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On L^r-intuitionistic fuzzy Henstock–Kurzweil integral with application to intuitionistic fuzzy Laplace transform

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Abstract: This article presents the concept of L^r-Henstock–Kurzweil integral of intuitionistic fuzzy number-valued function. First, we define the L^r-intuitionistic fuzzy Henstock–Kurzweil integral, explore its properties, demonstrate L^r-continuity of the primitive, and provide a convergence theorem. Furthermore, we show that this integral generalizes intuitionistic fuzzy Henstock–Kurzweil integral, has a broader scope and give a numerical example. We also introduce the proposed integral over an infinite interval and prove that the α- and β-cuts of the integral are Henstock–Kurzweil integrable. Finally, as an application, we define intuitionistic fuzzy Laplace transform based on L^r-intuitionistic fuzzy Henstock–Kurzweil integral and investigate the existence of intuitionistic fuzzy Laplace transform.

Keywords: Intuitionistic fuzzy set, L^r-intuitionistic fuzzy Henstock–Kurzweil integral, L^r-derivative, L^r-continuous, Intuitionistic fuzzy Laplace transform.
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1 Introduction

Atanassov [5] introduced the concept of intuitionistic fuzzy set which generalizes Zadeh's fuzzy set theory [46]. In [46] Zadeh's fuzzy set theory, $1 - \mu_{\tilde{A}}$ is the non-membership function so the sum of membership and non-membership function is always 1 which may be inadequate to deal with imprecision in models. As a result, intuitionistic fuzzy set is more flexible in handling uncertainty due to imprecision knowledge. Later more studies were conducted by [6–8] to further develop the theory. Since real-world models often contain partial data, fuzzy set and intuitionistic fuzzy set theory have found applications across various fields such as decision making, medical diagnosis, etc in [1, 2, 10, 19, 31, 33].

Henstock [29] and Kurzweil [32] separately introduced the Henstock–Kurzweil (HK) integral for real valued functions which is a Reimann-type integral, it is simpler and generalizes Lebesgue integral, Reimann integral, improper Reimann integral and Newton integral. The HK-type integral integrates all derivatives. Later authors in [15] introduced Henstock–Kurzweil integral on an unbounded interval and presented applications of HK integral to differential and integral equations. A generalized derivative, namely, L^r-derivative used for solving partial differential equations was introduced by Calderón and Zygmund [16]. In 1967, [27] Gordon introduced the concept of Dini derivatives in metric L^r, also introduced Perron's integral (P_r- integral) to solve L^r-derivative and further applied P_r-integral and L^r-derivative to Fourier series. Later, the notion of L^r-Henstock–Kurzweil integral was introduced by Musial and Sagher in 2004 and gave the definition of ACG_r- functions [35]. In 2012, Tikare gave the notion of Henstock-Stieltjes integral in L^r to recover function from its L^r-derivative and proved some properties [42]. In [37, 38], Musial and Tulone discussed the product rule for L^r-derivative, gave a formula for integration by parts, defined a norm on the space of L^r-Henstock–Kurzweil integrable functions and studied the dual as well as the completion of this space. Very recently the authors in [36, 43] defined L^r-variational integral, showed that it is equivalent to L^r-Henstock–Kurzweil integral and the L^r-Henstock–Kurzweil integral is not contained in the P_r- integral.

Ever since the introduction of calculus, integral theory has come a long way and there have been many generalizations. One such is the Henstock integral of fuzzy valued functions introduced by Wu and Gong in 2001 [44]. The authors proposed some properties and in addition gave necessary and sufficient conditions of integrability. Later, authors in [21, 24–26] studied the continuity, integrability, differentiability. Fuzzy Henstock integral on infinite interval was discussed in [23] to calculate the expectation of fuzzy random variables. A decomposition theorem for fuzzy Henstock integral was given by Bongiorno et al. which is that a fuzzy valued function is fuzzy Henstock integrable if and only if it can be written as a sum of fuzzy McShane integrable and of fuzzy Henstock integrable generated by a Henstock integrable function [14]. Herawan et al. [30] presented in their paper the Cauchy criterion for Henstock integrability of fuzzy number valued functions and gave the comparison of fuzzy Reimann and fuzzy Henstock integral. Henstock Stieltjes integral for fuzzy valued functions was introduced by [22], discussed some properties, differentiability and continuity. Also showed that the primitive of fuzzy Henstock Stieltjes integrable is not α- differentiable almost everywhere. In 2023, Shao et al. introduced the concept of L^r-Henstock–Kurzweil integral of fuzzy valued functions. At first the authors introduced fuzzy L^r-derivative then introduced L^r-fuzzy Henstock–Kurzweil integral to recover function from its fuzzy L^r-derivative and lastly, applied to study fuzzy Fourier series [41]. Fuzzy Integral theory was further extended to intuitionistic fuzzy which generalizes fuzzy integral theory in [4] and also the notion of intuitionistic fuzzy Riemann integral was discussed in [28].

Laplace transform plays an important role in solving differential equations which in turn have applications in physics, biology, engineering and other fields. Laplace transform based on Henstock–Kurzweil integral was given by Sánchez-Perales et al. [40] and showed that Laplace transform of a convolution is a pointwise product of Laplace transforms (Riemann-Lebesgue lemma). Fuzzy Laplace transform was first introduced by Allahviranloo et al. defined based on improper fuzzy Reimann integral to solve fuzzy differential equations [3]. Later in 2019, fuzzy Laplace transform based on Henstock integral was introduced by [20]. For the first time the existence of fuzzy Laplace transform by use of Henstock integral on infinite interval was shown which was not available in previous existing literatures and further applied in solving fuzzy initial value problems. The notion of intuitionistic fuzzy Laplace transform was given by Mondal et al. in 2015 [34]. The authors introduced generalized intuitionistic fuzzy Laplace transform based on intuitionistic fuzzy Riemann integral, showed some properties and applied to solve electric circuit theory problem. Further many researchers applied intuitionistic fuzzy Laplace transform to solve intuitionistic fuzzy partial differential equations [11], intuitionistic fuzzy transport equations [13], wave equations [12], higher order intuitionistic fuzzy differential equations [17, 18].

If we look at the present context, only a sizeable amount of work on intuitionistic integral theory is available. The authors in [45] extended intuitionistic fuzzy L^r-derivative (IFL^r-dervative) and applied to solve intuitionistic fuzzy initial value problem. This motivated us to work on L^r-intuitionistic fuzzy Henstock–Kurzweil integral theory to recover function from intuitionistic fuzzy L^r-derivative.

This paper is organised as follows: In Section 2, we provide some preliminaries that we will use all along this paper. In Section 3, we introduce L^r-intuitionistic fuzzy Henstock–Kurzweil integral, show that primitive is unique and L^r-continuous everywhere, give Cauchy criteria for integrability and a convergence theorem. In Section 4, we define L^r-intuitionistic fuzzy Henstock–Kurzweil integral for infinite interval and prove a theorem. In Section 5, we define intuitionistic fuzzy Laplace transform based on L^r-intuitionistic fuzzy Henstock–Kurzweil integral. Lastly in Section 6, we give conclusion of this paper, summarize the results obtained and a brief overview on the future study.

2 Preliminaries

Definition 1. [5] Let $\tilde{A} = \{(\varsigma, \mu_{\tilde{A}}(\varsigma), \nu_{\tilde{A}}(\varsigma)); \varsigma \in \aleph\}$ be an intuitionistic fuzzy set (IFS) where $\mu_{\tilde{A}}(\varsigma)$ and $\nu_{\tilde{A}}(\varsigma)$ are the membership function and non-membership function respectively.

$$\mu_{\tilde{A}}(\varsigma), \nu_{\tilde{A}}(\varsigma) : \aleph \to [0, 1] \text{ and } 0 \le \mu_{\tilde{A}}(\varsigma) + \nu_{\tilde{A}}(\varsigma) \le 1.$$

Intuitionistic fuzzy set generalizes Zadeh's fuzzy set which can be written as

$$\tilde{A} = \{(\varsigma, \mu_{\tilde{A}}(\varsigma), \nu_{\tilde{A}}(\varsigma)); \varsigma \in \aleph\} \text{ where } \nu_{\tilde{A}}(\varsigma) = 0.$$

Note 1. *$IF(\aleph)$ denotes the set of all intuitionistic fuzzy sets in $\aleph$. Then, $IF(\mathbb{R}^n)$ denotes the set of all intuitionistic fuzzy sets in $\mathbb{R}^n$.*

Definition 2. [5] Let $\tilde{A} \in IF(\aleph)$. The α-cut of $\tilde{A}$ is defined as follows:
For $\alpha \in (0,1]$, $\tilde{A}(\alpha) = \{\varsigma : \varsigma \in \aleph, \mu_{\tilde{A}}(\varsigma) \geq \alpha\}$.
For $\alpha = 0$, $\tilde{A}(0) = \overline{\bigcup_{\alpha \in (0,1]} \tilde{A}(\alpha)}$.

Definition 3. [5] Let $\tilde{A} \in IF(\aleph)$. The β-cut of $\tilde{A}$ is defined as follows:
For $\beta \in [0,1)$,$\tilde{A}^*(\beta) = \{\varsigma : \varsigma \in \aleph, \nu_{\tilde{A}}(\varsigma) \leq \beta\}$.
For $\beta = 1$, $\tilde{A}^*(1) = \overline{\bigcup_{\beta \in [0,1)} \tilde{A}^*(\beta)}$.

Definition 4. [9] If $\tilde{A} \in IF(\mathbb{R}^n)$ satisfies the following conditions then it is called intuitionistic fuzzy number in $\mathbb{R}^n$:

(i) $\tilde{A}$ is a normal set, i.e., there exists $\varsigma_0 \in \mathbb{R}^n$ such that $\mu_{\tilde{A}}(\varsigma_0) = 1$ (hence $\nu_{\tilde{A}}(\varsigma_0) = 0$).

(ii) $\tilde{A}(0)$ and $\tilde{A}^*(1)$ are bounded sets in $\mathbb{R}^n$.

