23000 Beni Mellal, Morocco

Corresponding author: melliani@fstbm.ac.ma

Abstract: In this paper, the LU-factorization is extended to the intuitionistic fuzzy square matrix with respect to the max-min product composition operator. Initially The algorithms are given to find two intuitionistic fuzzy (lower and upper) triangular matrices L and U for an intuitionistic fuzzy square matrix A such $A = LU$ and a result for the existence and uniqueness of this decomposition. An algorithm is also proposed to find the solution set of a square system of Intuitionistic Fuzzy Relation Equations (IFRE) using the LU-factorization. Finally an example is given to illustrate our work.
Keywords: Intuitionistic fuzzy sets, Intuitionistic fuzzy relation equation, Intuitionistic fuzzy matrix, Max-min product operator, LU-factorization, Intuitionistic fuzzy number.
AMS Classification: 03E72, 08A72.

1 Introduction

The LU-factorization method in the linear algebra with the summation and product operations is very interesting and useful for the reduction of a non-sparse square matrix to the product of two sparse square matrices called lower and upper triangular matrices. The method is applied to solve a non-sparse square linear system. Using the LU-factorization method, the non-sparse square linear system is easily reduced to two sparse square linear systems as lower and upper triangular systems. Hence, we can easily solve the systems by the forward and backward substitutions. This procedure is called the resolution of the system by the LU-factorization.

Ali Abbasi Molai in [4] extended the decomposition LU to the fuzzy square matrix with respect to the max-product composition operator called LU-factorization. An algorithm is presented to find the matrices L and U. Furthermore, some necessary and sufficient conditions are proposed for the existence and uniqueness of the LU-factorization for a given fuzzy square matrix A.

Let IF_1 the space of intuitionistic fuzzy set endowed by the following two binary operations

$$(u,v)+(u',v')=\Big(\max(u,v'),\min(u,v')\Big)$$

and

$$(u,v).(u',v')=\Big(\min(u,v'),\max(u,v')\Big)$$

by [6] these two operations give birth to an intuitionistic algebra. More work is apparent from the concept of algebra, and among the major give addition and multiplication of two matrices of meaning in the intuitionistic fuzzy theory.

The authors of the article [7] give the conditions for which the system of intuitionistic fuzzy linear equations (IFLEs) be solvable and the method Chybechev happens approached a matrix in which the system is not solvable by another whose system solvent. Therefore, the study of the LU-factorization for the systems of IFREs with the max-min product operator is motivated.

This paper is organized as follows. Section 2 studies the LU-factorization of an intuitionistic fuzzy matrix with respect to the max-min product composition. An algorithms also presented to find the matrices L and U. Section 3 presents the necessary and sufficient conditions for the existence and uniqueness of the LU-factorization for a given intuitionistic fuzzy square matrix. In Section 4, an algorithm is proposed to solve the system of the max-min product IFREs using the LU-factorization. The algorithm is outlined and illustrated by an example. The conclusions are finally given in Section 5.

2 Preliminaries

Let $(\text{IF})_{n\times n}$ (be the set of all intuitionistic fuzzy matrices of order $n\times n$. First we shall represent $A\in(\text{IF})_{n\times n}$ as Cartesian product of fuzzy matrices. The Cartesian product of any two matrices $A=(a_{ij})_{n\times n}$ $B=(b_{ij})_{n\times n}$, denoted as $\langle A,B\rangle$ is defined as the matrix whose ij-th entry is the ordered pair $\langle A,B\rangle=(\langle a_{ij},b_{ij}\rangle)$.
For $A=(a_{ij})_{n\times n}=(\langle a_{ij\mu},a_{ij,\nu}\rangle)$, we define $A_\mu=(a_{ij,\mu})\in(\text{IF})^M_{n\times n}$ as the membership part of A and $A_\nu=(a_{ij,\nu})\in(\text{IF})^N_{n\times n}$ as the non membership part of A. Thus A is the Cartesian product of A_μ and A_ν written as $A=\langle A_\mu,A_\nu\rangle$ with $A_\mu\in(\text{IF})^M_{n\times n}$, $A_\nu\in(\text{IF})^N_{n\times n}$.
We shall follow the matrix operations on intuitionistic fuzzy matrices as defined in our earlier works [5],[6]
For A, $B\in(\text{IF})_{n\times n}$, if $A=\langle A_\mu,A_\nu\rangle$ and $B=\langle B_\mu,B_\nu\rangle$, then

$$A+B=\langle A_\mu+B_\mu,A_\nu+B_\nu\rangle=\langle\max\{a_{ij,\mu},b_{ij,\mu}\},\min\{a_{ij,\nu},b_{ij,\nu}\}\rangle \quad (1)$$

$$A.B = \langle A_\mu.B_\mu, A_\nu.B_\nu \rangle = \langle \max_k \min\{a_{ik,\mu}, b_{kj,\mu}\}, \min_k \max\{a_{ik,\nu}, b_{kj,\nu}\} \rangle \tag{2}$$

We define the order relation on $(\mathrm{IF})_{n\times n}$ as,

$$A \leq B \Leftrightarrow a_{ij,\mu} \leq b_{ij,\mu} \quad \text{and} \quad a_{ij,\nu} \geq b_{ij,\nu} \quad \forall i,\ j \tag{3}$$

Let us consider the intuitionistic fuzzy relational equation of the form $Ax = b$, where A is an intuitionistic fuzzy matrix of order $n \times n$ and b is the intuitionistic fuzzy vector and $\Omega(A, b)$ be the set of all solutions of the equation $Ax = b$

Lemma 1. *[7] Let us consider the system of intuitionistic fuzzy relation equation $Ax = b$. If $\max_j(\langle a_{ij,\mu}, a_{ij,\nu}\rangle) < (\langle b_{k,\mu}, b_{k,\nu}\rangle)$ for some k , then $\Omega(A, b) = \varnothing$, that is the system is not solvable.*

Definition 1. *Any element $\overline{x}$ of $\Omega(A, b)$ is called a maximum solution of the system $Ax = b$ if for all $x \in \Omega(A, b)$, $x \geq \overline{x}$ implies $x = \overline{x}$.*

3 LU-factorization of the intuitionistic fuzzy matrix with respect to the max-min composition operator

Let $A = [a_{ij}]$ be an $n \times n$ dimensional intuitionistic fuzzy matrix where $a_{ij} = \langle a_{ij,\mu}, a_{ij,\nu}\rangle$. We define $A_\mu = (a_{ij,\mu})$ as the membership part of A and $A_\nu = (a_{ij,\nu})$ as the non membership part of A such that $0 \leq a_{ij,\mu} + a_{ij,\nu} \leq 1$. Thus A is the Cartesian product of A_μ and A_ν written as $A = \langle A_\mu, A_\nu\rangle$. We will show how to decompose a square intuitionistic fuzzy matrix A into two intuitionistic fuzzy (lower and upper) triangular matrices L and U such that $A = LU$.

