Notes on Intuitionistic Fuzzy Sets
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Modified level operator $N_{\gamma_1}^{\gamma_2}$ applied over interval valued intuitionistic fuzzy sets

Vassia Atanassova

Bioinformatics and Mathematical Modelling Department
Institute of Biophysics and Biomedical Engineering
Bulgarian Academy of Sciences
105 Acad. Georgi Bonchev Str., Sofia 1113, Bulgaria
e-mail: vassia.atanassova@gmail.com

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Abstract. The recently proposed intuitionistic fuzzy level operator N_γ generates a subset of an intuitionistic fuzzy set A, where the elements of the subset are those elements of A, for which the ratio of their degrees of membership to their degrees of non-membership is greater than or equal to a given constant $\gamma > 0$. Here we propose a continuation of this idea from the case of intuitionistic fuzzy sets to the case of interval-valued intuitionistic fuzzy sets. This modification requires us to introduce a second constant, i.e. γ_1, $\gamma_2 > 0$. We show that there are twenty possible scenarios for the mutual position of the intervalized level operator $N_{\gamma_1}^{\gamma_2}$ and the element of the interval-valued intuitionistic fuzzy set, and give the respective formulas which calculate in each case the membership and non-membership degrees with which the IVIFS element belongs to the set defined by the operator $N_{\gamma_1}^{\gamma_2}$. These twenty scenarios are graphically interpreted in the intuitionistic fuzzy interpretational triangle, and the respective formulas have been derived. In conclusion, further ideas of research have been suggested.

Keywords: Interval valued intuitionistic fuzzy sets, Intuitionistic fuzzy sets, Level operator, Decision making under uncertainty.
2010 Mathematics Subject Classification: 03E72.

1 Introduction

Extending the concept of fuzzy sets of level α, in [3] K. Atanassov introduced the concept of (α,β)-set, generated by the intuitionistic fuzzy set A in a universe E, where α, β are fixed numbers in the [0, 1]-interval, for which $\alpha + \beta \leq 1$. The formal notation of the operator that produces this (α,β)-set is the following:

$$N_{\alpha,\beta}(A) = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E \ \&\ \mu_A(x) \geq \alpha \ \&\ \nu_A(x) \leq \beta\}.$$

Hence, this operator reduces the number of elements of the set A, retrieving only those elements whose degrees of membership are above a given level (threshold) α and their degrees of non-membership are below a given level β. A series of properties of the operator $N_{\alpha,\beta}$ are checked, involving the set-theoretic operations "negation", "union", "intersection" and the relation "inclusion". $N_{\alpha,\beta}$ is also called to be a 'level operator'.

In continuation of this idea, in [5] the author proposed a new level operator N_γ, which employs the ratio γ of the membership to non-membership values of the elements of the set, instead of thresholds for these functions. Thus the new operator returns in the resultant subset only those elements of the set that maintain a ratio greater than or equal to a predefined number γ. As it was noted in [5], the reader may find it interesting that the idea about this new operator was inspired by the theory of the American psychologist John M. Gottman, stating that the marital relationships are likely to be stable if they exhibit the "magic ratio" of 5:1 of positive to negative interactions between the partners (see [7]). In comparison with the resultant set of the level operator $N_{\alpha,\beta}$ that of the new level operator N_γ, the elements of the γ-set may exhibit higher uncertainty, as long as the ratio between their membership and non-membership values are also maintained high enough (see Figure 1.1).