(iii) $\mu_{\tilde{A}}$ is upper semi-continuous: $\forall t \in [0,1]$, the set $\{\varsigma : \varsigma \in \mathbb{R}^n, \mu_{\tilde{A}}(\varsigma) < t\}$ is open.

(iv) $\nu_{\tilde{A}}$ is lower semi-continuous: $\forall t \in [0,1]$, the set $\{\varsigma : \varsigma \in \mathbb{R}^n, \nu_{\tilde{A}}(\varsigma) > t\}$ is open.

(v) The membership function $\mu_{\tilde{A}}$ is quasi-concave:
$\mu_{\tilde{A}}(\rho\varsigma + (1-\rho)\varrho) \geq \min\{\mu_{\tilde{A}}(\varsigma), \mu_{\tilde{A}}(\varrho)\}$, $\forall \varsigma, \varrho \in \mathbb{R}^n$, $\rho \in [0,1]$.

(vi) The non-membership function $\nu_{\tilde{A}}$ is quasi-convex:
$\nu_{\tilde{A}}(\rho\varsigma + (1-\rho)\varrho) \leq \max\{\nu_{\tilde{A}}(\varsigma), \nu_{\tilde{A}}(\varrho)\}$, $\forall \varsigma, \varrho \in \mathbb{R}^n$, $\rho \in [0,1]$.

Definition 5. [9] Let $\tilde{A} \in IF(\mathbb{R}^n)$ and α, $\beta \in [0,1]$ be such that its α- and β- cuts are given by $\tilde{A}(\alpha) = \{\varsigma : \varsigma \in \aleph, \mu_{\tilde{A}}(\varsigma) \geq \alpha\}$ and $\tilde{A}^*(\beta) = \{\varsigma : \varsigma \in \aleph, \nu_{\tilde{A}}(\varsigma) \leq \beta\}$. Then the following conditions hold:

(i) $\tilde{A}(\alpha)$ and $\tilde{A}^*(\beta)$ are non-empty compact and convex sets in $\mathbb{R}^n$.

(ii) If $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, then $\tilde{A}(\alpha_2) \subseteq \tilde{A}(\alpha_1)$.

(iii) If $0 \leq \beta_1 \leq \beta_2 \leq 1$, then $\tilde{A}^*(\beta_1) \subseteq \tilde{A}^*(\beta_2)$.

(iv) If (α_n) is a non-decreasing sequence converging to α, then $\bigcap_{n=1}^{\infty} \tilde{A}(\alpha_n) = \tilde{A}(\alpha)$.

(v) If (β_n) is a non-increasing sequence converging to β, then $\bigcap_{n=1}^{\infty} \tilde{A}^*(\beta_n) = \tilde{A}^*(\beta)$.

(vi) If (α_n) is a non-increasing sequence converging to 0, then $\tilde{A}(0) = \overline{\bigcup_{n=1}^{\infty} \tilde{A}(\alpha_n)}$.

(vii) If (β_n) is a non-decreasing sequence converging to 1, then $\tilde{A}^*(1) = \overline{\bigcup_{n=1}^{\infty} \tilde{A}^*(\beta_n)}$.

Note 2. *$IF_N(\mathbb{R}^n)$ denotes the set of intuitionistic fuzzy numbers.*

Definition 6. [45] The distance of intuitionistic fuzzy numbers $A, B \in IF_N(\mathbb{R}^n)$ with respect to their α- and β- cuts is denoted by $\mathbb{D}(A, B)$ and is defined as

$$\mathbb{D}_1(A, B) = \sup_{\alpha \in [0,1]} \max \left\{ \mid A(\alpha)^- - B(\alpha)^- \mid, \mid A(\alpha)^+ - B(\alpha)^+ \mid \right\}.$$
$$\mathbb{D}_2(A, B) = \sup_{\beta \in [0,1]} \max \left\{ \mid A(\beta)^- - B(\beta)^- \mid, \mid A(\beta)^+ - B(\beta)^+ \mid \right\}.$$
$$\mathbb{D}(A, B) = \max \ \{\mathbb{D}_1(A, \ B), \ \mathbb{D}_2(A, \ B)\}.$$

The function $\mathbb{D}(A, B)$ defines a metric on $IF_N(\mathbb{R}^n)$.

Note 3. *$(IF_N(\mathbb{R}^n), \mathbb{D})$ is a complete metric space.*

Lemma 1. [45] *For $A, B, C, D \in IF_N(\mathbb{R}^n)$:*

(i) $\mathbb{D}(A + C, B + C) = \mathbb{D}(A, B)$.

(ii) $\mathbb{D}(\lambda.A, \lambda.B) = \mid \lambda \mid \mathbb{D}(A, B)$, $\lambda \in \mathbb{R}$.

(iii) $\mathbb{D}(A + B, C + D) \leq \mathbb{D}(A, C) + \mathbb{D}(B, D)$.

(iv) $\mathbb{D}(\tau.A, \omega.A) = \mid \tau - \omega \mid \mathbb{D}(A, 0)$, *for* $\tau\omega > 0$.

(v) $A \leq B$ *if and only if* $A(\alpha) \leq B(\alpha)$, $\alpha, \beta \in [0, 1]$ *if and only if* $A(\alpha)^+ \leq B(\alpha)^+$; $A(\alpha)^- \leq B(\alpha)^-$ *and* $A^*(\beta)^+ \leq B^*(\beta)^+$; $A^*(\beta)^- \leq B^*(\beta)^-$.

Definition 7. [39] Let $\delta(\varsigma)$ be a positive real valued function on $[c, d]$. A partition $P = \{[u, v]; \xi\}$ is said to be δ-fine partition if it satisfies the following conditions:

(i) $c = \varsigma_0 < \varsigma_1 < \varsigma_2 < \cdots < \varsigma_n = d$;

(ii) $\xi_i \in [\varsigma_{i-1}, \varsigma_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$, $i = 1, 2, 3, \ldots, n$, where ξ_i is the associated point and ς_i is the division point of $[\varsigma_{i-1}, \varsigma_i]$.

Definition 8. [35] For $1 \leq r < \infty$, a real valued function f is L^r-Henstock–Kurzweil integral if there exists a function $F \in L^r[c, d]$ for all $\epsilon > 0$, there exists a gauge function $\delta(\varsigma) > 0$ such that for any δ-fine partition $P = \{[u, v]; \varsigma_i\}$ of $[c, d]$, we get
$\sum_{i=1}^n \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [D(f(\varsigma_i)(\varrho - \varsigma_i) + F(\varsigma_i), F(\varrho))]^r d\varrho \right)^{1/r} < \epsilon$.

3 L^r-Intuitionistic fuzzy Henstock–Kurzweil integral

Definition 9. An intuitionistic fuzzy number valued function $\tilde{f}$ is said to be L^r-Henstock–Kurzweil (L^r-*IFHK*) integrable on $[c, d]$ if there exists $\tilde{F} \in L^r[c, d]$ for all $\epsilon > 0$, there exists a gauge function $\delta(\varsigma) > 0$ such that for any δ-fine partition $P = \{[u, v]; \varsigma\}$ of $[c, d]$, we get

$$\sum_{i=1}^n \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{F}(\varrho))]^r d\varrho \right)^{1/r} < \epsilon. \quad (1)$$

$$\tilde{F}(\varsigma) = (L^r\text{-}IFHK) \int_c^\varsigma \tilde{f}(t) dt \ \text{ and } \ \tilde{f} \in L^r\text{-}IFHK[c, d].$$

Theorem 1. *For $1 \leq r < \infty$, if $\tilde{f} \in L^r$-IFHK$[c,d]$, then the primitive of $\tilde{f}$ is unique.*

Proof. Let $\tilde{F}_1$ and $\tilde{F}_2$ be two primitives of $\tilde{f}$ and $\tilde{W} = \tilde{F}_1 \ominus_H \tilde{F}_2$. Therefore for any $\epsilon > 0$, there exists a gauge $\delta(\varsigma) > 0$ such that for any δ-fine partition P,

$$\sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}_1(\varsigma_i), \tilde{F}_1(\varrho))]^r d\varrho \right)^{1/r} < \frac{\epsilon}{2}, \tag{2}$$

$$\sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}_2(\varsigma_i), \tilde{F}_2(\varrho))]^r d\varrho \right)^{1/r} < \frac{\epsilon}{2}. \tag{3}$$

Now,

$$\begin{aligned} &\sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\mathbb{D}(\tilde{W}(\varrho), \tilde{W}(\varsigma_i))]^r d\varrho \right)^{1/r} \\ &= \sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\mathbb{D}(\tilde{F}_1(\varrho) \ominus_H \tilde{F}_2(\varrho), \tilde{F}_1(\varsigma_i) \ominus_H \tilde{F}_2(\varsigma_i))]^r d\varrho \right)^{1/r} \end{aligned} \tag{4}$$

Then by Minkowski's inequality and Lemma 1:

$$\begin{aligned} &= \sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\mathbb{D}(\tilde{F}_1(\varrho) \ominus_H \tilde{F}_2(\varrho) + \tilde{f}(\varsigma_i)(\varrho - \varsigma_i), \right. \\ &\qquad \left. \tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}_1(\varsigma_i) \ominus_H \tilde{F}_2(\varsigma_i))]^r d\varrho \right)^{1/r} \\ &= \sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\mathbb{D}(\tilde{F}_2(\varsigma_i) \ominus_H \tilde{F}_2(\varrho) + \tilde{f}(\varsigma_i)(\varrho - \varsigma_i), \right. \\ &\qquad \left. \tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}_1(\varsigma_i) \ominus_H \tilde{F}_1(\varrho))]^r d\varrho \right)^{1/r} \\ &\leq \sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}_2(\varsigma_i) \ominus_H \tilde{F}_2(\varrho)) + \right. \\ &\qquad \left. \mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}_1(\varsigma_i) \ominus_H \tilde{F}_1(\varrho))]^r d\varrho \right)^{1/r} \\ &\leq \sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}_2(\varsigma_i) \ominus_H \tilde{F}_2(\varrho))]^r d\varrho \right)^{1/r} \\ &\quad + \sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}_1(\varsigma_i) \ominus_H \tilde{F}_1(\varrho))]^r d\varrho \right)^{1/r} \\ &= \sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}_2(\varsigma_i), \tilde{F}_2(\varrho))]^r d\varrho \right)^{1/r} \\ &\quad + \sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}_1(\varsigma_i), \tilde{F}_1(\varrho))]^r d\varrho \right)^{1/r} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{5}$$