In other words, we try to find two intuitionistic fuzzy (lower) triangular matrices $L = [l_{ij}]_{n\times n} = \langle l_{ij,\mu}, l_{ij,\nu}\rangle_{n\times n}$ and $U = [u_{ij}]_{n\times n} = \langle u_{ij,\mu}, u_{ij,\nu}\rangle_{n\times n}$ such that $0 \leq l_{ij,\mu} + l_{ij,\nu} \leq 1$ and $0 \leq u_{ij,\mu} + u_{ij,\nu} \leq 1$

$$\langle l_{ij,\mu}, l_{ij,\nu}\rangle = \begin{cases} \langle l_{ij,\mu}, l_{ij,\nu}\rangle & i > j \\ \langle 1, 0\rangle & i = j \\ \langle 0, 1\rangle & i < j \end{cases} \tag{4}$$

and

$$\langle u_{ij,\mu}, u_{ij,\nu}\rangle = \begin{cases} \langle u_{ij,\mu}, u_{ij,\nu}\rangle & i \leq j \\ \langle 0, 1\rangle & i > j \end{cases} \tag{5}$$

$$a_{ij,\mu} = \max_{t\in\underline{n}} \Big(\min(l_{it,\mu}, u_{tj,\mu}) \Big), \quad a_{ij,\nu} = \min_{t\in\underline{n}} \Big(\max(l_{it,\nu}, u_{tj,\nu}) \Big) \quad \forall 0 \leq i, j \leq 1$$

where $\underline{n} := \{1, 2, \ldots, n\}$.

With regard to the special structure of the intuitionistic fuzzy lower and upper matrices L and U and the max-min, min-max composition operators, we can write:

$$
\begin{aligned}
&(S_1^\mu) \quad a_{1j,\mu} = \min\Big(l_{11,\mu}, u_{1j,\mu}\Big) \quad \forall j \in \underline{n}\\
&(S_2^\mu) \quad a_{21,\mu} = \min\Big(l_{21,\mu}, u_{11,\mu}\Big);\\
&\qquad\quad a_{2j,\mu} = \max\Big(\min\left(l_{21,\mu}, u_{1j,\mu}\right), \min(l_{22,\mu}, u_{2j,\mu})\Big) \quad j = 2, \dots, n;\\
&(S_3^\mu) \quad a_{31,\mu} = \min\left(l_{31,\mu}, u_{11,\mu}\right);\\
&\qquad\quad a_{32,\mu} = \max\{\min\left(l_{31,\mu}, u_{12,\mu}\right), \min(l_{32,\mu}, u_{22,\mu})\};\\
&\qquad\quad a_{3j,\mu} = \max\Big\{\min(l_{31\mu}, u_{1j\mu}), \min(l_{32\mu}, u_{2j\mu}), \min(l_{33\mu}, u_{3j\mu})\Big\} \quad j = 3, \dots, n\\
&(S_4^\mu) \quad a_{41,\mu} = \min(l_{41,\mu}, u_{11,\mu});\\
&\qquad\quad a_{42,\mu} = \max\Big\{\min(l_{41,\mu}, u_{12,\mu}), \min(l_{42,\mu}, u_{22,\mu})\Big\};\\
&\qquad\quad a_{43,\mu} = \max\Big\{\min(l_{41,\mu}, u_{13,\mu}), \min(l_{42,\mu}, u_{23,\mu}), \min(l_{43,\mu}, u_{33,\mu})\Big\};\\
&\qquad\quad a_{4j,\mu} = \max\Big\{\min(l_{41,\mu}, u_{1j,\mu}), \min(l_{42,\mu}, u_{2j,\mu}), \min(l_{43,\mu}, u_{3j,\mu}), \min(l_{44,\mu}, u_{4j,\mu})\Big\},\\
&\qquad\quad j = 4, \dots, n\\
&(S_n^\mu) \quad a_{n1,\mu} = \min(l_{n1,\mu}, u_{11,\mu});\\
&\qquad\quad a_{n2,\mu} = \max\Big\{\min(l_{n1,\mu}, u_{12,\mu}), \min(l_{n2,\mu}, u_{22,\mu})\Big\};\\
&\qquad\quad \vdots\\
&\qquad\quad a_{n,n-1,\mu} = \max\Big\{\min(l_{n1,\mu}, u_{1,n-1,\mu}), \min(l_{n2,\mu}, u_{2,n-1,\mu}), \dots, \min(l_{n,n-1,\mu}, u_{n-1,n-1,\mu})\Big\},\\
&\qquad\quad a_{nn,\mu} = \max\Big\{\min(l_{n1,\mu}, u_{1n,\mu}), \min(l_{n2,\mu}, u_{2n,\mu}), \dots, \min(l_{nn,\mu}, u_{nn,\mu})\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
&(S_1^\nu) \quad a_{1j,\nu} = \max(l_{11,\nu}, u_{1j,\nu}) \quad j \in \underline{n}\\
&(S_2^\nu) \quad a_{21,\nu} = \max(l_{21\nu}, u_{11\nu});\\
&\qquad\quad a_{2j,\nu} = \min\Big\{\max(l_{21,\nu}, u_{1j,\nu}), \max(l_{22,\nu}, u_{2j,\nu})\Big\} \quad j = 2, \dots, n;\\
&(S_3^\nu) \quad a_{31,\nu} = \max(l_{31,\nu}, u_{11,\nu});\\
&\qquad\quad a_{32,\nu} = \min\Big\{\max(l_{31,\nu}, u_{12,\nu}), \max(l_{32,\nu}, u_{22,\nu})\Big\};\\
&\qquad\quad a_{3j,\nu} = \min\Big\{\max(l_{31,\nu}, u_{1j,\nu}), \max(l_{32,\nu}, u_{2j,\nu}), \max(l_{33,\nu}, u_{3j,\nu})\Big\} \quad j = 3, \dots, n\\
&(S_4^\nu) \quad a_{41,\nu} = \max(l_{41,\nu}, u_{11,\nu});\\
&\qquad\quad a_{42,\nu} = \min\Big\{\max(l_{41,\nu}, u_{12,\nu}), \max(l_{42,\nu}, u_{22,\nu})\Big\};\\
&\qquad\quad a_{43,\nu} = \min\Big\{\max(l_{41,\nu}, u_{13,\nu}), \max(l_{42,\nu}, u_{23,\nu}), \max(l_{43,\nu}, u_{33,\nu})\Big\};\\
&\qquad\quad a_{4j,\nu} = \min\Big\{\max(l_{41,\nu}, u_{1j,\nu}), \max(l_{42,\nu}, u_{2j,\nu}), \max(l_{43,\nu}, u_{3j,\nu}), \max(l_{44,\nu}, u_{4j,\nu})\Big\},\\
&\qquad\quad j = 4, \dots, n\\
&(S_n^\nu) \quad a_{n1,\nu} = \max(l_{n1,\nu}, u_{11,\nu});\\
&\qquad\quad a_{n2,\nu} = \min\Big\{\max(l_{n1,\nu}, u_{12,\nu}), \max(l_{n2,\nu}, u_{22,\nu})\Big\};\\
&\qquad\quad \vdots\\
&\qquad\quad a_{n,n-1,\nu} = \min\Big\{\max(l_{n1,\nu}, u_{1,n-1,\nu}), \max(l_{n2,\nu}, u_{2,n-1,\nu}), \dots, \max(l_{n,n-1,\nu}, u_{n-1,n-1,\nu})\Big\};\\
&\qquad\quad a_{nn\nu} = \min\Big\{\max(l_{n1,\nu}, u_{1n,\nu}), \max(l_{n2,\nu}, u_{2n,\nu}), \dots, \max(l_{nn,\nu}, u_{nn,\nu})\Big\}
\end{aligned}
$$