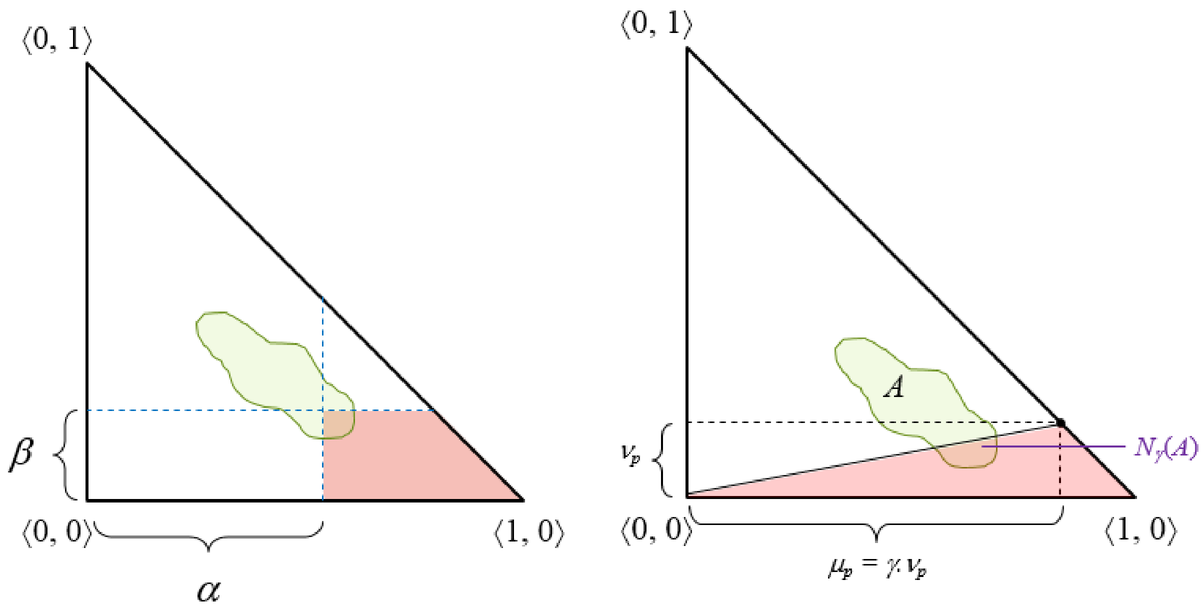

Figure 1.1. Comparison between the results
of the level operators $N_{\alpha,\beta}$ (left) and N_γ (right).

In this paper, the idea of modifying the level operator N_γ from the case of intuitionistic fuzzy sets to the case of interval-valued intuitionistic fuzzy sets. Section 2 gives some preliminaries and Section 3 gives the main results of the paper.

2 Preliminaries

Here we will remind the reader of some definitions from the area of intuitionistic fuzzy sets (see [1], Chapter 1 in [3]) and interval-valued intuitionistic fuzzy sets (see [4], Chapter 2 in [3]).

Definition 1. An intuitionistic fuzzy set (IFS) A over E is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\}$, where the functions $\mu_A : E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ define the degrees of membership and function of non-membership, respectively, of each element $x \in E$, and the condition holds: $\mu_A(x) + \nu_A(x) \leq 1$. The value $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the degree of uncertainty of the element $x \in E$ to the IFS A.

Definition 2. An interval valued intuitionistic fuzzy set (IVIFS) A^* over E is defined as an object of the form

$$A^* = \{\langle x, M_{A^*}(x), N_{A^*}(x)\rangle \mid x \in E\},$$

where $M_{A^*}(x) \subset [0, 1]$ and $N_{A^*}(x) \subset [0, 1]$ are intervals and for all $x \in E$, and the condition holds

$$\sup M_{A^*}(x) + \sup N_{A^*}(x) \leq 1.$$

Analogously to the IFS case, in case of $\sup M_{A^*}(x) + \sup N_{A^*}(x) < 1$, this gives rise to the interval $P_{A^*}(x)$, whose length equals $1 - \sup M_{A^*}(x) - \sup N_{A^*}(x)$, which attributes to the uncertainty of the element $x \in E$ to the IVIFS A^*.

Obviously, the definition of IVIFS is constructed analogously to the definition of an IFS. The geometrical interpretation is slightly more complex than that of an IFS (see [2]), and both geometrical interpretations are given below in Figure 2.1.

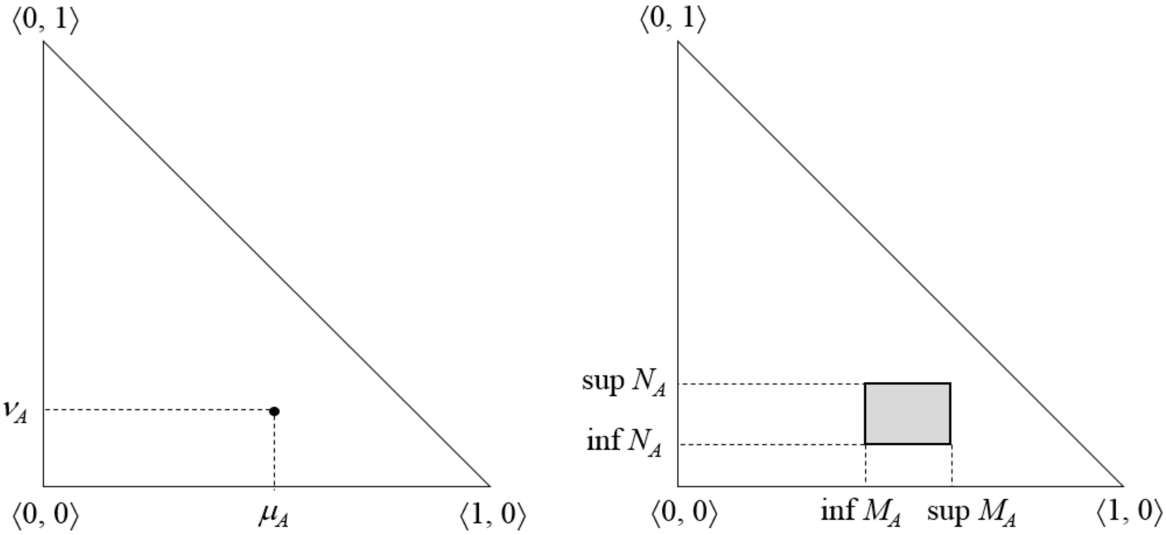

Figure 2.1. Geometrical interpretations of an element of an IFS (left) and an element of an IVIFS (right)

The other definition which we will be working with throughout this paper is that of the level operator N_γ., which we will modify as a next step of research.