Therefore, the primitive is unique. □

Theorem 2. *$\tilde{f} \in L^r$-IFHK$[c,d]$ if and only if for every $\epsilon > 0$ there exists a gauge function $\delta(\varsigma) > 0$ such that for any two δ-fine partitions $P_1 = \{[u,v];\varsigma\}$ and $P_2 = \{[u^*,v^*];\varsigma^*\}$ of $[c,d]$, we have*

$$\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{f}(\varsigma_i{}^*)(\varrho - \varsigma_i{}^*) + \tilde{F}(\varsigma_i{}^*)) < \epsilon. \tag{6}$$

Proof. First let us prove the necessary condition. Let $\tilde{f}$ be L^r-*IFHK* integrable on $[c,d]$ with its primitive $\tilde{F}$ and fix $\epsilon > 0$. Then there is a gauge $\delta(\varsigma) > 0$ on $[c,d]$ such that for any δ-fine partition P_1 of $[c,d]$, we have

$$\sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{F}(\varrho))]^r d\varrho \right)^{1/r} < \frac{\epsilon}{2}. \tag{7}$$

Similarly, for any δ-fine partition P_2 of $[c,d]$, we have

$$\sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\mathbb{D}(\tilde{f}(\varsigma_i{}^*)(\varrho - \varsigma_i{}^*) + \tilde{F}(\varsigma_i{}^*), \tilde{F}(\varrho))]^r d\varrho \right)^{1/r} < \frac{\epsilon}{2}. \tag{8}$$

Now,

$$\begin{aligned}
&\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{f}(\varsigma_i{}^*)(\varrho - \varsigma_i{}^*) + \tilde{F}(\varsigma_i{}^*)) \\
&= \mathbb{D}\left(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i) \ominus_H \tilde{F}(\varrho), \tilde{f}(\varsigma_i{}^*)(\varrho - \varsigma_i{}^*) + \tilde{F}(\varsigma_i{}^*) \ominus_H \tilde{F}(\varrho) \right) \\
&= \mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{F}(\varrho)) + \mathbb{D}(\tilde{f}(\varsigma_i{}^*)(\varrho - \varsigma_i{}^*) + \tilde{F}(\varsigma_i{}^*), \tilde{F}(\varrho)) \\
&\leq \sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{F}(\varrho))]^r d\varrho \right)^{1/r} \\
&\quad + \sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\mathbb{D}(\tilde{f}(\varsigma_i{}^*)(\varrho - \varsigma_i{}^*) + \tilde{F}(\varsigma_i{}^*), \tilde{F}(\varrho))]^r d\varrho \right)^{1/r} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \\
&\Rightarrow \mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{f}(\varsigma_i{}^*)(\varrho - \varsigma_i{}^*) + \tilde{F}(\varsigma_i{}^*)) < \epsilon.
\end{aligned}$$

Let us now prove the sufficient condition. For every $\epsilon > 0$ there exist a gauge function δ such that for any two δ-fine partitions $P_1 = \{[u,v];\varsigma\}$ and $P_2 = \{[u^*,v^*];\varsigma^*\}$ of $[c,d]$, we have

$$\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{f}(\varsigma_i{}^*)(\varrho - \varsigma_i{}^*) + \tilde{F}(\varsigma_i{}^*)) < \epsilon.$$

Now, we construct a decreasing monotonic intuitionistic fuzzy number sequence $\{\tilde{\epsilon}_n\}_{n\in\mathbb{N}}$. Let us choose $\epsilon_1 = 1$, there exists a gauge δ_1 on $[c,d]$ such that

$$\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{f}(\varsigma_i{}^*)(\varrho - \varsigma_i{}^*) + \tilde{F}(\varsigma_i{}^*)) < 1, \tag{9}$$

whenever P_1 and P_2 are δ_1-fine partitions.

Take $\epsilon_2 = 1/2$, there exists a gauge δ_2 on $[c,d]$ so that $\delta_2 \leq \delta_1$ for every $\varsigma \in [c,d]$ and satisfying

$$\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{f}(\varsigma_i{}^*)(\varrho - \varsigma_i{}^*) + \tilde{F}(\varsigma_i{}^*)) < 1/2, \tag{10}$$

whenever P_1 and P_2 are δ_2-fine partitions.

Continuing in this way, we get a sequence of gauges on $[c, d]$ such that $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{n-1} \geq \delta_n$ satisfying

$$\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{f}(\varsigma_i{}^*)(\varrho - \varsigma_i{}^*) + \tilde{F}(\varsigma_i{}^*)) < 1/n, \tag{11}$$

whenever P_1 and P_2 are δ_n-fine partitions.
Then for every $n \in \mathbb{N}$, let us define

$$\tilde{S}_n = (\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i))_n \tag{12}$$

be the sum for δ_n-fine partition P.

Let us choose a positive integer $m \in \mathbb{N}$, such that $m \geq n$. It is obvious that δ_m-fine partition is δ_n-fine partition.

Then for any δ_m-fine partition $P' = \{[u, v]; \varsigma\}$ and any δ_n-fine partition $P'' = \{[u^*, v^*]; \varsigma^*\}$ of $[c, d]$, the result holds true

$$\begin{aligned}\mathbb{D}(\tilde{S}_m, \tilde{S}_n) &= \mathbb{D}\left([\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i)]_m, [\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i)]_n\right) \\ &< 1/n. \end{aligned} \tag{13}$$

Let $\epsilon > 0$, then there exists $n_0 \in \mathbb{N}$, such that $\frac{1}{n_0} < \frac{\epsilon}{2}$. Then, for $m \geq n \geq n_0$, we have

$$\mathbb{D}(\tilde{S}_m, \tilde{S}_n) < \frac{1}{n} \leq \frac{1}{n_0} < \frac{\epsilon}{2}. \tag{14}$$

Therefore, $\tilde{S}_n$ is a Cauchy sequence.

Since $IF_N(\mathbb{R}^n)$ is a complete metric space, there exists a unique primitive $\tilde{F}(\varrho)$ such that $\lim_{n\to\infty} \tilde{S}_n = \tilde{F}(\varrho)$. Consequently,

$$\begin{aligned}&\sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{F}(\varrho))]^r d\varrho \right)^{1/r} \\ &\leq \sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} \left[\frac{\epsilon}{2}\right]^r d\varrho \right)^{1/r} \\ &\leq \sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \left[\frac{\epsilon}{2}\right]^r (v_i - u_i) \right)^{1/r} \\ &< \frac{\epsilon}{2}.\end{aligned}$$

Therefore,

$$\sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{F}(\varrho))]^r d\varrho \right)^{1/r} < \frac{\epsilon}{2}. \tag{15}$$

Hence, $\tilde{f} \in L^r$-*IFHK*$[c, d]$. □

Theorem 3. *If $\tilde{f} \in L^r$-IFHK$[c, d]$, then $\tilde{f} \in L^r$-IFHK$[a, b]$, for any $[a, b] \subset [c, d]$.*

Proof. Given, $\tilde{f} \in L^r$-*IFHK*$[c, d]$. Then by Theorem 2, for every $\epsilon > 0$, there exists a gauge $\delta(\varsigma) > 0$ such that for any two δ-fine partitions $P = \{[u, v]; \varsigma\}$ and $Q = \{[u^*, v^*]; \varsigma^*\}$ of $[c, d]$, we have

$$\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{f}(\varsigma_i{}^*)(\varrho - \varsigma_i{}^*) + \tilde{F}(\varsigma_i{}^*)) < \epsilon. \tag{16}$$

Given $[a,b]$ be any subinterval of $[c,d]$. Then consider the restriction of δ to $[a,b]$ be a gauge δ_1. It is clear that δ-fine partition is δ_1-fine partition.

Now, let P_1 and Q_1 be two δ_1-fine partitions of $[a,b]$ such that

$$\begin{aligned}
&\mathbb{D}\Big((\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i))_1,(\tilde{f}(\varsigma_i{}^*)(\varrho-\varsigma_i{}^*)+\tilde{F}(\varsigma_i{}^*))_1\Big)\\
&=\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i),\tilde{f}(\varsigma_i{}^*)(\varrho-\varsigma_i{}^*)+\tilde{F}(\varsigma_i{}^*))<\epsilon\\
\Rightarrow\quad&\mathbb{D}\Big((\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i))_1,(\tilde{f}(\varsigma_i{}^*)(\varrho-\varsigma_i{}^*)+\tilde{F}(\varsigma_i{}^*))_1\Big)<\epsilon. \qquad (17)
\end{aligned}$$

Therefore, $\tilde{f}\in L^r$-*IFHK*$[a,b]$. □

Remark 1. *Given three points $c<a<d$. If $\tilde{f}\in L^r$-IFHK$[c,a]$ and $\tilde{f}\in L^r$-IFHK$[a,d]$ then $\tilde{f}\in L^r$-IFHK$[c,d]$.*

Theorem 4. *If $\tilde{f}\in L^r$-IFHK$[c,d]$ and $\tilde{g}\in L^r$-IFHK$[c,d]$, then*

(i) $\tilde{f}+\tilde{g}\in L^r$-*IFHK*$[c,d]$.