If we can solve the systems $(S_1^\mu) - (S_n^\mu)$ and $(S_1^\nu) - (S_n^\nu)$ with respect to the components l_{ij} for $i > j$ and u_{ij} for $i \leq j$ where $i, j \in \underline{n}$ then at least an LU-factorization is found for the square intuitonistic fuzzy matrix $A = [a_{ij}]$.

Now we present two algorithms to find the components l_{ij} for $i > j$, and u_{ij} for $i \leq j$, where $i, j \in \underline{n}$. The first one is an algorithm for computing the matrices L_μ and U_μ.

Algorithm 3.1 Let $A_\mu = [a_{ij\mu}], 0 \leq a_{ij\mu} \leq 1$, be an $n \times n$-dimensional fuzzy matrix.

Step 1. Let $l_{ii\mu} = 1$, for each $i \in \underline{n}$ and $u_{1j\mu} = a_{1j\mu}$ for each $j \in \underline{n}$.

Step 2. Run **Proc2**$(a_{21\mu}, 0, u_{11\mu})$; Let $l_{21\mu}$ = **Proc2**$(a_{21\mu}, 0, u_{11\mu})$;
For $j = 2$ to n
$\{$Run **Proc1**$(a_{2j\mu}, \min\{l_{21\mu}, u_{1j\mu}\})$; Let $u_{2j\mu}$ = **Proc1**$(a_{2j\mu}, \min\{l_{21\mu}, u_{1j\mu}\})$;$\}$

Step 3. Run **Proc2**$(a_{31\mu}, 0, u_{11\mu})$; Let $l_{31\mu}$ = **Proc2**$(a_{31\mu}, 0, u_{11\mu})$;
Run **Proc2**$(a_{32\mu}, \min\{l_{31\mu}, u_{12\mu}\}, u_{22\mu})$; Let $l_{32\mu}$ = **Proc2**$(a_{32\mu}, \min\{l_{31\mu}, u_{12\mu}\}, u_{22\mu})$;
For $j = 3$ to n
$\{$Let max=$\max\{\min\{l_{31\mu}, u_{1j\mu}\}, \min\{l_{32\mu}, u_{2j\mu}\}\}$;
Run **Proc1**$(a_{3j}$,max); Let $u_{3j\mu}$ = **Proc1**$(a_{3j}$,max)$\}$;

Step 4. Run **Proc2**$(a_{41\mu}, 0, u_{11\mu})$; Let $l_{41\mu}$ = **Proc2**$(a_{41\mu}, 0, u_{11\mu})$;
Run **Proc2**$(a_{42\mu}, \min\{l_{41\mu}, u_{12\mu}\}, u_{22\mu})$; Let $l_{42\mu}$ = **Proc2**$(a_{42\mu}, \min\{l_{41\mu}, u_{12\mu}\}, u_{22\mu})$;
Let max=$\max\{\min\{l_{41\mu}, u_{13\mu}\}, \min\{l_{42\mu}, u_{23\mu}\}\}$;
Run **Proc2**$(a_{43\mu}$,max,$u_{33\mu})$; Let $l_{43\mu}$ = **Proc2**$(a_{43\mu}$,max,$u_{33\mu})$;
For $j = 4$ to n
$\{$ Let max=$\max\{\min\{l_{41\mu}, u_{1j\mu}\}, \min\{l_{42\mu}, u_{2j\mu}\}, \min\{l_{43\mu}, u_{3j\mu}\}\}$
Run **Proc1**$(a_{4j}$,max); Let $u_{4j\mu}$ = **Proc1**$(a_{4j}$,max); $\}$

$\vdots$

Step n. Run **Proc2**$(a_{n1\mu}, 0, u_{11\mu})$; Let $l_{n1\mu}$ = **Proc2**$(a_{n1\mu}, 0, u_{11\mu})$;
Run **Proc2**$(a_{n2\mu}, \min\{l_{n1\mu}, u_{12\mu}\}, u_{22\mu})$; Let $l_{n2\mu}$ = **Proc2**$(a_{n2\mu}, \min\{l_{n1\mu}, u_{12\mu}\}, u_{22\mu})$;
...
Let max= $\max\{\min\{l_{n1\mu}, u_{1,n-1\mu}\}, ..., \min\{l_{n,n-2\mu}, u_{n-2,n-1\mu}\}\}$
Run **Proc2**$(a_{n,n-1\mu}$,max,$u_{n-1,n-1\mu})$; Let $l_{n,n-1\mu}$ = **Proc2**$(a_{n,n-1\mu}$,max,$u_{n-1,n-1\mu})$;
Let max= $\max\{\min\{l_{n1\mu}, u_{1,n\mu}\}, \min\{l_{n2\mu}, u_{2n\mu}\}, .., \min\{l_{n,n-1\mu}, u_{n-1,n\mu}\}\}$
Run **Proc1**$(a_{nn\mu}$,max); Let $u_{nn\mu}$ = **Proc1**$(a_{nn\mu}$,max); $\}$

End.

Furthermore, the used procedures in the algorithm are presented below:

Procedure Proc1(a, lu)
Begin
if $a > lu$ then let $u = a$;
if $a = lu$ then choose the value u from $[0, a]$, arbitrarily;
if $a < lu$ then the matrix A_μ has no LU-factorization and $u = \emptyset$;
Return(l);
End.

Procedure Proc2(a, lu, u)
Begin
if $lu < a \leq u$ then let $l = a$ and Return(l);
if $lu = a \leq u$ then let $l \in [0, a]$ and Return(l);
Else the matrix A_μ has no LU-factorization and $l = \emptyset$; and Return(l);
End.

An algorithm for computing the matrices L_ν and U_ν

Algorithm 3.2 Let $A_\nu = [a_{ij\nu}], 0 \leq a_{ij\nu} \leq 1$, be an $n \times n$-dimensional fuzzy matrix.

Step 1. Let $l_{ii\nu} = 0$, for each $i \in \underline{n}$ and $u_{1j\nu} = a_{1j\nu}$ for each $j \in \underline{n}$.