Definition 3 (see [7]): Let us call an IFS A *ν-positive*, if for each IFS A we have $(\forall x \in E)(\nu_A(x) > 0)$. Let us define for each ν-positive IFS A the following operator

$$N_\gamma(A) = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E \ \& \ \frac{\mu_A(x)}{\nu_A(x)} \geq \gamma\},$$

where γ is an arbitrary non-negative real number.

3 Main results

Here we asked ourselves, what if we consider not just one segment that represents the threshold γ forming the N_γ operator, but a whole interval $[\gamma_1; \gamma_2]$, whose interpretation in the intuitionistic fuzzy triangle will be that of an angle. We will only remark, that so far, we have used the traditional notations in literature, where in the context of operators over IFS N was used for the level operator $N_{\alpha,\beta}$ and respectively N_γ, while in the context of IVIFS N denotes the interval within the non-membership function takes its values. In order to avoid duplicate notation and confusion, we will denote here the modified level operator over IVIFS as $N_{\gamma_1}^{\gamma_2}$ where γ_1 stays for the upper line closer to the $\langle 0,1\rangle$ point, or the logical *Falsity*, and γ_2 stays for the lower line closer to the $\langle 1,0\rangle$ point, or the logical *Truth.* In the Figures 3.1–3.20 below, the lines will be denoted respectively by u and l (for "upper" and "lower").

The careful investigation of the intersection between an element of an IVIFS (the rectangle) and the angle formed by "intervalized" operator $N_{\gamma_1}^{\gamma_2}$ shows that there are exactly 20 possible cases, which are different from each other and unrepresentable by each other. What is interesting here are the surfaces of the figures (triangles, trapezoids or pentagons), which occur as a results of the intersection of the rectangle, representing and IVIFS element, with the two lines, representing the level operator. The proportion of the surfaces of the figures thus obtained to the surface of the rectangle can be used as an intuitionistic fuzzy measure of the membership and non-membership (as well as uncertainty) of the IVIFS element to the set defined by the level operator $N_{\gamma_1}^{\gamma_2}$.

Why 20 cases? If we take one line and one rectangle with the desired properties, inscribed in the intuitionistic fuzzy interpretational triangle, we see that there are 6 possible positions of the line against the rectangle. Given that the two lines needed for representing the "intervalized" level operator are strictly arranged (upper and lower), this gives 21 combinations. One of these however is impossible, when the two lines intersect, due to their intersection in the $\langle 0,0\rangle$ point, i.e. cannot again intersect within the rectangle. For all these 20 cases we provide below the graphic representation and the formulas for the surfaces of the figures, obtained as a result of section between the rectangle and the two components of the $N_{\gamma_1}^{\gamma_2}$ operator, as interpreted as intuitionistic fuzzy values.

Let us first start with the notations used. The upper line u and the lower line l have the respective equations:

$$u: \quad y = \frac{p}{q}x; \quad l: \quad y = \frac{r}{s}x$$

For the sake of brevity and readability, let us also introduce the following notations:

$$\inf M = a, \quad \sup M = b, \quad \inf N = c, \quad \inf N = d.$$

Last but not least, from this point forward, by $\mu_{\gamma_1}^{\gamma_2}$ and $\nu_{\gamma_1}^{\gamma_2}$ we will denote the ratios of the surfaces of the figures, cut from the rectangle, located respectively under line l and above line u, to the surface of the rectangle. Obviously, these ratios are numbers in the [0, 1]-interval, hence can be treated as a representation of the membership and the non-membership of the IVIFS element to the set, generated of the level operator $N_{\gamma_1}^{\gamma_2}$.