(ii) $k\tilde{f}\in L^r$-*IFHK*$[c,d]$*, where k is real.*

Proof. Since $\tilde{f}\in L^r$-*IFHK*$[c,d]$ and $\tilde{g}\in L^r$-*IFHK*$[c,d]$, for every $\epsilon>0$ there exists a gauge $\delta(\varsigma)>0$ such that for any δ-fine partition $P=\{[u,v];\varsigma\}$ of $[c,d]$, the following conditions hold

$$\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i),\tilde{F}(\varrho))]^r d\varrho\right)^{1/r}<\frac{\epsilon}{2} \qquad (18)$$

and

$$\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{g}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{g}(\varsigma_i),\tilde{G}(\varrho))]^r d\varrho\right)^{1/r}<\frac{\epsilon}{2} \qquad (19)$$

where $\tilde{F}(\varrho)$ and $\tilde{G}(\varrho)$ are the primitives of $\tilde{f}$ and $\tilde{g}$ respectively. Now,

$$\begin{aligned}
&\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{g}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i)+\tilde{G}(\varsigma_i),\tilde{F}(\varrho)+\tilde{G}(\varrho))]^r d\varrho\right)^{1/r}\\
&\leq\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i),\tilde{F}(\varrho))+\mathbb{D}(\tilde{g}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{G}(\varsigma_i),\tilde{G}(\varrho))]^r d\varrho\right)^{1/r}\\
&\leq\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i),\tilde{F}(\varrho))]^r d\varrho\right)^{1/r}\\
&\quad+\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{g}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{G}(\varsigma_i),\tilde{G}(\varrho))]^r d\varrho\right)^{1/r}\\
&<\frac{\epsilon}{2}+\frac{\epsilon}{2}\\
\Rightarrow\quad&\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{g}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i)+\tilde{G}(\varsigma_i),\tilde{F}(\varrho)+\tilde{G}(\varrho))]^r d\varrho\right)^{1/r}\\
&<\epsilon. \qquad (20)
\end{aligned}$$

Therefore, $\tilde{f}+\tilde{g}\in L^r$-*IFHK*$[c,d]$. Similarly, $k\tilde{f}\in L^r$-*IFHK*$[c,d]$. □

Theorem 5. *If $\tilde{f} \in IFHK[c,d]$ then $\tilde{f} \in L^r$-IFHK$[c,d]$.*

Proof. Since $\tilde{f} \in IFHK[c,d]$, for every $\epsilon > 0$ there exists a gauge $\delta(\varsigma) > 0$ such that for any δ-fine partition $P = \{[u,v];\varsigma\}$ of $[c,d]$, we have

$$\sum_{i=1}^{n} \mathbb{D}\left(\tilde{f}(\varsigma_i)(\varrho_i - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{F}(\varrho_i)\right) < \epsilon. \tag{21}$$

For each i, set on every interval $[u_i, v_i]$, there exists $\varsigma_i \neq \varrho_i$ such that $\forall\ \varrho \in [u_i, v_i]$,

$$\mathbb{D}\left(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{F}(\varrho)\right) \leq \mathbb{D}\left(\tilde{f}(\varsigma_i)(\varrho_i - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{F}(\varrho_i)\right). \tag{22}$$

We define $P' = \{[\min(\varsigma_i, \varrho_i), \max(\varsigma_i, \varrho_i)]; \varsigma_i\}$, then $P' < \delta$. Now consider,

$$\sum_{i=1}^{n}\left(\frac{1}{v_i - u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{F}(\varrho))]^r d\varrho\right)^{1/r}$$

$$\leq \sum_{i=1}^{n}\left(\frac{1}{v_i - u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho_i - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{F}(\varrho_i))]^r d\varrho\right)^{1/r}$$

$$\Rightarrow \sum_{i=1}^{n}\left(\frac{1}{v_i - u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{F}(\varrho))]^r d\varrho\right)^{1/r} < \epsilon. \tag{23}$$

Hence, $\tilde{f} \in L^r$-IFHK$[c,d]$. □

Remark 2. *If $\tilde{f} = \tilde{g}$ almost everywhere on $[c,d]$ and $\tilde{f} \in L^r$-IFHK$[c,d]$; then $\tilde{g} \in L^r$-IFHK$[c,d]$ and*

$$(L^r\text{-IFHK})\int_c^d \tilde{f}(t)dt = (L^r\text{-IFHK})\int_c^d \tilde{g}(t)dt. \tag{24}$$

Let us consider $\tilde{f} = \tilde{g} + \tilde{h}$ where $\tilde{h} = 0$ almost everywhere on $[c,d]$. Then $\tilde{h} \in IFHK[c,d]$ and by the previous theorem $\tilde{h} \in L^r$-IFHK$[c,d]$.

$$(L^r\text{-IFHK})\int_c^d \tilde{h}(t)dt = 0. \tag{25}$$

By additivity of L^r-IFHK integral, we have $\tilde{g} \in L^r$-IFHK$[c,d]$, and the L^r-IFHK integrals of $\tilde{f}$ and $\tilde{g}$ are equal.

$$(L^r\text{-IFHK})\int_c^d \tilde{f}(t)dt = (L^r\text{-IFHK})\int_c^d \tilde{g}(t)dt.$$

Definition 10. [41] For $1 \leq r < \infty$. $\tilde{f} \in L^r[c,d]$ is said to be L^r-continuous at $\varsigma_0 \in [c,d]$ if

$$\int_\theta [\mathbb{D}(\tilde{f}(\varsigma), \tilde{f}(\varsigma_0))]^r d\varsigma = o(h), \tag{26}$$

where $\theta = [c,d] \cap [\varsigma_0 - h, \varsigma_0 + h]$.

Theorem 6. *For $1 \le r < \infty$. If $\tilde{f} \in L^r$-IFHK$[c,d]$, then $\tilde{F}$ is L^r-continuous everywhere.*

Proof. Since $\tilde{f} \in L^r$-*IFHK*$[c,d]$, for every $\epsilon > 0$ there exists a gauge $\delta(\varsigma) > 0$ such that for any δ-fine partition $P = \{[u,v];\varsigma\}$ of $[c,d]$, $\mathbb{D}(\tilde{f}(\varsigma),0)\delta < \epsilon$ holds true. Then,

$$\left(\frac{1}{2\delta}\int_{\varsigma-\delta}^{\varsigma+\delta}[\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i),\tilde{F}(\varrho))]^r d\varrho\right)^{1/r} \le \epsilon. \tag{27}$$

Now, consider

$$\begin{aligned}
&\left(\int_{\varsigma-\delta}^{\varsigma+\delta}[\mathbb{D}(\tilde{F}(\varrho),\tilde{F}(\varsigma))]^r d\varrho\right)^{1/r}\\
&\le \left(\int_{\varsigma-\delta}^{\varsigma+\delta}[\mathbb{D}(\tilde{f}(\varsigma)(\varrho-\varsigma)+\tilde{F}(\varrho),\tilde{f}(\varsigma)(\varrho-\varsigma)+\tilde{F}(\varsigma))]^r d\varrho\right)^{1/r}\\
&\le \left(\int_{\varsigma-\delta}^{\varsigma+\delta}[\mathbb{D}(\tilde{f}(\varsigma)(\varrho-\varsigma)+\tilde{F}(\varsigma),\tilde{F}(\varrho)]^r d\varrho\right)^{1/r} + \left(\int_{\varsigma-\delta}^{\varsigma+\delta}[\mathbb{D}(\tilde{f}(\varsigma)(\varrho-\varsigma),0)]^r d\varrho\right)^{1/r}\\
&< \epsilon(2\delta)^{1/r} + \mathbb{D}(\tilde{f}(\varsigma),0)\left(\int_{\varsigma-\delta}^{\varsigma+\delta}[\varrho-\varsigma]^r d\varrho\right)^{1/r}\\
&\le \epsilon(2\delta)^{1/r} + \mathbb{D}(\tilde{f}(\varsigma),0)(2\delta^{r+1})^{1/r}\\
&\le \epsilon(2\delta)^{1/r} + \mathbb{D}(\tilde{f}(\varsigma),0)\delta(2\delta)^{1/r}\\
&< \epsilon(2\delta)^{1/r} + \epsilon(2\delta)^{1/r} = 2\epsilon(2\delta)^{1/r}.
\end{aligned} \tag{28}$$

Thus, $\tilde{F}$ is L^r-continuous everywhere. □

Theorem 7. *For $1 \le r < \infty$. Let $\{\tilde{f}_n\}$ be a sequence of L^r-IFHK integrable functions on $[c,d]$ and $\tilde{f}_n \to \tilde{f}$. Suppose $\tilde{F}_n(\varsigma) = (L^r$-IFHK$)\int_c^\varsigma \tilde{f}_n(t)dt$ and $\tilde{F}_n \to \tilde{F}$. Then $\tilde{f} \in L^r$-IFHK$[c,d]$.*

Proof. Let $\epsilon > 0$. Choose n so that $\mathbb{D}(\tilde{f}_n(\varsigma),\tilde{f}(\varsigma)) < \epsilon$ and $\mathbb{D}(\tilde{F}_n(\varsigma),\tilde{F}(\varsigma)) < \epsilon$. Since $\tilde{f}_n \in L^r$-*IFHK*$[c,d]$, for every $\epsilon > 0$ there exists a gauge $\delta(\varsigma) > 0$ such that for any δ-fine partition $P = \{[u,v];\varsigma\}$ of $[c,d]$, we get

$$\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{f}_n(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}_n(\varsigma_i),\tilde{F}_n(\varrho))]^r d\varrho\right)^{1/r} < \epsilon. \tag{29}$$