Step 2. Run **Proc4**$(a_{21\nu}, 1, u_{11\nu})$; Let $l_{21\nu}$ = **Proc4**$(a_{21\nu}, 1, u_{11\nu})$;
For $j = 2$ to n
$\{$Run **Proc3**$(a_{2j\nu}, \max\{l_{21\nu}, u_{1j\nu}\})$; Let $u_{2j\nu}$ = **Proc3**$(a_{2j\nu}, \max\{l_{21\nu}, u_{1j\nu}\})$;$\}$

Step 3. Run **Proc4**$(a_{31\nu}, 1, u_{11\nu})$; Let $l_{31\nu}$ = **Proc4**$(a_{31\nu}, 1, u_{11\nu})$;
Run **Proc4**$(a_{32\nu}, \max\{l_{31\nu}, u_{12\nu}\}, u_{22\nu})$; Let $l_{32\nu}$ = **Proc4**$(a_{32\nu}, \max\{l_{31\nu}, u_{12\nu}\}, u_{22\nu})$;
For $j = 3$ to n
$\{$Let min=$\min\{\max\{l_{31\nu}, u_{1j\nu}\}, \max\{l_{32\nu}, u_{2j\nu}\}\}$;
Run **Proc3**(a_{3j},min); Let $u_{3j\nu}$ = **Proc3**(a_{3j},min)$\}$;

Step 4. Run **Proc4**$(a_{41\nu}, 1, u_{11\nu})$; Let $l_{41\nu}$ = **Proc4**$(a_{41\nu}, 1, u_{11\nu})$;
Run **Proc4**$(a_{42\nu}, \max\{l_{41\nu}, u_{12\nu}\}, u_{22\nu})$; Let $l_{42\nu}$ = **Proc4**$(a_{42\nu}, \max\{l_{41\nu}, u_{12\nu}\}, u_{22\nu})$;
Let min=$\min\{\max\{l_{41\nu}, u_{13\nu}\}, \max\{l_{42\nu}, u_{23\nu}\}\}$;
Run **Proc4**($a_{43\nu}$,min,$u_{33\nu}$); Let $l_{43\nu}$ = **Proc4**($a_{43\nu}$,min,$u_{33\nu}$);
For $j = 4$ to n
$\{$ Let min=$\min\{\max\{l_{41\nu}, u_{1j\nu}\}, \max\{l_{42\nu}, u_{2j\nu}\}, \max\{l_{43\nu}, u_{3j\nu}\}\}$
Run **Proc3**(a_{4j},min); Let $u_{4j\nu}$ = **Proc3**(a_{4j},min); $\}$

⋮

Step n. Run **Proc4**($a_{n1\nu}, 1, u_{11\nu}$); Let $l_{n1\nu}$ = **Proc4**($a_{n1\nu}, 1, u_{11\nu}$);
Run **Proc4**($a_{n2\nu}, \max\{l_{n1\nu}, u_{12\nu}\}, u_{22\nu}$); Let $l_{n2\nu}$ = **Proc4**($a_{n2\nu}, \max\{l_{n1\nu}, u_{12\nu}\}, u_{22\nu}$);
...
Let min= $\min\{\max\{l_{n1\nu}, u_{1,n-1\nu}\}, ..., \max\{l_{n,n-2\nu}, u_{n-2,n-1\nu}\}\}$
Run **Proc4**($a_{n,n-1\nu}$,min,$u_{n-1,n-1\nu}$); Let $l_{n,n-1\nu}$ = **Proc4**($a_{n,n-1\nu}$,min,$u_{n-1,n-1\nu}$);
Let min= $\min\{\max\{l_{n1\nu}, u_{1,n\nu}\}, \max\{l_{n2\nu}, u_{2n\nu}\}, ..., \max\{l_{n,n-1\nu}, u_{n-1,n\nu}\}\}$
Run **Proc3**($a_{nn\nu}$,min); Let $u_{nn\nu}$ = **Proc3**($a_{nn\nu}$,min); }

End.

Furthermore, the used procedures in the algorithm are presented below:

Procedure Proc3(a, lu)
Begin
if $a < lu$ then let $u = a$;
if $a = lu$ then choose the value u from $[a, 1]$, arbitrarily;
if $a > lu$ then the matrix A_ν has no LU-factorization and $u = \emptyset$;
Return(l);
End.

Procedure Proc4(a, lu, u)
Begin
if $u \leq a < lu$ then let $l = a$ and Return(l);
if $u \leq a = lu$ then let $l \in [a, 1]$ and Return(l);
Else the matrix A_ν has no LU-factorization and $l = \emptyset$; and Return(l);
End.

Example
Consider the intuitionistic fuzzy matrix $A = \langle A_\mu, A_\nu \rangle$ as follows:

$$A = \begin{pmatrix} \langle 0.7, 0.3 \rangle & \langle 0.6, 0.4 \rangle & \langle 0.7, 0.2 \rangle \\ \langle 0.6, 0.3 \rangle & \langle 0.8, 0.1 \rangle & \langle 0.9, 0.1 \rangle \\ \langle 0.3, 0.5 \rangle & \langle 0.4, 0.6 \rangle & \langle 0.5, 0.3 \rangle \end{pmatrix}$$

with

$$A_\mu = \begin{pmatrix} 0.7 & 0.6 & 0.7 \\ 0.6 & 0.8 & 0.9 \\ 0.3 & 0.4 & 0.5 \end{pmatrix},$$

where the resultant matrices L_μ and U_μ are as follows:

$$L_\mu = \begin{pmatrix} 1 & 0 & 0 \\ 0.6 & 1 & 0 \\ 0.3 & 0.4 & 1 \end{pmatrix}, U_\mu = \begin{pmatrix} 0.7 & 0.6 & 0.7 \\ 0 & 0.8 & 0.9 \\ 0 & 0 & 0.5 \end{pmatrix};$$

and

$$A_\nu = \begin{pmatrix} 0.3 & 0.4 & 0.2 \\ 0.3 & 0.1 & 0.1 \\ 0.5 & 0.6 & 0.3 \end{pmatrix},$$

where the resultant matrices L_ν and U_ν are as follows:

$$L_\nu = \begin{pmatrix} 0 & 1 & 1 \\ 0.3 & 0 & 1 \\ 0.3 & 0.4 & 0 \end{pmatrix}, U_\nu = \begin{pmatrix} 0.3 & 0.4 & 0.2 \\ 1 & 0.1 & 0.1 \\ 1 & 1 & 0.3 \end{pmatrix}.$$

Then, the resulted matrices $L = \langle L_\mu, L_\nu \rangle$ and $U = \langle U_\mu, U_\nu \rangle$ are as follows:

$$L = \begin{pmatrix} \langle 1, 0 \rangle & \langle 0, 1 \rangle & \langle 0, 1 \rangle \\ \langle 0.6, 0.3 \rangle & \langle 1, 0 \rangle & \langle 0, 1 \rangle \\ \langle 0.3, 0.3 \rangle & \langle 0.4, 0.4 \rangle & \langle 1, 0 \rangle \end{pmatrix}, U = \begin{pmatrix} \langle 0.7, 0.3 \rangle & \langle 0.6, 0.4 \rangle & \langle 0.7, 0.2 \rangle \\ \langle 0, 1 \rangle & \langle 0.8, 0.1 \rangle & \langle 0.9, 0.1 \rangle \\ \langle 0, 1 \rangle & \langle 0, 1 \rangle & \langle 0.5, 0.3 \rangle \end{pmatrix}.$$

4 Existence and uniqueness of LU-factorization

The following theorems express the necessary and sufficient conditions for the existence and uniqueness of the LU-factorization for a given intuitionistic fuzzy square matrix.

