**Notes on Intuitionistic Fuzzy Sets**
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# Solution of $n$-th order intuitionistic fuzzy differential equation by variational iteration method

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**Abstract:** In this paper, the variational iteration method proposed by Ji-Huan He is applied to solve $n$-th order intuitionistic fuzzy differential equations with intuitionistic fuzzy initial conditions. Several numerical examples are given to illustrate the efficiency of the presented method.
**Keywords:** Intuitionistic fuzzy number, Intuitionistic fuzzy differential equation, Variational iteration method.
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## 1 Introduction

One of the generalizations of fuzzy sets theory [28] is the theory of intuitionistic fuzzy set (IFS). Atanassov generalized the concept of fuzzy sets and introduced the idea of intuitionistic fuzzy sets ([2, 3, 4]). They are very necessary and powerful tool in modelling imprecision, valuable applications of IFSs have flourished in many different fields ([10, 14, 15, 22, 26, 27]).

For intuitionistic fuzzy differential equations concepts, recently the authors [8, 9, 16, 18, 19, 23, 24] established, respectively, the Cauchy problem for intuitionistic fuzzy differential equations, intuitionistic fuzzy functional differential equations, intuitionistic fuzzy differential equations, intuitionistic fuzzy fractional equations, intuitionistic fuzzy differential equations with nonlocal condition, first order homogeneous ordinary differential equations with initial value as triangular intuitionistic fuzzy numbers, and system of differential equations with initial value as triangular intuitionistic fuzzy numbers and their application. They proved the existence and uniqueness of the intuitionistic fuzzy solution for these intuitionistic fuzzy differential equations using different concepts. There are only few applications of numerical methods such as the intuitionistic fuzzy Euler and Taylor methods, Runge–Kutta of order four, Adams–Bashforth, Adams–Moulton and Predictor-Corrector methods in intuitionistic fuzzy differential equations, presented in [1, 5, 6, 7, 20, 21]. In this paper, intuitionistic fuzzy Cauchy problem is solved numerically by Runge–Kutta–Gill method of order four.

In 1999, the Variational Iteration Method (VIM) was developed by He in ([11, 12, 13]). In recent years, a great deal of attention has been devoted to the study of the method. The method introduces a reliable and efficient process for a wide variety of scientific and engineering applications because it gives rapidly convergent successive approximations of the exact solution if such a solution exists. The aim of this paper is to extend the VIM for solving linear intuisionistic fuzzy differential equations.

The paper is organized as follows. In Section 2, some basic definitions and results are brought. Section 3 contains an intuitionistic fuzzy differential equation whose numerical solution is the main focus of this paper. Solving numerically the intuitionistic fuzzy differential equation by using VIM in Section 4. An example is presented, and in the final section conclusion is drawn.

# 2 Preliminairies

## 2.1 Definition and notation

An intuitionistic fuzzy set (IFS) $A \in X$ is defined as an object of the following form

$$A = \{(x, \mu_A(x), \nu_A(x), x \in X)\},$$

where the functions $\mu_A, \nu_A(x) : X \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in X$ to the set $A$, respectively, and for every $x \in X$

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

Obviously, each ordinary fuzzy set can be written as

$$\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}.$$

**Definition 1.** *The value of*

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$$

*is called the degree of non-determinacy (or uncertainty) of the element $x \in X$ to the intuitionistic fuzzy set $A$.*

**Remark 1.** *Clearly, in the case of ordinary fuzzy sets, $\pi_A(x) = 0$ for every $x \in X$.*

The collection of all intuitionistic fuzzy numbers is denoted by

$$\mathbb{F}_1 = \{\langle u, v\rangle \ : \ \mathbb{R} \to [0,1]^2, \ \ \forall x \in \mathbb{R} \ \ 0 \leq u(x) + v(x) \leq 1\}.$$

An element $\langle u, v\rangle$ of $\mathbb{F}_1$ is said to be an intuitionistic fuzzy number if it satisfies the following conditions:

(i) $\langle u, v\rangle$ is normal, i.e., there exist $x_0$, $x_1$ such that $u(x_0) = 1$ and $v(x_1) = 1$;

(ii) $u$ is fuzzy convex and $v$ is fuzzy concave;

(iii) $u$ is upper semi-continuous and $v$ is lower semi-continuous;

(iv) $supp(u) = cl\{x \in \mathbb{R} : v(x) < 1\}$ is bounded.