Now, consider

$$\begin{aligned}
&\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i),\tilde{F}(\varrho))]^r d\varrho\right)^{1/r}\\
&= \sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i)+\tilde{F}_n(\varrho)+\tilde{F}_n(\varsigma_i)+\tilde{f}_n(\varsigma_i)(\varrho-\varsigma_i),\right.\\
&\qquad\qquad \left.\tilde{F}(\varrho)+\tilde{F}_n(\varrho)+\tilde{F}_n(\varsigma_i)+\tilde{f}_n(\varsigma_i)(\varrho-\varsigma_i))]^r d\varrho\right)^{1/r}
\end{aligned}$$

$$\leq \sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{f}_n(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}_n(\varsigma_i),\tilde{F}_n(\varrho))]^r d\varrho\right)^{1/r}$$
$$+\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{f}_n(\varsigma_i)(\varrho-\varsigma_i))]^r d\varrho\right)^{1/r}$$
$$+\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{F}(\varsigma_i)+\tilde{F}_n(\varrho),\tilde{F}_n(\varsigma_i)+\tilde{F}(\varrho))]^r d\varrho\right)^{1/r}$$
$$< \epsilon+\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{f}_n(\varsigma_i),\tilde{f}(\varsigma_i))\mid \varrho-\varsigma_i\mid]^r d\varrho\right)^{1/r}$$
$$+\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{F}_n(\varrho),\tilde{F}(\varrho))]^r d\varrho\right)^{1/r}$$
$$+\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{F}_n(\varsigma_i),\tilde{F}(\varsigma_i))]^r d\varrho\right)^{1/r}$$
$$< \epsilon+\sum_{i=1}^{n}\mathbb{D}(\tilde{f}_n(\varsigma_i),\tilde{f}(\varsigma_i))\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}\mid \varrho-\varsigma_i\mid^r d\varrho\right)^{1/r}+\epsilon+\epsilon$$
$$< 3\epsilon+(d-c)\epsilon. \tag{30}$$

Therefore, $\tilde{f} \in L^r\text{-}IFHK[c,d]$. □

Example 1. Let $\tilde{a}$ be an intuitionistic fuzzy number and $\tilde{f}:[1,d]\to IF_N(\mathbb{R})$ be an intuitionistic fuzzy number valued function defined as $\tilde{f}(\varsigma)=\frac{\tilde{a}}{\varsigma^2}=(0,0,\frac{1}{\varsigma^2});(0,0,\frac{2}{\varsigma^2})$. Its α- and β-cuts are given by, $[\tilde{f}(\varsigma)]_\alpha=[\tilde{f}_\alpha^-,\tilde{f}_\alpha^+]=[0,\frac{1-\alpha}{\varsigma^2}]=(1-\alpha)[0,\frac{1}{\varsigma^2}]$ and $[\tilde{f}(\varsigma)]_\beta=[\tilde{f}_\beta^-,\tilde{f}_\beta^+]=[0,\frac{2\beta}{\varsigma^2}]=\beta[0,\frac{2}{\varsigma^2}]$, respectively.

$$\int_1^d \tilde{f}_\alpha^- d\varsigma=0,\int_1^d \tilde{f}_\alpha^+ d\varsigma=(1-\alpha)\left(1-\frac{1}{d}\right);\int_1^d \tilde{f}_\beta^- d\varsigma=0,\int_1^d \tilde{f}_\beta^+ d\varsigma=\beta\left(2-\frac{2}{d}\right).$$

For $1\leq r<\infty$ and $\epsilon>0$, let us assume $\delta(\varsigma)=\frac{\epsilon}{4n}\varsigma$ be a gauge function. Now,

$$\mathbb{D}_1(\tilde{f}(\varsigma_i)(y-\varsigma_i)+\tilde{F}(\varsigma_i),\tilde{F}(\varrho))$$
$$=\sup_{\alpha\in[0,1]}\max\{|\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i)-\tilde{F}(\varrho)|_\alpha^-,|\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i)-\tilde{F}(\varrho)|_\alpha^+\}$$
$$=\sup_{\alpha\in[0,1]}\max\left\{0,\left|\frac{1}{{\varsigma_i}^2}(\varrho-\varsigma_i)+(1-\alpha)\left(1-\frac{1}{\varsigma_i}\right)-(1-\alpha)\left(1-\frac{1}{d}\right)\right|\right\}$$
$$=\sup_{\alpha\in[0,1]}\left\{\left|\frac{1}{{\varsigma_i}^2}(\varrho-\varsigma_i)+(1-\alpha)\left(1-\frac{1}{\varsigma_i}\right)-(1-\alpha)\left(1-\frac{1}{d}\right)\right|\right\}$$
$$=\left|\frac{1}{{\varsigma_i}^2}(\varrho-\varsigma_i)+\left(1-\frac{1}{\varsigma_i}\right)-\left(1-\frac{1}{d}\right)\right|$$
$$=\left|\frac{1}{{\varsigma_i}^2}(\varrho-\varsigma_i)-\frac{1}{\varsigma_i}+\frac{1}{d}\right|$$
$$=\left|\frac{1}{{\varsigma_i}^2}\frac{\epsilon}{4n}\varsigma_i-\frac{1}{\varsigma_i}+\frac{1}{d}\right|=\left|\frac{1}{\varsigma_i}\frac{\epsilon}{4n}-\frac{1}{\varsigma_i}+\frac{1}{d}\right|$$
$$\Rightarrow \mathbb{D}_1(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i),\tilde{F}(\varrho))\leq\left|\frac{\epsilon}{4n}-\frac{1}{\varsigma_i}+\frac{1}{d}\right|.$$

Similarly, $\mathbb{D}_2(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i),\tilde{F}(\varrho)) \leq 2\left|\frac{\epsilon}{4n}-\frac{1}{\varsigma_i}+\frac{1}{d}\right|$.

Then,

$$
\begin{aligned}
&\sum_{i=1}^{n}(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i),\tilde{F}(\varrho))]^r d\varrho)^{\frac{1}{r}} \\
&=\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\max\{\mathbb{D}_1(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i),\tilde{F}(\varrho)),\mathbb{D}_2(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i),\tilde{F}(\varrho))\}]^r d\varrho\right)^{\frac{1}{r}} \\
&\leq \sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}\left[\max\left\{|\frac{\epsilon}{4n}+\frac{1}{d}-\frac{1}{\varsigma_i}|,2|\frac{\epsilon}{4n}+\frac{1}{d}-\frac{1}{\varsigma_i}|\right\}\right]^r d\varrho\right)^{\frac{1}{r}} \\
&=\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[2|\frac{\epsilon}{4n}+\frac{1}{d}-\frac{1}{\varsigma_i}|]^r d\varrho\right)^{\frac{1}{r}} \\
&=\sum_{i=1}^{n}2|\frac{\epsilon}{4n}+\frac{1}{d}-\frac{1}{\varsigma_i}|\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}d\varrho\right)^{\frac{1}{r}} \\
&\leq 2\frac{\epsilon}{4n}n+2\sum_{i=1}^{n}|\frac{1}{d}-\frac{1}{\varsigma_i}| \\
&=\frac{\epsilon}{2}+2\sum_{i=1}^{n}\frac{|\varsigma_i-d|}{d\varsigma_i} \\
&=\frac{\epsilon}{2}+2\sum_{i=1}^{n}\frac{1}{d\varsigma_i}\frac{\epsilon}{4n}\varsigma_i \\
&=\frac{\epsilon}{2}+\frac{\epsilon}{2d}\leq\frac{\epsilon}{2}+\frac{\epsilon}{2}\leq\epsilon \\
&\therefore \sum_{i=1}^{n}(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i),\tilde{F}(\varrho))]^r d\varrho)^{\frac{1}{r}}\leq\epsilon.
\end{aligned}
$$

Thus, $\tilde{f}(\varsigma)$ is L^r-*IFHK*-integrable.

4 On infinite intervals

In this section we shall define L^r-intuitionistic fuzzy Henstock–Kurzweil integral on infinite interval. Let $\tilde{f}$ be an intuitionistic fuzzy number valued function on $[c,\infty)$. We define $\tilde{f}(\infty)=0$, and $0(\infty)=0$.

Let $\delta:[0,\infty]\to\mathbb{R}^+$ be a positive real valued function. A partition $P=\{[u,v];\xi\}$ is said to be δ-fine partition if it satisfies the following conditions (see [23]):

(i) $c=\varsigma_0<\varsigma_1<\varsigma_2<\cdots<\varsigma_{n-1}=d<\varsigma_n=\infty$;

(ii) $\xi_i\in[\varsigma_{i-1},\varsigma_i]\subset(\xi_i-\delta(\xi_i),\xi_i+\delta(\xi_i))$, $i=1,2,3,\ldots,n-1$;

(iii) $(\xi_n-\delta(\xi_n),\xi_n+\delta(\xi_n))=[d,\infty]$.

Definition 11. An intuitionistic fuzzy number valued function $\tilde{f}$ is said to be L^r-intuitionistic fuzzy Henstock–Kurzweil (L^r-*IFHK*) integrable on $[c,\infty]$ if there exists $\tilde{F}\in L^r[c,\infty]$ $\forall\,\epsilon>0$, there exists a gauge function $\delta(\varsigma) > 0$ on $[c, \infty]$ such that for any δ-fine partition $P = \{[u, v]; \varsigma\}$ of $[c, \infty]$, we get

$$\sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{F}(\varrho))]^r d\varrho \right)^{1/r} < \epsilon.$$

$$\tilde{F}(\varsigma) = (L^r\text{-}IFHK) \int_c^{\infty} \tilde{f}(t)dt \quad \text{and} \quad \tilde{f} \in L^r\text{-}IFHK[c, \infty]. \tag{31}$$

The definition is similar for $\tilde{f} \in L^r$-$IFHK(-\infty, c]$. So evidently, $\tilde{f} \in L^r$-$IFHK(-\infty, \infty)$ if and only if $\tilde{f} \in L^r$-$IFHK(-\infty, c]$ and $\tilde{f} \in L^r$-$IFHK[c, \infty)$. Furthermore,

$$(L^r\text{-}IFHK) \int_{-\infty}^{\infty} \tilde{f}(t)dt = (L^r\text{-}IFHK) \int_{-\infty}^{c} \tilde{f}(t)dt + (L^r\text{-}IFHK) \int_c^{\infty} \tilde{f}(t)dt. \tag{32}$$

Theorem 8. *Let $\tilde{f}$ be an intuitionistic fuzzy number valued function on $[c, \infty)$. Then the following statements are equivalent:*

(i) $\tilde{f} \in L^r$-$IFHK[c, \infty)$, $\tilde{F}(\varsigma) = \int_c^{\infty} \tilde{f}(t)dt$.