Theorem 1. *Let $A = [a_{ij}] = [\langle a_{ij\mu}, a_{ij\nu} \rangle], 0 \leq a_{ij\mu} + a_{ij\nu} \leq 1$, be an $n \times n$ dimensional intuitionistic fuzzy matrix. Then there exist(s) at least an LU-factorization for the intuitionistic fuzzy matrix A with respect to the max-min , min-max compositions if and only if there exist(s) at least a pair of $n \times n$-dimensional intuitionistic fuzzy (lower and upper) triangular matrices as $L = [\langle l_{ij,\mu}, l_{ij,\nu} \rangle]$ and $U = [\langle u_{ij,\mu}, u_{ij,\nu} \rangle]$ such that their components satisfy the following conditions.*

(C_1^μ) $u_{11,\mu} \geq a_{21,\mu} = \min(l_{21,\mu}, u_{11,\mu})$

(C_2^μ) $a_{2j,\mu} \geq \min(l_{21,\mu}, u_{1j,\mu}) \quad j = 2, \ldots, n$

(C_3^μ) $u_{22,\mu} \geq a_{32,\mu} \geq \min(l_{31,\mu}, u_{12,\mu})$

$a_{3j,\mu} \geq \max\Big(\min(l_{31,\mu}, u_{1j,\mu}), \min(l_{32,\mu}, u_{2j,\mu})\Big) \quad j = 3, \ldots, n$

(C_n^μ) $u_{22,\mu} \geq a_{n2,\mu} \geq \min(l_{n1,\mu}, u_{12,\mu})$

$u_{33,\mu} \geq a_{n3,\mu} \geq \max\Big(\min(l_{n1,\mu}, u_{13,\mu}), \min(l_{n2,\mu}, u_{23,\mu})\Big)$ *and*

$u_{44,\mu} \geq a_{n4,\mu} \geq \max\Big(\min(l_{n1,\mu}, u_{14,\mu}), \min(l_{n2,\mu}, u_{24,\mu}), \min(l_{n3,\mu}, u_{34,\mu})\Big)$

$u_{n-1,n-1\mu} \geq a_{n,n-1,\mu} \geq \max\Big(\min(l_{n1,\mu}, u_{1,n-1,\mu}), \ldots, \min(l_{n,n-2,\mu}, u_{n-2,n-1,\mu})\Big)$

$a_{n,n\mu} \geq \max\Big(\min(l_{n1\mu}, u_{1,n\mu}), \min(l_{n2\mu}, u_{2,n\mu}), \ldots, \min(l_{n,n-1\mu}, u_{n-1,n\mu})\Big)$

and

(C_1^ν) $u_{11,\nu} \leq a_{21,\nu} = \max(l_{21,\nu}, u_{11,\nu})$

(C_2^ν) $a_{2j,\nu} \leq \max(l_{21,\nu}, u_{1j,\nu}) \quad j = 2, \ldots, n$

(C_3^ν) $u_{22,\nu} \leq a_{32,\nu} \leq \max(l_{31,\nu}, u_{12,\nu})$ *and*

$a_{3j,\nu} \leq \min\Big(\max(l_{31,\nu}, u_{1j,\nu}), \max(l_{32,\nu}, u_{2j,\nu})\Big) \quad j = 3, \ldots, n$

$\vdots$

(C_n^ν) $u_{22,\nu} \le a_{n2,\nu} \le \max(l_{n1\nu}, u_{12\nu})$ *and*
$u_{33,\nu} \le a_{n3,\nu} \le \min\Big(\max(l_{n1,\nu}, u_{13,\nu}), \max(l_{n2,\nu}, u_{23,\nu})\Big)$ *and*
$u_{44,\nu} \le a_{n4,\nu} \le \min\Big(\max(l_{n1,\nu}, u_{14,\nu}), \max(l_{n2,\nu}, u_{24,\nu}), \max(l_{n3,\nu}, u_{34,\nu})\Big) \dots$
$u_{n-1,n-1,\nu} \le a_{n,n-1,\nu} \le \min\Big(\max(l_{n1\nu}, u_{1,n-1\nu}), \dots, \max(l_{n,n-2\nu}, u_{n-2,n-1\nu})\Big)$ *and*
$a_{n,n,\nu} \le \min\Big(\max(l_{n1,\nu}, u_{1n,\nu}), \max(l_{n2,\nu}, u_{2n,\nu}), \dots, \max(l_{n,n-1,\nu}, u_{n-1,n,\nu})\Big)$

Proof. If the components L and U satisfy the relations $(C_1^\mu) - (C_n^\mu)$ and $(C_1^\nu) - (C_n^\nu)$, then it is easily shown that the solution set of the systems $(S_1^\mu) - (S_n^\mu)$ and $(S_1^\nu) - (S_n^\nu)$ is non-empty and the intuitionistic fuzzy matrix $A = [\langle a_{ij,\mu}, a_{ij,\nu}\rangle]_{n\times n}$ can be decomposed into two intuitionistic fuzzy lower and upper triangular matrices $L = [\langle l_{ij,\mu}, l_{ij,\nu}\rangle]_{n\times n}$ and $U = [\langle u_{ij,\mu}, u_{ij,\nu}\rangle]_{n\times n}$ such that $A = LU$. Conversely, if the intuitionistic fuzzy matrix $A = [\langle a_{ij,\mu}, a_{ij,\nu}\rangle]_{n\times n}$ is decomposed into two intuitionistic fuzzy lower and upper triangular matrices L and U such that $A = LU$, then by extending the relation $A = LU$ with the max-min, min-max compositions, we can obtain the relations $(C_1^\mu) - (C_n^\mu)$ and $(C_1^\nu) - (C_n^\nu)$. Since there exist(s) at least an LU-factorization for the intuitionistic fuzzy matrix A, the solution set of the system $(S_1^\mu) - (S_n^\mu)$ and $(S_1^\nu) - (S_n^\nu)$ is non-empty. On other hand, since the matrices L and U are intuitionistic fuzzy matrices, the existence conditions of solution for the systems $(S_1^\mu) - (S_n^\mu)$ and $(S_1^\nu) - (S_n^\nu)$ are as $(C_1^\mu) - (C_n^\mu)$ and $(C_1^\nu) - (C_n^\nu)$. □