Case 1. $\mu_{\gamma_1}^{\gamma_2} = 1$

$\nu_{\gamma_1}^{\gamma_2} = 0$

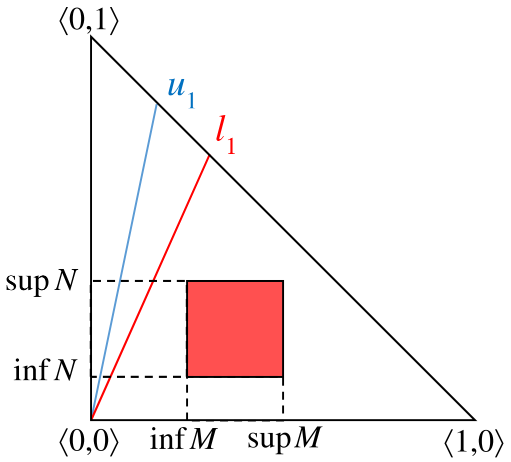

Figure 3.1 Illustration of Case 1

Case 2. $\mu_{\gamma_1}^{\gamma_2} = (b-a)(d-c) - \frac{1}{2rs}(sd - ra)^2$

$\nu_{\gamma_1}^{\gamma_2} = 0$

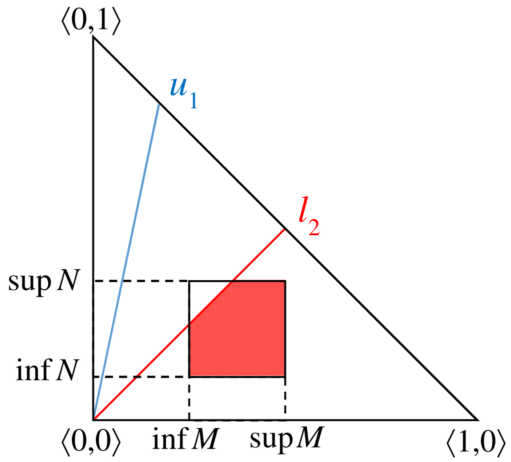

Figure 3.2 Illustration of Case 2

Case 3. $\mu_{\gamma_1}^{\gamma_2} = \frac{b-a}{2s}(r(a+b) - 2sc)$

$\nu_{\gamma_1}^{\gamma_2} = 0$

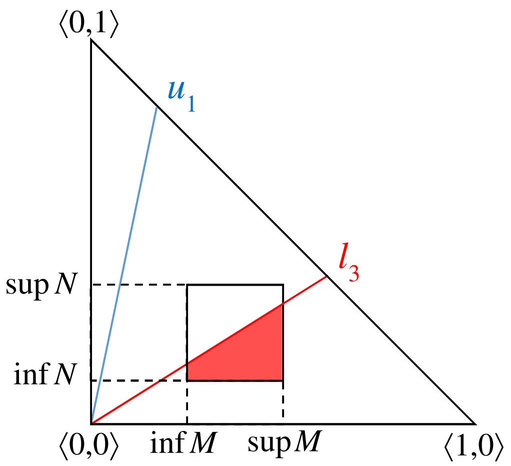

Figure 3.3 Illustration of Case 3

Case 4. $\mu_{\gamma_1}^{\gamma_2} = \frac{d-c}{2r}(2rb - s(c+d))$

$\nu_{\gamma_1}^{\gamma_2} = 0$

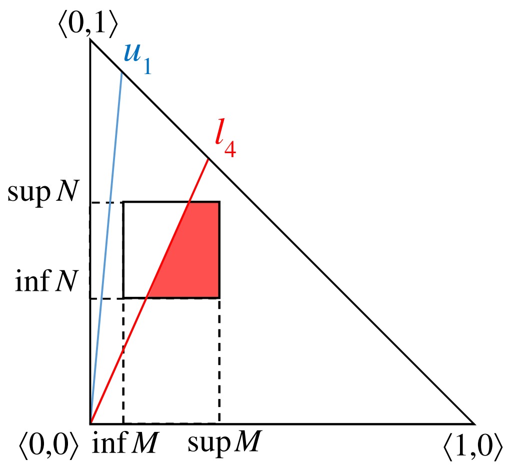

Figure 3.4 Illustration of Case 4

Case 5. $\mu_{\gamma_1}^{\gamma_2} = \frac{1}{2rs}(rb - sc)^2$

$\nu_{\gamma_1}^{\gamma_2} = 0$

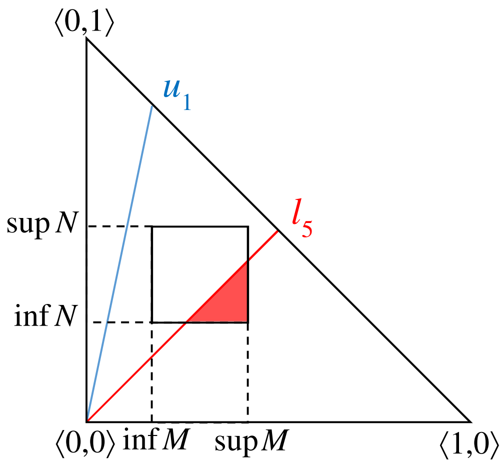

Figure 3.5 Illustration of Case 5

Case 6. $\mu_{\gamma_1}^{\gamma_2} = 0$

$\nu_{\gamma_1}^{\gamma_2} = 0$

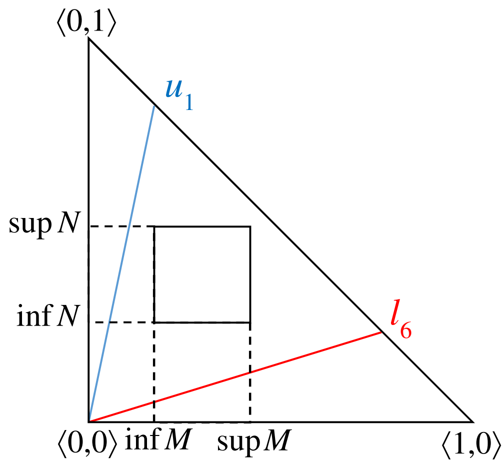