(ii) for any α, $\beta \in [0, 1]$; f_α^-, f_α^+, f_β^-, f_β^+ are L^r-Henstock–Kurzweil integrable uniformly with respect to α, $\beta \in [0, 1]$ on $[c, \infty)$ $\{\delta(\varsigma)$ is independent of α and $\beta\}$, and

$$F_\alpha(\varsigma) = \left(\int_c^{\infty} f_\alpha{}^-(t)dt, \int_c^{\infty} f_\alpha{}^+(t)dt \right); F_\beta(\varsigma) = \left(\int_c^{\infty} f_\beta{}^-(t)dt, \int_c^{\infty} f_\beta{}^+(t)dt \right)$$

where $F_\alpha(\varsigma)$ and $F_\beta(\varsigma)$ are the α- and β- cuts of $\tilde{F}(\varsigma)$.

(iii) for any $d > c$, $\tilde{f} \in L^r$-$IFHK[c, d]$, $\lim_{d\to\infty} \int_c^{\infty} \tilde{f}(t)dt$ exists as an intuitionistic fuzzy number and

$$\lim_{d\to\infty} \int_c^{d} \tilde{f}(t)dt = \int_c^{\infty} \tilde{f}(t)dt.$$

Proof. (i) $\Rightarrow$ (ii). If $\tilde{F}(\varsigma) = \int_c^{\infty} \tilde{f}(t)dt$. Then for $\epsilon > 0$, there exists a gauge function $\delta(\varsigma) > 0$ on $[c, \infty)$ such that for any δ-fine partition $P = \{[u, v]; \varsigma\}$ of $[c, \infty)$, we get

$$\sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{F}(\varrho))]^r d\varrho \right)^{1/r} < \epsilon. \tag{33}$$

Now,

$$\begin{aligned}
&\max \left\{ \mathbb{D}_1(\tilde{f}_\alpha(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}_\alpha(\varsigma_i), \tilde{F}_\alpha(\varrho)), \mathbb{D}_2(\tilde{f}_\beta(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}_\beta(\varsigma_i), \tilde{F}_\beta(\varrho)) \right\} \\
&\leq \sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\max\{\mathbb{D}_1(\tilde{f}_\alpha(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}_\alpha(\varsigma_i), \tilde{F}_\alpha(\varrho)), \mathbb{D}_2(\tilde{f}_\beta(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}_\beta(\varsigma_i), \right. \\
&\qquad\qquad \left. \tilde{F}_\beta(\varrho))\}]^r d\varrho \right)^{1/r} \\
&\leq \sum_{i=1}^{n} \left(\frac{1}{v_i - u_i} \int_{u_i}^{v_i} [\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}(\varsigma_i), \tilde{F}(\varrho))]^r d\varrho \right)^{1/r} < \epsilon. \\
&\Rightarrow \max \left\{ \mathbb{D}_1(\tilde{f}_\alpha(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}_\alpha(\varsigma_i), \tilde{F}_\alpha(\varrho)), \mathbb{D}_2(\tilde{f}_\beta(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}_\beta(\varsigma_i), \tilde{F}_\beta(\varrho)) \right\} < \epsilon \\
&\Rightarrow \mathbb{D}_1(\tilde{f}_\alpha(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}_\alpha(\varsigma_i), \tilde{F}_\alpha(\varrho)) < \epsilon, \mathbb{D}_2(\tilde{f}_\beta(\varsigma_i)(\varrho - \varsigma_i) + \tilde{F}_\beta(\varsigma_i), \tilde{F}_\beta(\varrho)) < \epsilon.
\end{aligned} \tag{34}$$

Then for the α-cut,

$$\mathbb{D}_1(\tilde{f}_\alpha(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}_\alpha(\varsigma_i),\tilde{F}_\alpha(\varrho))<\epsilon$$

$$\Rightarrow \sup_{\alpha\in[0,1]}\max\Big\{\mid \tilde{f}^-_\alpha(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}^-_\alpha(\varsigma_i),\tilde{F}^-_\alpha(\varrho)\mid,\mid \tilde{f}^+_\alpha(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}^+_\alpha(\varsigma_i),\tilde{F}^+_\alpha(\varrho)\mid\Big\}<\epsilon.$$

Thus,

$$\mid \tilde{f}^-_\alpha(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}^-_\alpha(\varsigma_i),\tilde{F}^-_\alpha(\varrho)\mid<\epsilon;\mid \tilde{f}^+_\alpha(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}^+_\alpha(\varsigma_i),\tilde{F}^+_\alpha(\varrho)\mid<\epsilon. \quad (35)$$

Furthermore,

$$\mid \tilde{f}^-_\alpha(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}^-_\alpha(\varsigma_i),\tilde{F}^-_\alpha(\varrho)\mid<\epsilon$$

Therefore,

$$\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}\mid \tilde{f}^-_\alpha(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}^-_\alpha(\varsigma_i),\tilde{F}^-_\alpha(\varrho)\mid^r d\varrho\right)^{1/r}<\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}\epsilon^r d\varrho\right)^{1/r}\leq\epsilon.$$

Thus,

$$\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}\mid \tilde{f}^-_\alpha(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}^-_\alpha(\varsigma_i),\tilde{F}^-_\alpha(\varrho)\mid^r d\varrho\right)^{1/r}<\epsilon. \quad (36)$$

Hence, f^-_α is L^r-Henstock–Kurzweil integrable uniformly with respect to $\alpha\in[0,1]$ on $[c,\infty)$. We apply similar steps to f^+_α, f^-_β, f^+_β; we get f^+_α, f^-_β, f^+_β are L^r-Henstock–Kurzweil integrable uniformly with respect to α, $\beta\in[0,1]$ on $[c,\infty)$ and

$$F_\alpha(\varsigma)=\left(\int_c^\infty f_\alpha{}^-(t)dt,\int_c^\infty f_\alpha{}^+(t)dt\right);F_\beta(\varsigma)=\left(\int_c^\infty f_\beta{}^-(t)dt,\int_c^\infty f_\beta{}^+(t)dt\right). \quad (37)$$

(ii) $\Rightarrow$ (i). Since f^-_α, f^+_α, f^-_β, f^+_β are L^r-Henstock–Kurzweil integrable uniformly with respect to α, $\beta\in[0,1]$ on $[c,\infty)$, for $\delta>0$, there exists a gauge function $\delta(\varsigma)>0$ on $[c,\infty)$ such that for any δ-fine partition $P=\{[u,v];\varsigma\}$ of $[c,\infty)$, we get

$$\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}\mid f^-_\alpha(\varsigma_i)(\varrho-\varsigma_i)+F^-_\alpha(\varsigma_i),F^-_\alpha(\varrho)\mid^r d\varrho\right)^{1/r}<\epsilon, \quad (38)$$

$$\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}\mid f^+_\alpha(\varsigma_i)(\varrho-\varsigma_i)+F^+_\alpha(\varsigma_i),F^+_\alpha(\varrho)\mid^r d\varrho\right)^{1/r}<\epsilon, \quad (39)$$

$$\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}\mid f^-_\beta(\varsigma_i)(\varrho-\varsigma_i)+F^-_\beta(\varsigma_i),F^-_\beta(\varrho)\mid^r d\varrho\right)^{1/r}<\epsilon, \quad (40)$$

$$\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}\mid f^+_\beta(\varsigma_i)(\varrho-\varsigma_i)+F^+_\beta(\varsigma_i),F^+_\beta(\varrho)\mid^r d\varrho\right)^{1/r}<\epsilon. \quad (41)$$

Next, we prove that the class of closed intervals $\{[F^-_\alpha,F^+_\alpha]:\alpha\in[0,1]\}$ and $\{[F^-_\beta,F^+_\beta]:\beta\in[0,1]\}$ determines an intuitionistic fuzzy number. $[F^-_\alpha,F^+_\alpha]$ and $[F^-_\beta,F^+_\beta]$ satisfies the conditions of Definition 5.

1) Since $f_\alpha^- \le f_\alpha^+$, $f_\beta^- \le f_\beta^+$ for $\alpha, \beta \in [0,1]$ implies $F_\alpha^- \le F_\alpha^+$, $F_\beta^- \le F_\beta^+$ i.e., $[F_\alpha^-, F_\alpha^+]$ and $[F_\beta^-, F_\beta^+]$ are closed intervals for α, $\beta \in [0,1]$.

2) For any $0 \le \alpha_1 \le \alpha_2 \le 1$, $f_{\alpha_1}^- \le f_{\alpha_2}^- \le f_{\alpha_2}^+ \le f_{\alpha_1}^+$

$$\int_c^\infty f_{\alpha_1}^-(t)dt \le \int_c^\infty f_{\alpha_2}^-(t)dt \le \int_c^\infty f_{\alpha_2}^+(t)dt \le \int_c^\infty f_{\alpha_1}^+(t)dt.$$
$$\Rightarrow [F_{\alpha_2}^-, F_{\alpha_2}^+] \subset [F_{\alpha_1}^-, F_{\alpha_1}^+]. \tag{42}$$

For any $0 \le \beta_1 \le \beta_2 \le 1$, $f_{\beta_2}^- \le f_{\beta_1}^- \le f_{\beta_1}^+ \le f_{\beta_2}^+$

$$\int_c^\infty f_{\beta_2}^-(t)dt \le \int_c^\infty f_{\beta_1}^-(t)dt \le \int_c^\infty f_{\beta_1}^+(t)dt \le \int_c^\infty f_{\beta_2}^+(t)dt.$$
$$\Rightarrow [F_{\beta_1}^-, F_{\beta_1}^+] \subset [F_{\beta_2}^-, F_{\beta_2}^+]. \tag{43}$$

3) Let $\{\alpha_n\}$ be a non-decreasing sequence converging to $\alpha \in [0,1]$ such that

$$\bigcap_{n=1}^{\infty} [\tilde{f}]_{\alpha_n} = [\tilde{f}]_\alpha \Rightarrow \bigcap_{n=1}^{\infty} [f_{\alpha_n}^-, f_{\alpha_n}^+] = [f_\alpha^-, f_\alpha^+]$$
$$\Rightarrow \lim_{n\to\infty} f_{\alpha_n}^- = f_\alpha^-, \lim_{n\to\infty} f_{\alpha_n}^+ = f_\alpha^+$$
$$\Rightarrow 0 \le f_{\alpha_n}^- - f_0^- \le f_1^- - f_0^- \qquad [\text{Since } f_0^- \le f_{\alpha_n}^- \le f_1^-],$$
$$0 \le f_{\alpha_n}^+ - f_1^+ \le f_0^+ - f_1^+ \qquad [\text{Since } f_1^+ \le f_{\alpha_n}^+ \le f_0^+]. \tag{44}$$