Theorem 2. *Let $A = [a_{ij}] = [\langle a_{ij,\mu}, a_{ij,\nu}\rangle]$, $0 \le a_{ij,\mu} + a_{ij,\nu} \le 1$, be an $n \times n$ dimensional intuitionistic fuzzy matrix and has at least an LU-factorization with respect to the max-min, min-max compositions. If the following conditions are satisfied, then the LU-factorization is unique.*

(Q_1^μ) $u_{22\mu} \ge a_{32\mu} > \min\{l_{31\mu}, u_{12\mu}\}$ *and*
$a_{3j\mu} > \max\{\min\{l_{31\mu}, u_{1j\mu}\}, \min\{l_{32\mu}, u_{2j\mu}\}\}\ \forall j \in \{3, \dots, n\}$ *and*

(Q_2^μ) $u_{22\mu} \ge a_{42\mu} > \min\{l_{41\mu}, u_{12\mu}\}$ *and*
$u_{33\mu} \ge a_{43\mu} > \max\{\min\{l_{41\mu}, u_{13\mu}\}, \min\{l_{42\mu}, u_{23\mu}\}\}$ *and*
$a_{4j\mu} > \max\{\min\{l_{41\mu}, u_{1j\mu}\}, \min\{l_{42\mu}, u_{2j\mu}\}, \min\{l_{43\mu}, u_{3j\mu}\}\}\ \forall j \in \{4, ..., n\}$ *and*

$\vdots$

(Q_{n-1}^μ) $u_{22\mu} \ge a_{n2\mu} > \min\{l_{n1\mu}, u_{12\mu}\}$ *and*
$u_{33\mu} \ge a_{n3\mu} > \max\{\min\{l_{n1\mu}, u_{13\mu}\}, \min\{l_{n2\mu}, u_{23\mu}\}\}$ *and*
$u_{n-1,n-1\mu} \ge a_{n,n-1\mu} > \max\{\min\{l_{n1\mu}, u_{1,n-1\mu}\}, \dots, \min\{l_{n,n-2\mu}, u_{n-2,n-1\mu}\}\}$ *and*
$a_{n,n\mu} > \max\{\min\{l_{n1\mu}, u_{1,n\mu}\}, \min\{l_{n2\mu}, u_{2,n\mu}\}, ..., \min\{l_{n,n-1\mu}, u_{n-1,n\mu}\}\}$

and

(Q_1^ν) $u_{22\nu} \le a_{32\nu} < \max\{l_{31\nu}, u_{12\nu}\}$ *and*
$a_{3j\nu} < \min\{\max\{l_{31\nu}, u_{1j\nu}\}, \max\{l_{32\nu}, u_{2j\nu}\}\}\ \forall j \in \{3, \dots, n\}$

(Q_2^ν) $u_{22\nu} \le a_{42\nu} < \max\{l_{41\nu}, u_{12\nu}\}$ *and*
$a_{3j\nu} < \min\{\max\{l_{31\nu}, u_{1j\nu}\}, \max\{l_{32\nu}, u_{2j\nu}\}\}\ \forall j \in \{3, \dots, n\}$ *and*
$u_{33\nu} \le a_{43\nu} < \min\{\max\{l_{41\nu}, u_{13\nu}\}, \max\{l_{42\nu}, u_{23\nu}\}\}$ *and*
$a_{4j\nu} < \min\{\max\{l_{41\nu}, u_{1j\nu}\}, \max\{l_{42\nu}, u_{2j\nu}\}, \max\{l_{43\nu}, u_{3j\nu}\}\}\ \forall j \in \{4, \dots, n\}$ *and*

$\vdots$

(Q^{ν}_{n-1}) $\quad u_{22\nu} \leq a_{n2\nu} < \max\{l_{n1\nu}, u_{12\nu}\}$

$u_{33\nu} \leq a_{n3\nu} < \min\{\max\{l_{n1\nu}, u_{13\nu}\}, \max\{l_{n2\nu}, u_{23\nu}\}\}$

$u_{44\nu} \leq a_{n4\nu} < \min\{\max\{l_{n1\nu}, u_{14\nu}\}, \max\{l_{n2\nu}, u_{24\nu}\}, \max\{l_{n3\nu}, u_{34\nu}\}\}$

$\vdots$

$u_{n-1,n-1\nu} \leq a_{n,n-1\nu} < \min\{\max\{l_{n1\nu}, u_{1,n-1\nu}\}, \ldots, \max\{l_{n,n-2\nu}, u_{n-2,n-1\nu}\}\}$ *and*

$a_{n,n\nu} < \min\{\max\{l_{n1\nu}, u_{1,n\nu}\}, \ldots, \max\{l_{n,n-1\nu}, u_{n-1,n\nu}\}\}$

Proof. Since the matrix A has at least an LU-factorization and the conditions $(Q^{\mu}_1) - (Q^{\mu}_{n-1})$ and $(Q^{\nu}_1) - (Q^{\nu}_{n-1})$ are satisfied, the systems $(S^{\mu}_1) - (S^{\mu}_n)$ and $(S^{\nu}_1) - (S^{\nu}_n)$ have an LU-factorization.

To show it, suppose that the conditions (Q^{μ}_1) and (Q^{ν}_1) are satisfied, i.e., $u_{22\mu} \geq a_{32\mu} > \min\{l_{31\mu}, u_{12\mu}\}$ and $u_{22\nu} \leq a_{32\nu} < \max\{l_{31\nu}, u_{12\nu}\}$ then the statements $a_{32\mu} = \max\{\min\{l_{31\mu}, u_{12\mu}\}, \min\{l_{32\mu}, u_{22\mu}\}\}$ and $a_{32\nu} = \min\{\max\{l_{31\nu}, u_{12\nu}\}, \max\{l_{32\nu}, u_{22\nu}\}\}$ are reduced to $a_{32\mu} = \min\{l_{32\mu}, u_{22\mu}\}$ and $a_{32\nu} = \max\{l_{32\nu}, u_{22\nu}\}$.