**Definition 2.** [17] *For* $\alpha \in [0,1]$, *we define the upper and lower* $\alpha$*-cut by*

$$\Big[\langle u, v\rangle\Big]_\alpha = \Big\{x \in \mathbb{R} \ : \ u(x) \geq \alpha\Big\},$$
$$\Big[\langle u, v\rangle\Big]^\alpha = \Big\{x \in \mathbb{R} \ : \ v(x) \leq 1 - \alpha\Big\}.$$

**Remark 2.** *If* $\langle u, v\rangle$ *is a fuzzy number, we can see* $[\langle u, v\rangle]_\alpha$ *as* $[u]^\alpha$ *and* $[\langle u, v\rangle]^\alpha$ *as* $[1 - v]^\alpha$

A triangular intuitionistic fuzzy number (TIFN) $\langle u, v\rangle$ is an intuitionistic fuzzy set in $\mathbb{R}$ with the following membership function $u$ and non-membership function $v$:

$$u(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & \text{if } \ a_1 \leq x \leq a_2 \\[2ex] \dfrac{a_3 - x}{a_3 - a_2} & \text{if } \ a_2 \leq x \leq a_3 \\[2ex] 0 & \text{otherwise} \end{cases},$$

$$v(x) = \begin{cases} \dfrac{a_2 - x}{a_2 - a_1'} & \text{if } \ a_1' \leq x \leq a_2 \\[2ex] \dfrac{x - a_2}{a_3' - a_2} & \text{if } \ a_2 \leq x \leq a_3' \\[2ex] 1 & \text{otherwise} \end{cases},$$

where $a_1' \leq a_1 \leq a_2 \leq a_3 \leq a_3'$ and $u(x), v(x) \leq 0.5$ for $u(x) = v(x), \ \ \forall x \in \mathbb{R}$.
This TIFN is denoted by $\langle u, v\rangle = \langle a_1, a_2, a_3; a_1', a_2, a_3'\rangle$ where,

$$\Big[\langle u, v\rangle\Big]_\alpha = \Big[a_1 + \alpha(a_2 - a_1), a_3 - \alpha(a_3 - a_2)\Big],$$

$$\Big[\langle u, v\rangle\Big]^\alpha = \Big[a_1' + \alpha(a_2 - a_1'), a_3' - \alpha(a_3' - a_2)\Big].$$

**Definition 3.** [17] *Let $\langle u, v\rangle$ be an element of $I\!F_1$ and $\alpha \in [0,1]$. Then, we define the following sets:*

$$\Big[\langle u,v\rangle\Big]_l^+(\alpha) = \inf\{x \in \mathbb{R} \mid u(x) \geq \alpha\}, \quad \Big[\langle u,v\rangle\Big]_r^+(\alpha) = \sup\{x \in \mathbb{R} \mid u(x) \geq \alpha\}$$
$$\Big[\langle u,v\rangle\Big]_l^-(\alpha) = \inf\{x \in \mathbb{R} \mid v(x) \leq 1-\alpha\}, \quad \Big[\langle u,v\rangle\Big]_r^-(\alpha) = \sup\{x \in \mathbb{R} \mid v(x) \leq 1-\alpha\}$$

**Remark 3.**

$$\Big[\langle u,v\rangle\Big]_\alpha = \Big[\Big[\langle u,v\rangle\Big]_l^+(\alpha), \Big[\langle u,v\rangle\Big]_r^+(\alpha)\Big], \quad \Big[\langle u,v\rangle\Big]^\alpha = \Big[\Big[\langle u,v\rangle\Big]_l^-(\alpha), \Big[\langle u,v\rangle\Big]_r^-(\alpha)\Big]$$

We define the following operations by

$$\Big[\langle u,v\rangle \oplus \langle z,w\rangle\Big]^\alpha = \Big[\langle u,v\rangle\Big]^\alpha + \Big[\langle z,w\rangle\Big]^\alpha, \quad \Big[\lambda\langle u,v\rangle\Big]^\alpha = \lambda\Big[\langle u,v\rangle\Big]^\alpha,$$
$$\Big[\langle u,v\rangle \oplus \langle z,w\rangle\Big]_\alpha = \Big[\langle u,v\rangle\Big]_\alpha + \Big[\langle z,w\rangle\Big]_\alpha, \quad \Big[\lambda\langle u,v\rangle\Big]_\alpha = \lambda\Big[\langle u,v\rangle\Big]_\alpha,$$

where $\langle u,v\rangle$, $\langle z,w\rangle \in \mathbb{F}_1$ and $\lambda \in \mathbb{R}$.

**Definition 4.** *Let $\langle u,v\rangle$ and $\langle u',v'\rangle \in I\!F_1$. Then, the H-difference is the IFN $\langle z,w\rangle \in I\!F_1$, if it exists, such that*

$$\langle u,v\rangle \ominus \langle u',v'\rangle = \langle z,w\rangle \Longleftrightarrow \langle u,v\rangle = \langle u',v'\rangle \oplus \langle z,w\rangle.$$

On the space $\mathbb{F}_1$ we will consider the following $L_p$-metric.