Figure 3.6 Illustration of Case 6

Case 7. $\mu_{\gamma_1}^{\gamma_2} = (b-a)(d-c) - \frac{1}{2rs}(sd - ra)^2$

$\nu_{\gamma_1}^{\gamma_2} = \frac{1}{2pq}(qd - pa)^2$

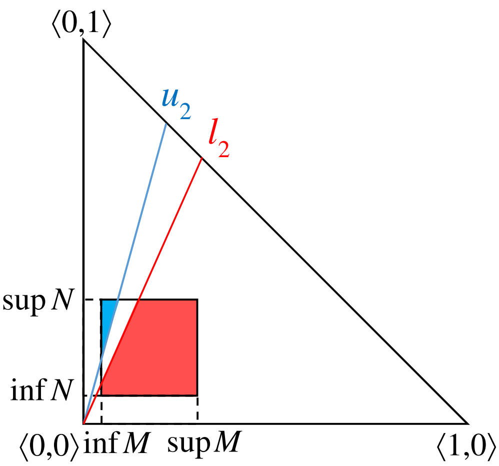

Figure 3.7 Illustration of Case 7

Case 8. $\mu_{\gamma_1}^{\gamma_2} = \frac{b-a}{2s}(r(a+b) - 2sc)$

$\nu_{\gamma_1}^{\gamma_2} = \frac{1}{2pq}(qd - pa)^2$

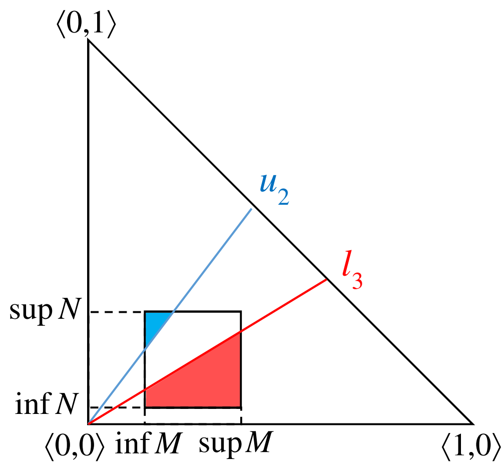

Figure 3.8 Illustration of Case 8

Case 9. $\mu_{\gamma_1}^{\gamma_2} = \dfrac{d-c}{2r}(2rb - s(c+d))$

$$\nu_{\gamma_1}^{\gamma_2} = \frac{1}{2pq}(qd - pa)^2$$

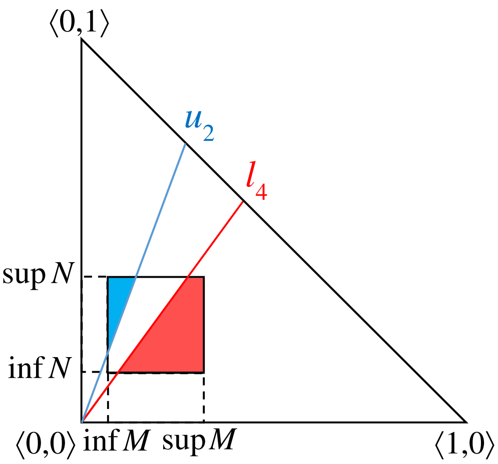

Figure 3.9 Illustration of Case 9

Case 10. $\mu_{\gamma_1}^{\gamma_2} = \dfrac{1}{2rs}(rb - sc)^2$

$$\nu_{\gamma_1}^{\gamma_2} = \frac{1}{2pq}(qd - pa)^2$$

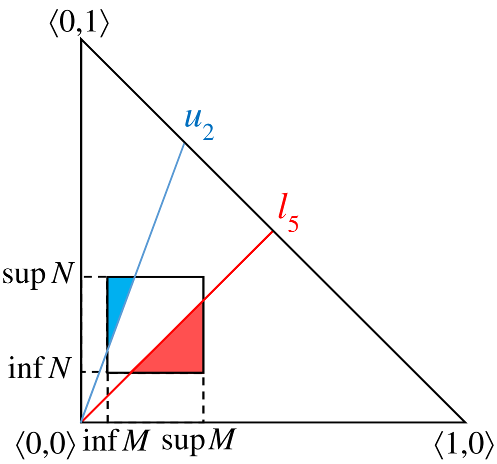

Figure 3.10 Illustration of Case 10

Case 11. $\mu_{\gamma_1}^{\gamma_2} = 0$

$$\nu_{\gamma_1}^{\gamma_2} = \frac{1}{2pq}(qd - pa)^2$$

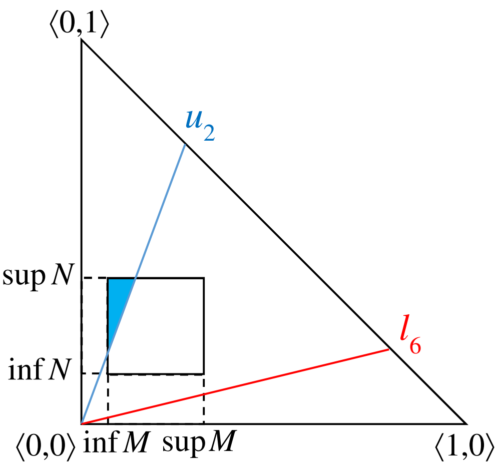