The equality results that $l_{32\mu} = a_{32\mu}$ and $l_{32\nu} = a_{32\nu}$ is only value for $l_{32} = \langle l_{32\mu}, l_{32\nu}\rangle$. It is necessary to note that the component $a_{32} = \langle a_{32\mu}, a_{32\nu}\rangle$ is known and $u_{22} = \langle u_{22\mu}, u_{22\nu}\rangle$ has also been determined uniquely from the previous equations in the systems $((S^{\mu}_1) - (S^{\mu}_n)$ and $(S^{\nu}_1) - (S^{\nu}_n)$. Similarly, if each of the conditions $(Q^{\mu}_1) - (Q^{\mu}_{n-1})$ and $(Q^{\nu}_1) - (Q^{\nu}_{n-1})$ are satisfied, then we can show that the values $l_{ij} = \langle l_{ij\mu}, l_{ij\nu}\rangle$, for $i > j$, and $u_{ij} = \langle u_{ij\mu}, u_{ij\nu}\rangle$, for $i \leq j$, are determined uniquely. Therefore, under the conditions $(Q^{\mu}_1) - (Q^{\mu}_{n-1})$ and $(Q^{\nu}_1) - (Q^{\nu}_{n-1})$, the matrices L and U are determined uniquely. □

5 Resolution of the system of the max-min product intuitionistic fuzzy relation equations using the LU-factorization

Let $A = \langle A_{\mu}, A_{\nu}\rangle = [\langle a_{ij\mu}, a_{ij\nu}\rangle], 0 \leq a_{ij\mu}+a_{ij\nu} \leq 1$,be an $n\times n$ dimensional Intuitionistic fuzzy matrix and $b = [\langle b_{1\mu}, b_{1\nu}\rangle, \ldots, \langle b_{n\mu}, b_{n\nu}\rangle]^t$, $0 \leq b_{j\mu} + b_{j\nu} \leq 1$, be an n-dimensional intuitionistic fuzzy vector. Then the following system of intuitionistic fuzzy relation equations is defined by A and b as follows:

$$Ax = b \tag{6}$$

that is

$$\langle A_{\mu}, A_{\nu}\rangle\langle x_{\mu}, x_{\nu}\rangle = \langle b_{\mu}, b_{\nu}\rangle$$

In other words, we try to find a solution vector $x = [\langle x_{1\mu}, x_{1\nu}\rangle, \ldots, \langle x_{n\mu}, x_{n\nu}\rangle]^t$ with $0 \leq x_{j\mu} + x_{j\nu} \leq 1$, such that

$$\max_j \min\{a_{ij\mu}, x_{j\mu}\} = b_{j\mu}, \quad \forall i \in \underline{n}$$

$$\min_j \max\{a_{ij\nu}, x_{j\nu}\} = b_{j\nu}, \quad \forall i \in \underline{n}.$$

Remark 1. *Let A, B, C be an $n \times n$ dimensional Intuitionistic fuzzy matrices. Then we have*

$$(AB)C = A(BC),$$

i.e the operators max-min product composition and min-max product composition are associative.

We now focus on the resolution of the system of intuitionistic fuzzy relation equations with the intuitionistic fuzzy square matrix A. Assume that the intuitionistic fuzzy matrix A has at least an LU-factorization. Then we can rewrite the system (6) as follows:

$$(LU)x = b$$

that is

$$\big(\langle L_\mu, L_\nu\rangle\langle U_\mu, U_\nu\rangle\big)\langle x_\mu, x_\nu >=< b_\mu, b_\nu\rangle$$

Using Remark 5.1,

$$L(Ux) = b$$

that is

$$\langle L_\mu, L_\nu\rangle\big(\langle U_\mu, U_\nu\rangle\langle x_\mu, x_\nu\rangle\big) = \langle b_\mu, b_\nu\rangle$$

Now, let $Ux = y$ where $y = \langle y_\mu, y_\nu\rangle = [\langle y_{1\mu}, y_{1\nu}\rangle, \langle y_{2\mu}, y_{2\nu}\rangle, \ldots, \langle y_{n\mu}, y_{n\nu}\rangle]^t$ Then the system (6) is reduced to the following system:

$$Ly = b \tag{7}$$

with regard to the special structure of the matrix L, we can easily solve the system (7). Here the component $y_1 = \langle y_{1\mu}, y_{1\nu}\rangle$ is obtained from the first equation and then inserting its value in the second equation, $y_2 = \langle y_{2\mu}, y_{2\nu}\rangle$ is obtained, and so on. This process is called the forward substitution. Then the system $Ux = y$ is similarly solved. At first, the component $x_n = \langle x_{n\mu}, x_{n\nu}\rangle$ is computed from the last equation. Then by inserting its value in the $(n-1)$-th equation, the component $x_{n-1} = \langle x_{n-1\mu}, x_{n-1\nu}\rangle$ is obtained and so on. This process is called the backward substitution. Using the two substitutions, the solution set of the system (6) is found. Now, we express this method by the following algorithm.

5.1 An algorithm for the resolution of the system (6)

Algorithm 1

Step 1. Compute the maximum solution of the system (6) by the following formula:

$$\hat{x} = \langle \min_{k\in \underline{n}} \sigma(a_{jk,\mu}, b_{k,\mu}), \max_{k\in \underline{n}} \sigma(a_{jk,\nu}, b_{k,\nu})\rangle$$

where

$$\sigma(a_{jk,\mu}, b_{k,\mu}) = \begin{cases} b_{k\mu} & \text{if } a_{jk,\mu} > b_{k,\mu}, \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad \sigma(a_{jk,\nu}, b_{k,\nu}) = \begin{cases} b_{k\nu} & \text{if } a_{jk,\mu} < b_{k,\mu} \\ 0 & \text{otherwise} \end{cases}$$

Step 2. Check its feasibility by verifying whether $A\hat{x} = b$. If it is infeasible, then stop and $\Omega(A, b) = \emptyset$.

Step 3. Run Algorithm 3.1 and Algorithm 3.2 to find an LU-factorization for the intuitionistic fuzzy matrix A. If there is no LU-factorization for the intuitionistic fuzzy matrix A, then stop!

Step 4. Find the solution set of the system $Ly = b$ ie of the systems $L_\mu y_\mu = b_\mu$ and $L_\nu y_\nu = b_\nu$ by the forward substitution.

Step 5. Find the solution set of the system $Ux = y$ i.e of the systems $U_\mu x_\mu = y_\mu$ and $U_\nu x_\nu = y_\nu$by the backward substitution.

Step 6. End.

5.2 Numerical example

Now, the algorithm is illustrated by an example. As it is seen, the solution set of the system (6) is found. We only apply the maximum solution to check the feasibility of the system. Since the maximum solution has a special formula, its computation is easily done.

Example 2.
Consider the system of intuitionistic fuzzy relation equations as follows.

$$Ax = b$$

where

$$A = \begin{pmatrix} \langle 0.7, 0.3\rangle & \langle 0.6, 0.4\rangle & \langle 0.7, 0.2\rangle \\ \langle 0.6, 0.3\rangle & \langle 0.8, 0.1\rangle & \langle 0.9, 0.1\rangle \\ \langle 0.3, 0.5\rangle & \langle 0.4, 0.6\rangle & \langle 0.5, 0.3\rangle \end{pmatrix}$$

and

$$b = \begin{bmatrix} \langle 0.4, 0.4\rangle \\ \langle 0.4, 0.4\rangle \\ \langle 0.4, 0.4\rangle \end{bmatrix}$$

Algorithm 1 is applied to solve the example.

Step 1. The maximum solution of the system is as

$$\hat{x} = [\langle 0.4, 0.4\rangle \ \ \langle 0.4, 0.4\rangle \ \ \langle 0.4, 0.4\rangle]^t$$

Step 2. Since $A\hat{x} = b$, the system is feasible. Hence, go to **Step 3**.