**Theorem 1.** [17] *For $1 \leq p \leq \infty$*

$$\begin{aligned} d_p\Big(\langle u,v\rangle, \langle z,w\rangle\Big) = \frac{1}{4}\Big\{ \int_0^1 \Big|[\langle u,v\rangle]_r^+(\alpha) - [\langle z,w\rangle]_r^+(\alpha)\Big|^p d\alpha \\ + \int_0^1 \Big|[\langle u,v\rangle]_l^+(\alpha) - [\langle z,w\rangle]_l^+(\alpha)\Big|^p d\alpha + \int_0^1 \Big|[\langle u,v\rangle]_r^-(\alpha) - [\langle z,w\rangle]_r^-(\alpha)\Big|^p d\alpha \\ + \int_0^1 \Big|[\langle u,v\rangle]_l^-(\alpha) - [\langle z,w\rangle]_l^-(\alpha)\Big|^p d\alpha \Big\}^{\frac{1}{p}} \end{aligned}$$

*and for $p = \infty$*

$$\begin{aligned} d_\infty\Big(\langle u,v\rangle, \langle z,w\rangle\Big) = \frac{1}{4}\Big\{ \sup_{0<\alpha\leq 1} \Big|[\langle u,v\rangle]_r^+(\alpha) - [\langle z,w\rangle]_r^+(\alpha)\Big| \\ + \sup_{0<\alpha\leq 1} \Big|[\langle u,v\rangle]_l^+(\alpha) - [\langle z,w\rangle]_l^+(\alpha)\Big| + \sup_{0<\alpha\leq 1} \Big|[\langle u,v\rangle]_r^-(\alpha) - [\langle z,w\rangle]_r^-(\alpha)\Big| \\ + \sup_{0<\alpha\leq 1} \Big|[\langle u,v\rangle]_l^-(\alpha) - [\langle z,w\rangle]_l^-(\alpha)\Big| \Big\} \end{aligned}$$

*is a metric on $I\!F_1$.*

**Theorem 2.** [17] *The metric space $\Big(I\!F_1, d_\infty\Big)$ is complete.*

**Definition 5.** [18] *Let $F : [a, b] \to I\!F_1$ be an intuitionistic fuzzy valued mapping and $t_0 \in [a, b]$. Then $F$ is called intuitionistic fuzzy continuous in $t_0$ iff:*

$$\forall(\varepsilon > 0)\ (\exists\delta > 0)\ (\forall t \in [a, b]\ \ such\ \ that\ \ |t - t_0| < \delta) \Rightarrow d_p\Big(F(t), F(t_0)\Big) < \varepsilon.$$

**Definition 6.** [18] *$F$ is called intuitionistic fuzzy continuous iff is intuitionistic fuzzy continuous in every point of $[a, b]$*

**Definition 7.** *A mapping $F : [a, b] \to I\!F_1$ is said to be Hukuhara derivable at $t_0$ if there exists $F'(t_0) \in IF_1$ such that both limits:*

$$\lim_{\Delta t \to 0^+} \frac{F(t_0 + \Delta t) \ominus F(t_0)}{\Delta t}$$

*and*

$$\lim_{\Delta t \to 0^-} \frac{F(t_0) \ominus F(t_0 - \Delta t)}{\Delta t}$$

*exist and they are equal to $F'(t_0) = \langle u'(t_0), v'(t_0)\rangle$, which is called the Hukuhara derivative of $F$ at $t_0$.*

**Definition 8.** *Let $F : [a, b] \to I\!F_1$. We define the $n$-th order differential of $F$ as follows. Let $F : [a, b] \to I\!F_1$ and $t_0 \in [a, b]$. We say that $F$ is differentiable of the $n$-th order at $t_0$, if there exist elements $F^s(t_0) \in I\!F_1$, $\forall s = 1, 2, \dots, n$ such that both limits*

$$\lim_{\Delta t \to 0^+} \frac{F^{(s-1)}(t_0 + \Delta t) \ominus F^{(s-1)}(t_0)}{\Delta t}$$

*and*

$$\lim_{\Delta t \to 0^-} \frac{F^{(s-1)}(t_0) \ominus F^{(s-1)}(t_0 - \Delta t)}{\Delta t}$$

*exist and they are equal to $F^{(s)}(t_0) = \langle u^{(s)}(t_0), v^{(s)}(t_0)\rangle$.*

## 2.2 Variational iteration method

Here a description of the method (see [11, 12]) is given to handle the general nonlinear problem,

$$Lu(t) + Nu(t) = f(t), \tag{1}$$

where $L$ is a linear operator, $N$ is a nonlinear operator and $f(t)$ is given. We can construct a correction functional according to the variational method, as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)\Big(Lu(s) + N\tilde{u}(s) - f(s)\Big)ds, \tag{2}$$

where $\lambda$ is a general Lagrange multiplier, which can be identified optimally via variational theory, $u_n$ is the $n$-th approximate solution and $\tilde{u}_n$ denotes a restricted variation, which means $\delta\tilde{u}_n = 0$. Successive approximations, $u_{n+1}(t)$, will be obtained by applying the obtained Lagrange multiplier and a properly chosen initial approximation $u_0(t)$. Consequently, the solution is given by $u = \lim\limits_{n \longrightarrow \infty} u_n$.