Figure 3.11 Illustration of Case 11

Case 12. $\mu_{\gamma_1}^{\gamma_2} = \dfrac{b-a}{2s}(r(a+b) - 2sc)$

$$\nu_{\gamma_1}^{\gamma_2} = \frac{d-c}{2q}(2qd - p(a+b))$$

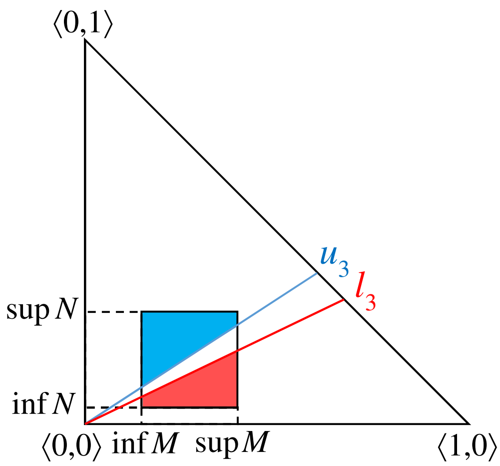

Figure 3.12 Illustration of Case 12

Case 13. $\mu_{\gamma_1}^{\gamma_2} = \dfrac{1}{2rs}(rb - sc)^2$

$$\nu_{\gamma_1}^{\gamma_2} = \frac{d-c}{2q}(2qd - p(a+b))$$

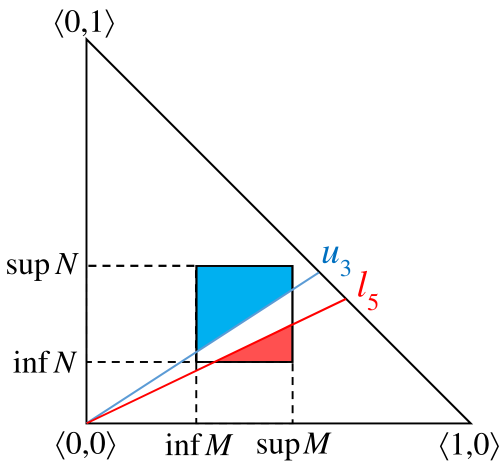

Figure 3.13 Illustration of Case 13

Case 14. $\mu_{\gamma_1}^{\gamma_2} = 0$

$$\nu_{\gamma_1}^{\gamma_2} = \frac{d-c}{2q}(2qd - p(a+b))$$

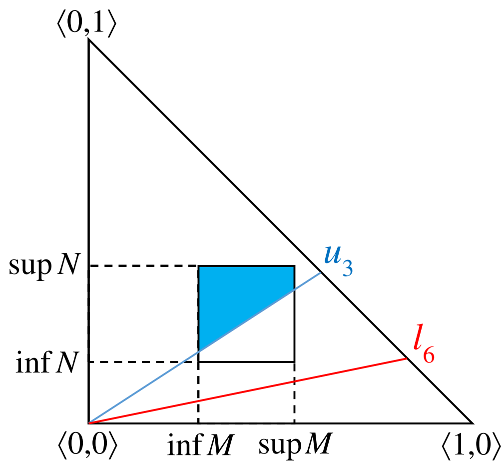

Figure 3.14 Illustration of Case 14

Case 15. $\mu_{\gamma_1}^{\gamma_2} = \dfrac{d-c}{2pr}(2prb - qrd - psc)$

$$\nu_{\gamma_1}^{\gamma_2} = \frac{d-c}{2pr}(qrd + psc - 2pra)$$

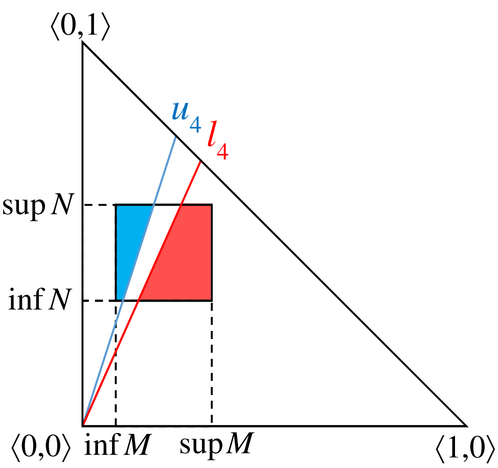

Figure 3.15 Illustration of Case 15

Case 16. $\mu_{\gamma_1}^{\gamma_2} = \dfrac{1}{2rs}(rb - sc)^2$

$$\nu_{\gamma_1}^{\gamma_2} = \frac{d-c}{2pr}(qrd + psc - 2pra)$$