Step 3. Applying Algorithm 3.1 and Algorithm 3.2, the LU-factorization of the matrix A is as follows:

$$A = LU$$

where,

$$L = \begin{pmatrix} \langle 1, 0\rangle & \langle 0, 1\rangle & \langle 0, 1\rangle \\ \langle 0.6, 0.3\rangle & \langle 1, 0\rangle & \langle 0, 1\rangle \\ \langle 0.3, 0.3\rangle & \langle 0.4, 0.4\rangle & \langle 1, 0\rangle \end{pmatrix}, U = \begin{pmatrix} \langle 0.7, 0.3\rangle & \langle 0.6, 0.4\rangle & \langle 0.7, 0.2\rangle \\ \langle 0, 1\rangle & \langle 0.8, 0.1\rangle & \langle 0.9, 0.1\rangle \\ \langle 0, 1\rangle & \langle 0, 1\rangle & \langle 0.5, 0.3\rangle \end{pmatrix}$$

Step 4. In this step, we find the solution set of the system $Ly = b$ by the forward substitution.

1. $\min\{1, y_{1\mu}\} = 0.4$, $\max\{0, y_{1\nu}\} = 0.4 \Rightarrow y_{1\mu} = 0.4$ and $y_{1\nu} = 0.4$.

2. $\max\{\min\{0.6, y_{1\mu}\}, \min\{1, y_{2\mu}\}\} = 0.4$, $\min\{\max\{0.3, y_{1\nu}\}, \max\{0, y_{2\nu}\}\} = 0.4$
i.e $\max\{0.4, y_{2\mu}\} = 0.4$, $\min\{0.4, y_{2\nu}\} = 0.4 \Rightarrow 0 \leq y_{2\mu} \leq 0.4$ and $0.4 \leq y_{2\nu} \leq 1$.

3. $\max\{\min\{0.3, y_{1\mu}\}, \min\{0.4, y_{2\mu}\}, \min\{1, y_{3\mu}\}\} = 0.4$,
$\min\{\max\{0.3, y_{1\nu}\}, \max\{0.4, y_{2\nu}\}, \max\{0, y_{3\nu}\}\} = 0.4$
i.e $\max\{0.3, y_{2\mu}, y_{3\mu}\} = 0.4$, $\min\{0.4, y_{2\nu}, y_{3\nu}\} = 0.4 \Rightarrow y_{3\mu} = 0.4$ and $0.4 \leq y_{3\nu} \leq 0.6$.

Therefore, the solution set of the system $Ly = b$ is as follows:

$$\Omega(L, b) = \{y = [\langle y_{1\mu}, y_{1\nu}\rangle, \langle y_{2\mu}, y_{2\nu}\rangle, \langle y_{3\mu}, y_{3\nu}\rangle] \in [0,1]^2 \times [0,1]^2 \times [0,1]^2$$

$$|y \in \{\langle 0.4, 0.4\rangle\} \times \{\langle [0, 0.4], [0.4, 1]\rangle\} \times \{\langle 0.4, [0.4, 0.6]\rangle\}\}.$$

Step 5. In this step, we find the solution set $Ux = y$ by the backward substitution.

1. $\min\{0.5, x_{3\mu}\} = 0.4$, $\max\{0.3, x_{3\nu}\} = [0.4, 0.6] \Rightarrow x_{3\mu} = 0.4$ and $0.4 \leq x_{3\nu} \leq 0.6$.

2. $\max\{\min\{0.8, x_{2\mu}\}, \min\{0.9, x_{3\mu}\}\} = [0, 0.4]$,
$\min\{\max\{0.1, x_{2\nu}\}, \max\{0.1, x_{3\nu}\}\} = [0.4, 1]$, i.e.
$\max\{x_{2\mu}, 0.4\} = [0, 0.4]$, $\min\{x_{2\nu}, x_{3\nu}\} = [0.4, 1] \Rightarrow 0 \leq x_{2\mu} \leq 0.4$ and$0.4 \leq x_{2\nu} \leq 0.6$.

3. $\max\{\min\{0.7, x_{1\mu}\}, \min\{0.6, x_{2\mu}\}, \min\{0.7, x_{3\mu}\}\} = 0.4$,
$\min\{\max\{0.3, x_{1\nu}\}, \max\{0.4, x_{2\nu}\}, \max\{0.2, x_{3\nu}\}\} = 0.4$, i.e.
$\max\{x_{1\mu}, x_{2\mu}, 0.4\} = 0.4$, $\min\{x_{1\nu}, x_{2\nu}, x_{3\nu}\} = 0.4 \Rightarrow 0 \leq x_{1\mu} \leq 0.4$ and $x_{1\nu} = 0.4$.

With regard to the above items, we conclude that the solution set $Ax = b$ is as follows :

$$\Omega(U, y) = \Big\{\langle [0, 0.4], 0.4\rangle\Big\} \times \Big\{\langle [0, 0.4], [0.4, 0.6]\rangle\Big\} \times \Big\{\langle 0.4, [0.4, 0.6]\rangle\Big\}$$

6 Conclusion

In this paper, the LU-factorization was extended to an intuitionistic fuzzy square matrix with respect to the max-min product composition operator called LU-factorization. The sufficient and necessary conditions of its existence and uniqueness for an intuitionistic fuzzy matrix were presented. Moreover, an algorithm was proposed to find the intuitionistic fuzzy lower and upper triangular matrices L and U , respectively.

Finally, applying the LU-factorization of an intuitionistic fuzzy matrix, an algorithm was suggested to solve a system of intuitionistic fuzzy relation equations. This algorithm uses the forward and backward substitution to solve the system.

References

[1] Atanassov, K., Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, Vol. 20, 1986, 87–96.

[2] Atanassov, K., *Intuitionistic Fuzzy Sets: Theory and Applications,* Physica-Verlag 1999.

[3] Atanassov, K. T., P. M. Vassilev, R. T. Tsvetkov, *Intuitionistic Fuzzy Sets, Measures and Integrals*, Bulgarian Academic Monographs (12), "Professor Marin Drinov" Academic Publishing House, Sofia, 2013.

[4] Molai, A. A. Resolution of a system of the max-product fuzzy relation equations using $L \circ U$-factorization, *Information Sciences*, Vol. 234, 2013, 86–96.

[5] Meenakshi, A. R., C. Inbam, Fuzzy matrices and Linear transformations on fuzzy vector spaces, *Int. J. Fuzzy Math.*, Vol. 11, 2003, 955–966.

[6] Meenakshi, A. R., T. Gandhimathi, Special Types of Intuitionistic Fuzzy Linear Transformations, *International Journal of Mathematical Sciences and Applications*, Vol. 1, 2011, No. 1.

[7] Pradhan, R., M. Pal, Solvability of system of intuitionistic fuzzy linear equations, *International Journal of Fuzzy Logic Systems (IJFLS)*, Vol. 4, 2014, No. 3, 13–24.