By the non-negativity and L^r-Henstock–Kurzweil integrability of $f_1^- - f_0^-$, $f_0^+ - f_1^+$, we know that $f_1^- - f_0^-$, $f_0^+ - f_1^+$ are Lebesgue integrable. Hence, $f_{\alpha_n}^- - f_0^-$, $f_{\alpha_n}^+ - f_1^+$ are Lebesgue integral, and

$$\lim_{n\to\infty} \int_c^\infty (f_{\alpha_n}^- - f_0^-)dt = \int_c^\infty (f_\alpha^- - f_0^-)dt, \lim_{n\to\infty} \int_c^\infty (f_{\alpha_n}^+ - f_1^+)dt = \int_c^\infty (f_\alpha^+ - f_1^+)dt.$$

$$\Rightarrow \lim_{n\to\infty} \int_c^\infty f_{\alpha_n}^- dt = \int_c^\infty f_\alpha^- dt, \lim_{n\to\infty} \int_c^\infty f_{\alpha_n}^+ dt = \int_c^\infty f_\alpha^+ dt.$$

Thus,

$$\bigcap_{n=1}^{\infty} [F_{\alpha_n}^-, F_{\alpha_n}^+] = [F_\alpha^-, F_\alpha^+]. \tag{45}$$

4) Let $\{\beta_n\}$ be a non-increasing sequence converging to $\beta \in [0,1]$ such that

$$\bigcap_{n=1}^{\infty} [\tilde{f}]_{\beta_n} = [\tilde{f}]_\beta \Rightarrow \bigcap_{n=1}^{\infty} [f_{\beta_n}^-, f_{\beta_n}^+] = [f_\beta^-, f_\beta^+]$$

Applying similar steps from above, we get

$$\bigcap_{n=1}^{\infty} [F_{\beta_n}^-, F_{\beta_n}^+] = [F_\beta^-, F_\beta^+]. \tag{46}$$

Combining Inequalities (38), (39), (40) and (41), we get

$$\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i),\tilde{F}(\varrho))]^r d\varrho\right)^{1/r}<\epsilon,$$

i.e.,

$$\tilde{f}\in L^r\text{-}IFHK[c,\infty),\tilde{F}(\varsigma)=\int_c^{\infty}\tilde{f}(t)dt. \tag{47}$$

<u>(i) ⇒ (iii).</u> Let $\epsilon>0$. Let $\tilde{f}\in L^r$-$IFHK[c,\infty)$, then there exists a function $\delta>0$ on $[c,\infty)$ such that

$$\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i),\tilde{F}(\varrho))]^r d\varrho\right)^{1/r}<\epsilon, \tag{48}$$

for any δ-fine partition $P=\{[u,v];\varsigma\}$ of $[c,\infty]$ and $\tilde{F}(\varsigma)=\int_c^{\infty}\tilde{f}(t)dt$.

From Theorem 2, if $\tilde{f}\in L^r$-$IFHK[c,d]$, then $\tilde{f}\in L^r$-$IFHK[c,d]$ for any $d>c$. There exists a function $\delta_1>0$ on $[c,d]$ such that

$$\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}_1(\varsigma_i),\tilde{F}_1(\varrho))]^r d\varrho\right)^{1/r}<\epsilon, \tag{49}$$

for any δ_1-fine partition $P=\{[u,v];\varsigma\}$ of $[c,d]$ and $\tilde{F}_1(\varsigma)=\int_c^{d}\tilde{f}(t)dt$.

We may assume that $\delta_1\le\delta$ for any $\varsigma\in[c,d]$. Then

$$\mathbb{D}\left(\int_c^{\infty}\tilde{f}(t)dt,\int_c^{d}\tilde{f}(t)dt\right)<\epsilon. \tag{50}$$

$$\text{Hence,}\quad\lim_{d\to\infty}\int_c^{d}\tilde{f}(t)dt=\int_c^{\infty}\tilde{f}(t)dt. \tag{51}$$

<u>(iii) ⇒ (i).</u> Let $\epsilon>0$. Choose a sequence $c=d_0<d_1<d_2<\cdots<d_k\uparrow+\infty$. Since $\tilde{f}\in L^r$-$IFHK[d_{k-1},d_k]$, $k=1,2,3,\dots$ there exists δ_k such that

$$\sum_{[d_{k-1}-d_k]}\left(\frac{1}{v-u}\int_{u}^{v}[\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{F}(\varsigma_i),\tilde{F}(\varrho))]^r d\varrho\right)^{1/r}<\frac{\epsilon}{2^{k+2}}, \tag{52}$$

for any δ_k-fine partition on $[d_{k-1},d_k]$, $k=1,2,3,\dots$.

Suppose, $\lim_{d\to\infty}\int_c^{d}\tilde{f}(t)dt=\tilde{W}$. Choose N such that $d>d_N$ and

$$\mathbb{D}\left(\int_c^{d}\tilde{f}(t)dt,\tilde{W}\right)<\frac{\epsilon}{2}. \tag{53}$$

Define

$$\delta=\begin{cases}\delta_1(\varsigma), & \varsigma\in[d_0,d_1)\\ \delta_k(\varsigma), & \varsigma\in(d_{k-1},d_k),k=1,2,3,\dots\\ \min\{\delta_k(d_k),\delta_{k+1}(d_k)\}, & \varsigma=d_k,k=1,2,3,\dots\end{cases} \tag{54}$$

For any δ-fine partition $P = \{[u,v];\varsigma\}$ of $[c,\infty]$, we have

$$\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}[\mathbb{D}(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{W}(\varsigma_i),\tilde{W}(\varrho))]^r d\varrho\right)^{1/r}$$
$$=\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}\left[\mathbb{D}\left(\int_c^d\tilde{f}+\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{W}(\varsigma_i),\int_c^d\tilde{f}+\tilde{W}(\varrho)\right)\right]^r d\varrho\right)^{1/r}$$
$$\leq\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}\left[\mathbb{D}\left(\int_c^d\tilde{f},\tilde{W}(\varrho)\right)\right]^r d\varrho\right)^{1/r}$$
$$+\sum_{i=1}^{n}\left(\frac{1}{v_i-u_i}\int_{u_i}^{v_i}\left[\mathbb{D}\left(\tilde{f}(\varsigma_i)(\varrho-\varsigma_i)+\tilde{W}(\varsigma_i),\int_c^d\tilde{f}\right)\right]^r d\varrho\right)^{1/r}$$
$$\leq\frac{\epsilon}{2}+\sum_{i=1}^{\infty}\frac{\epsilon}{2^{k+2}}<\epsilon. \tag{55}$$

Hence,

$$\tilde{f}\in L^r\text{-}IFHK[c,\infty)\quad\text{and}\quad\lim_{d\to\infty}\int_c^d\tilde{f}(t)dt=\int_c^\infty\tilde{f}(t)dt.\qquad\square$$

5 Application: Intuitionistic fuzzy Laplace transform

Definition 12. For $1\leq r<\infty$. Let $\tilde{f}(t)$ be a continuous intuitionistic fuzzy number valued function on $[0,\infty)$. Suppose $(L^r\text{-}IFHK)\int_0^\infty\tilde{f}(t)e^{-pt}dt$ exists, then $(L^r\text{-}IFHK)\int_0^\infty\tilde{f}(t)e^{-pt}dt$ is called intuitionistic fuzzy Laplace transform based on L^r-intuitionistic fuzzy Henstock–Kurzweil integral of $\tilde{f}$ at p and is denoted by

$$\hat{L}[\tilde{f}(t)]=(L^r\text{-}IFHK)\int_0^\infty\tilde{f}(t)e^{-pt}dt\quad(p>0\text{ and integer}). \tag{56}$$

Lemma 2. *Let $\tilde{f}\in L^r\text{-}IFHK[c,d]$, and put $\tilde{W}=(L^r\text{-}IFHK)\int_0^\varsigma\tilde{f}(t)e^{-pt}dt$. Then $\tilde{W}$ is continuous on $[c,d]$.*

Proof. Since $\tilde{f}\in L^r\text{-}IFHK[c,d]$, $\tilde{F}(\varsigma)=(L^r\text{-}IFHK)\int_c^\varsigma\tilde{f}(t)dt$ and from Theorem 6, $\tilde{F}$ is L^r-continuous everywhere. Also, observe that e^{-pt} is a continuous function of a real variable t. Therefore, $\tilde{W}$ is continuous on $[c,d]$. $\square$

Lemma 3. [20] *Let $f\in H[c,d]$ and let g be monotone. Then there exists $\vartheta\in[c,d]$ such that*

$$\int_c^d f(t)g(t)dt=g(c)\int_c^\vartheta f(t)dt+g(d)\int_\vartheta^d f(t)dt. \tag{57}$$

Lemma 4. [9] *Let $\tilde{f}$ be a continuous intuitionistic fuzzy number valued function on $[c,d]$. Then $\tilde{f}$ is bounded on $[c,d]$.*

Theorem 9. *Let $\tilde{f}\in L^r\text{-}IFHK[0,\infty)$, then $\hat{L}[\tilde{f}(t)]$ exists for any $p\in[0,\infty)$.*

Proof. Firstly, we shall prove that if $\tilde{f} \in L^r$-*IFHK*$[0,\infty)$, then

$$\tilde{W}(\varsigma) = (L^r\text{-}IFHK)\int_0^{\varsigma} \tilde{f}(t)e^{-pt}dt \tag{58}$$

is bounded on $[0,\infty)$. Let $\int_0^{\infty} \tilde{f}(t)dt = \tilde{F}$, then by Theorem 8(iii) we get

$$\lim_{d\to\infty} (L^r\text{-}IFHK)\int_0^{d} \tilde{f}(t)dt = \tilde{F}. \tag{59}$$