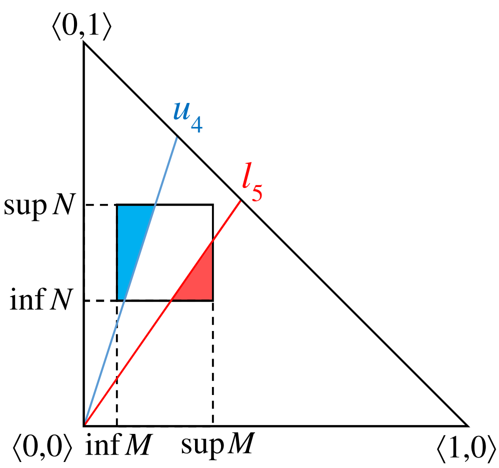

Figure 3.16 Illustration of Case 16

Case 17. $\mu_{\gamma_1}^{\gamma_2} = 0$

$$\nu_{\gamma_1}^{\gamma_2} = \frac{d-c}{2pr}(qrd + psc - 2pra)$$

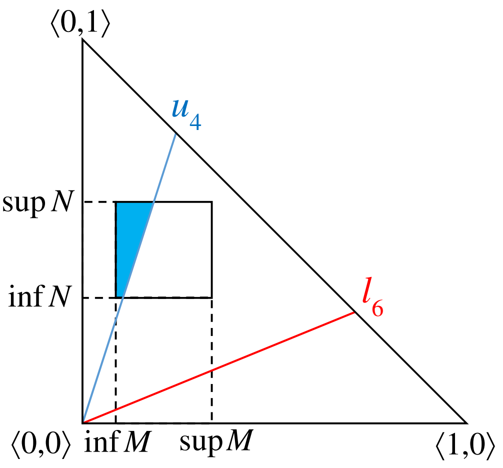

Figure 3.17 Illustration of Case 17

Case 18. $\mu_{\gamma_1}^{\gamma_2} = \frac{1}{2rs}(rb - sc)^2$

$$\nu_{\gamma_1}^{\gamma_2} = (b-a)(d-c) - \frac{1}{2pq}(pb - qc)^2$$

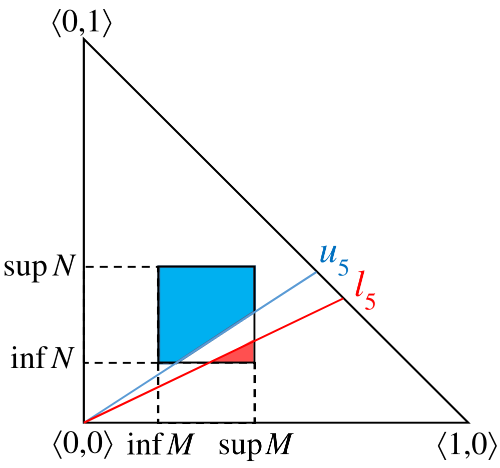

Figure 3.18 Illustration of Case 18

Case 19. $\mu_{\gamma_1}^{\gamma_2} = 0$

$$\nu_{\gamma_1}^{\gamma_2} = (b-a)(d-c) - \frac{1}{2pq}(pb - qc)^2$$

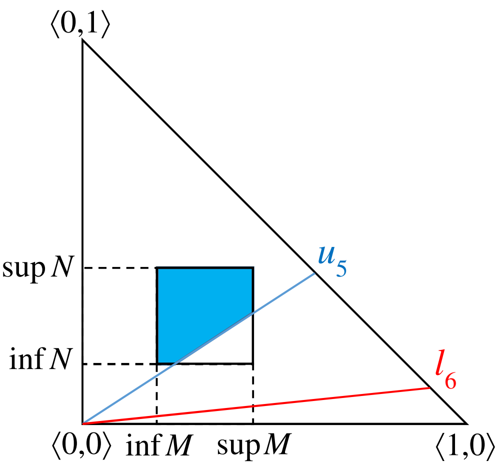

Figure 3.19 Illustration of Case 19

Case 20. $\mu_{\gamma_1}^{\gamma_2} = 0$

$$\nu_{\gamma_1}^{\gamma_2} = 1$$

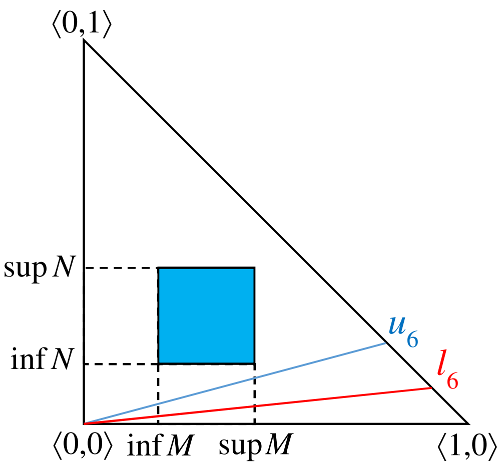

Figure 3.20 Illustration of Case 20

Now we are already ready, to give the formal definition of the level operator $N_{\gamma_1}^{\gamma_2}$ defined over an IVIFS. It has the form.

Definition. Let x be an element of an IVIFS A in universe E, defined by the intervals of membership $M_A(x)$ and non-membership $N_A(x)$, where inf $M = a$, sup $M = b$, inf $N = c$, sup $N = d$. For the real numbers γ_1, γ_2 ($\gamma_1 < \gamma_2$), are defined the lines $y_1 = \gamma_1.x$ and $y_2 = \gamma_2.x$. Then, the level operator $N_{\gamma_1}^{\gamma_2}$ over A is defined as

$$N_{\gamma_1}^{\gamma_2}(A) = \{\langle x,\ \mu_{\gamma_1}^{\gamma_2}(x),\ \nu_{\gamma_1}^{\gamma_2}(x)\rangle \mid x \in E\},$$

where $\mu_{\gamma_1}^{\gamma_2}$ is the surface of the segment of x cut off above the line y_1, normalized by the whole surface of x (= $(b - a).(d - c)$), corresponding to the membership, and $\nu_{\gamma_1}^{\gamma_2}$ is the surface of the segment of x cut off below the line y_2, normalized by the whole surface of x, corresponding to the non-membership. The surface of the segment of x cut off between the lines y_1 and y_2, normalized by the whole surface of x, corresponds to the uncertainty.

We can additionally make an observation concerning the formulas for $\mu_{\gamma_1}^{\gamma_2}$ and $\nu_{\gamma_1}^{\gamma_2}$. With their boundary values ranging from 0 to 1, we can arrange the twenty above described cases in a 6×6 table as shown on Table 1.1. It is easily seen the place of the "missing" 21-st case, explained in the beginning of the section.

$\nu = 1$	**Case 20**					
	Case 19	**Case 18**				
	Case 17	**Case 16**	**Case 15**			
	Case 14	**Case 13**		**Case 12**		
	Case 11	**Case 10**	**Case 9**	**Case 8**	**Case 7**	
$\nu = 0$	**Case 6**	**Case 5**	**Case 4**	**Case 3**	**Case 2**	**Case 1**
	$\mu = 0$					$\mu = 1$

Table 1.1. Arrangement of the 20 cases

4 Conclusion and next steps of research

The present paper is an attempt to modify and extend the recently proposed level operator N_γ over intuitionistic fuzzy sets for the case of interval-valued intuitionistic fuzzy sets. While the ordinary element of an intuitionistic fuzzy set is graphically interpreted as a point plotted on the intuitionistic fuzzy interpretational triangle, in the IVIFS case, the element of the set is graphically interpreted as a rectangle, defined by the intervals of its membership $M_A(x)$ and non-membership $N_A(x)$ for $M_A(x) \subset [0, 1]$, $N_A(x) \subset [0, 1]$ and for all $x \in E$, sup $M_A(x)$ + sup $N_A(x) \leq 1$. The "intervalization" of the level operator N_γ requires us to introduce an additional second constants and work with the interval of the membership-to-non-membership ratios γ_1, γ_2, interpreted graphically as an angle in the intuitionistic fuzzy interpretational triangle. Thus we define the extended level operator $N_{\gamma_1}^{\gamma_2}$. Depending on the mutual position of the IVIFS element and the two lines forming this angle, staying for γ_1, γ_2, we can possibly have 20 different cases, and for all of them the graphical interpretation and the respective formulas are given for the surfaces of the rectangle (IVIFS element) cut off by the lines. In the next step of research, we will investigate the properties of $N_{\gamma_1}^{\gamma_2}$ similarly to what has been done for the level operator N_γ in [5]. We will also research the usability of the new operator in particular decision making problems, for instance, problems handled with intercriteria analysis, as shown for the case of N_γ , [6].

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References

[1] Atanassov, K. T. (1983). Intuitionistic fuzzy sets. *VII ITKR Session, Sofia,* 20-23 June 1983 (Deposed in Centr. Sci.-Techn. Library of the Bulg. Acad. of Sci., 1697/84) (in Bulgarian). Reprinted: *Int. J. Bioautomation*, 2016, 20(S1), S1–S6.

[2] Atanassov K. T. (1989). Geometrical Interpretation of the Elements of the Intuitionistic Fuzzy Objects, *Mathematical Foundations of Artificial Intelligence Seminar, Sofia*, 1989, Preprint IM-MFAIS-1-89. Reprinted: *Int J Bioautomation*, 2016, 20(S1), S27–S42.

[3] Atanassov, K. T. (1999). *Intuitionistic Fuzzy Sets: Theory and Applications*. Physica-Verlag, Heidelberg.

[4] Atanassov, K. T., & Gargov, G. (1999). Interval valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 31(3), 343–349.

[5] Atanassova, V. (2017). New modified level operator N_γ over intuitionistic fuzzy sets. In: *Christiansen H., Jaudoin H., Chountas P., Andreasen T., Legind Larsen H. (eds) Flexible Query Answering Systems. FQAS 2017*. Lecture Notes in Computer Science, vol 10333. Springer, Cham, 209–214.

[6] Doukovska, L., Atanassova, V., Mavrov, D., & Radeva, I. (2018). Intercriteria Analysis of EU Competitiveness Using the Level Operator N_γ. In: *Kacprzyk J., Szmidt E., Zadrożny S., Atanassov K., Krawczak M. (eds) Advances in Fuzzy Logic and Technology 2017. EUSFLAT 2017, IWIFSGN 2017*. Advances in Intelligent Systems and Computing, Vol 641. Springer, Cham, 631–647.

[7] Gottman, J. (1995). *Why Marriages Succeed or Fail: And How You Can Make Yours Last*. Simon and Schuster.