Therefore, given $\epsilon_0 = 1$, there exists a real number $b \geq 0$ such that for any $\varsigma \geq b$, $\mathbb{D}(\tilde{W}(\varsigma), \tilde{F}) < 1$. i.e.,

$$\sup_{\alpha\in[0,1]} \max\left\{\left|\int_0^{\varsigma} f_{\alpha}{}^{-}(t)dt - F_{\alpha}{}^{-}\right|, \left|\int_0^{\varsigma} f_{\alpha}{}^{+}(t)dt - F_{\alpha}{}^{+}\right|\right\} < 1; \tag{60}$$

$$\sup_{\beta\in[0,1]} \max\left\{\left|\int_0^{\varsigma} f_{\beta}{}^{-}(t)dt - F_{\beta}{}^{-}\right|, \left|\int_0^{\varsigma} f_{\beta}{}^{+}(t)dt - F_{\beta}{}^{+}\right|\right\} < 1 \tag{61}$$

which implies,

$$F_{\alpha}^{-} - 1 < \int_0^{\varsigma} f_{\alpha}^{-}(t)dt < F_{\alpha}^{-} + 1, \;\; F_{\alpha}^{+} - 1 < \int_0^{\varsigma} f_{\alpha}^{+}(t)dt < F_{\alpha}^{+} + 1; \tag{62}$$

$$F_{\beta}^{-} - 1 < \int_0^{\varsigma} f_{\beta}^{-}(t)dt < F_{\beta}^{-} + 1, \;\; F_{\beta}^{+} - 1 < \int_0^{\varsigma} f_{\beta}^{+}(t)dt < F_{\beta}^{+} + 1 \tag{63}$$

for any $\alpha, \beta \in [0,1]$ and $\varsigma \geq b$. Thus,

$$F_0^{-} - 1 \leq F_{\alpha}^{-} - 1 < \int_0^{\varsigma} f_{\alpha}^{-}(t)dt < F_{\alpha}^{+} - 1 \leq F_{\alpha}^{+} + 1 \leq F_0^{+} + 1, \tag{64}$$

$$F_0^{-} - 1 \leq F_{\alpha}^{-} - 1 \leq F_{\alpha}^{+} - 1 < \int_0^{\varsigma} f_{\alpha}^{+}(t)dt < F_{\alpha}^{+} + 1 \leq F_0^{+} + 1, \tag{65}$$

$$F_0^{-} - 1 \leq F_{\beta}^{-} - 1 < \int_0^{\varsigma} f_{\beta}^{-}(t)dt < F_{\beta}^{+} - 1 \leq F_{\beta}^{+} + 1 \leq F_0^{+} + 1, \tag{66}$$

$$F_0^{-} - 1 \leq F_{\beta}^{-} - 1 \leq F_{\beta}^{+} - 1 < \int_0^{\varsigma} f_{\beta}^{+}(t)dt < F_{\beta}^{+} + 1 \leq F_0^{+} + 1. \tag{67}$$

Now, let

$$S_1 = \max\{|F_0^{-} - 1|, |F_0^{+} + 1|\}, \tag{68}$$

then for any $\alpha, \beta \in [0,1]$ and $\varsigma \geq b$, we have

$$\left|\int_0^{\varsigma} f_{\alpha}^{-}(t)dt\right| < S_1, \left|\int_0^{\varsigma} f_{\alpha}^{+}(t)dt\right| < S_1; \left|\int_0^{\varsigma} f_{\beta}^{-}(t)dt\right| < S_1, \left|\int_0^{\varsigma} f_{\beta}^{+}(t)dt\right| < S_1. \tag{69}$$

$$\Rightarrow \mathbb{D}(\tilde{W}(\varsigma), 0) \leq S_1 \text{ for any } \varsigma \geq b. \tag{70}$$

Since, $\tilde{f} \in L^r$-*IFHK*$[0,b]$ and from Lemma 2, $\tilde{W}$ is continuous on $[0,b]$. And according to Lemma 4, $\tilde{W}$ is bounded on $[0,b]$, thus there exists $S_2 > 0$ such that

$$\mathbb{D}(\tilde{W}(\varsigma), 0) \leq S_2 \text{ for any } \varsigma \in [0,b]. \tag{71}$$

$$\text{Put } S=\max\{S_1,S_2\},\ \text{then } \mathbb{D}(\tilde{W}(\varsigma),0)\leq S \text{ for any } \varsigma\in[0,\infty). \tag{72}$$

Finally, we shall prove that $\lim_{\varsigma\to\infty}\int_0^{\varsigma}\tilde{f}(t)e^{-p_0 t}dt$ exists. From above, there exists $S>0$ such that for any $\varsigma\geq 0$,

$$\mathbb{D}\left(\int_0^{\varsigma}\tilde{f}(t)dt,0\right)\leq S. \tag{73}$$

Therefore, for any $\alpha,\beta\in[0,1]$ and for any $\varsigma\geq 0$,

$$\left|\int_0^{\varsigma}f_{\alpha}^{-}(t)dt\right|<S,\left|\int_0^{\varsigma}f_{\alpha}^{+}(t)dt\right|<S;\left|\int_0^{\varsigma}f_{\beta}^{-}(t)dt\right|<S,\left|\int_0^{\varsigma}f_{\beta}^{+}(t)dt\right|<S. \tag{74}$$

Also, $e^{-p_0 t}$ is a positive and monotone function and $\lim_{t\to\infty}e^{-p_0 t}=0$. Therefore, for any $\epsilon>0$ there exists $b>0$. When $t>b$, we have

$$|e^{-p_0 t}|<\frac{\epsilon}{4S}. \tag{75}$$

Let $h>g>b$, for any $\alpha\in[0,1]$ and by Lemma 3, there exists $\vartheta\in[g,h]$ such that

$$\begin{aligned}\left|\int_g^h f_{\alpha}^{-}(t)e^{-p_0 t}dt\right|&=\left|e^{-p_0 g}\int_g^{\vartheta}f_{\alpha}^{-}(t)dt+e^{-p_0 h}\int_{\vartheta}^{h}f_{\alpha}^{-}(t)dt\right|\\&=|e^{-p_0 g}(F_{\alpha}^{-}(\vartheta)-F_{\alpha}^{-}(c))+e^{-p_0 h}(F_{\alpha}^{-}(d)-F_{\alpha}^{-}(\vartheta))|\\&\leq 2e^{-p_0 g}S+2e^{-p_0 h}S\leq\epsilon,\end{aligned} \tag{76}$$

where $F_{\alpha}^{-}(\varsigma)=\int_0^{\varsigma}f_{\alpha}^{-}(t)dt$.

Similarly, for $\alpha,\beta\in[0,1]$,

$$\left|\int_g^h f_{\alpha}^{+}(t)e^{-p_0 t}dt\right|\leq\epsilon,\quad\left|\int_g^h f_{\beta}^{-}(t)e^{-p_0 t}dt\right|\leq\epsilon,\quad\left|\int_g^h f_{\beta}^{+}(t)e^{-p_0 t}dt\right|\leq\epsilon. \tag{77}$$

Hence, when $h>g>b^*$,

$$\mathbb{D}\left((L^r\text{-}IFHK)\int_0^g\tilde{f}(t)e^{-p_0 t}dt,(L^r\text{-}IFHK)\int_0^h\tilde{f}(t)e^{-p_0 t}dt\right)\leq\epsilon. \tag{78}$$

Thus,

$$\lim_{\varsigma\to\infty}(L^r\text{-}IFHK)\int_0^{\varsigma}\tilde{f}(t)e^{-p_0 t}dt \tag{79}$$

exists and hence by Theorem 8(iii),

$$(L^r\text{-}IFHK)\int_0^{\infty}\tilde{f}(t)e^{-p_0 t}dt \tag{80}$$

exists. Since p_0 is arbitrary, $\hat{L}[\tilde{f}(t)]$ exists for any $p\in[0,\infty)$. □

Some basic properties of intuitionistic fuzzy Laplace transform based on L^r-*IFHK*

Let $\tilde{f}$ and $\tilde{g}$ be continuous intuitionistic fuzzy number valued functions on $[0,\infty)$, then:

(i) $\hat{L}[C_1\tilde{f}(t)+C_2\tilde{g}(t)]=C_1\hat{L}[\tilde{f}(t)]+C_2\hat{L}[\tilde{g}(t)]$, where C_1 and C_2 are constants.

(ii) $\hat{L}[\lambda\tilde{f}(t)]=\lambda\hat{L}[\tilde{f}(t)],\lambda\geq 0$.

(iii) $\hat{L}[\tilde{f}(\varsigma)]=\tilde{F}(p)$, then $\hat{L}[e^{a\varsigma}\tilde{f}(\varsigma)]=\tilde{F}(p-a)$ where $e^{a\varsigma}$ is a real valued function and $p-a>0$.

6 Conclusion

In this study, we aim to extend L^r-Henstock–Kurzweil integral for intuitionistic fuzzy number valued functions. Firstly, we propose the concept and show some properties of the integral such as the primitive is unique and L^r-continuous everywhere. We further establish that L^r-intuitionistic fuzzy Henstock–Kurzweil integral (L^r-$IFHK$) generalizes intuitionistic fuzzy Henstock–Kurzweil integral. Secondly, we extend L^r-$IFHK$ on infinite interval, show that the primitive exists as an intuitionistic fuzzy number and that the α- cut and β- cut of L^r-$IFHK$ integrable functions are L^r-Henstock–Kurzweil integrable. Lastly, as an application we define intuitionistic Laplace transform based on L^r-$IFHK$ and establish the existence of intuitionistic Laplace transform. Our results generalize intuitionistic fuzzy Laplace transform with the existing study of intuitionistic fuzzy Laplace transform based on HK-integral. For future work, we plan on applying intuitionistic fuzzy Laplace transform based on L^r-$IFHK$ to solve intuitionistic fuzzy initial value problems and intuitionistic fuzzy Volterra integro differential equations.

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