In other words, the correction functional (2) will give a sequence of approximations and the exact solution is obtained as the limit of the successive approximations. In fact, the solution of problem (1) is considered as a fixed point of the following functional under a suitable choice of the initial term $u_0(t)$,

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)\Big(Lu(s) + Nu(s) - f(s)\Big)ds \tag{3}$$

for convergence of the variational iteration method [25], as a well known powerful tool. We have

**Theorem 3. (Banach's Fixed Point Theorem)** *Assume that $X$ is a Banach space and $A : X \longrightarrow X$ is a nonlinear mapping, and suppose that $\|A[u] - A[v]\| \leq k\|u - v\|,\ \ u, v \in X$ for some constant $k < 1$. Then $A$ has an unique fixed point. Furthermore, the sequence $u_{n+1} = A[u_n]$ with an arbitrary choice of $u_0 \in X$ converges to the fixed point of $A$.*

According to the Theorem, for the nonlinear mapping

$$A[u(t)] = u(t) + \int_0^t \lambda(s)\Big(Lu(s) + Nu(s) - f(s)\Big)ds$$

a sufficient condition for convergence of the variational iteration method is a strict contraction of $A$. Furthermore, the sequence (3) converges to the fixed point of $A$, which is also the solution of problem (1).

# 3 The intuitionistic fuzzy differential equation

In this section, we consider the initial value problem for the intuitionistic fuzzy differential equation

$$\begin{cases} x'(t) &= f(t, x(t)),\ \ t \in I \\ x(0) &= \langle u_{t_0}, v_{t_0}\rangle \in \mathbb{F}_1 \end{cases}, \tag{4}$$

where $x \in \mathbb{F}_1$ is unknown, $I = [t_0, T]$, $f : I \times \mathbb{F}_1 \to \mathbb{F}_1$ and $x(0)$ is intuitionistic fuzzy number. Sufficient conditions for the existence of an unique solution to Eq. $(4)$ are:

1. Continuity of $f$.
2. Lipschitz condition: for any pair $\Big(t, \langle u, v\rangle\Big), \Big(t, \langle u', v'\rangle\Big) \in I \times \mathbb{F}_1$, we have

$$d_\infty\Big(f\big(t, \langle u, v\rangle\big), f\big(t, \langle u', v'\rangle\big)\Big) \leq K d_\infty\Big(\langle u, v\rangle, \langle u', v'\rangle\Big), \tag{5}$$

where $K > 0$ is a given constant.

**Theorem 4.** [8] *Let us suppose that the following conditions hold*

*(a) Let $R_0 = [t_0, t_0 + p] \times \overline{B}(\langle u, v\rangle_{t_0}, q)$, $p,\ q \geq 0$, $\langle u, v\rangle_{t_0} \in \mathbb{F}_1$ where $\overline{B}(\langle u, v\rangle_{t_0}, q) = \{\langle u, v\rangle \in IF_1 : d_\infty(\langle u, v\rangle, \langle u, v\rangle_{t_0}) \leq q\}$ denote a closed ball in $\mathbb{F}_1$ and let $f : R_0 \longrightarrow \mathbb{F}_1$ be a continuous function such that $d_\infty\Big(f(t, \langle u, v\rangle), 0_{(1,0)}\Big) \leq M$ for all $(t, \langle u, v\rangle) \in R_0$.*

*(b) Let* $g : [t_0, t_0+p]\times[0, q] \longrightarrow \mathbb{R}$ *such that* $g(t, 0) \equiv 0$ *and* $0 \leq g(t, x) \leq M_1$, $\forall t \in [t_0, t_0+p]$, $0 \leq x \leq q$ *such that* $g(t, x)$ *is non-decreasing in u and g is such that the initial value problem*

$$x'(t) = g(t, x(t)), \qquad x(t_0) = x_0. \tag{6}$$

*has only the solution* $x(t) \equiv 0$ *on* $[t_0, t_0 + p]$

*(c) We have*

$$d_\infty\Big(f\Big(t, \langle u, v\rangle\Big), f\Big(t, \langle z, w\rangle\Big)\Big) \leq g\Big(t, d_\infty\Big(\langle u, v\rangle, \langle z, w\rangle\Big)\Big)$$

$\forall(t, \langle u, v\rangle)$, $(t, \langle z, w\rangle) \in R_0$, *and* $d_\infty\Big(\langle u, v\rangle, \langle z, w\rangle\Big) \leq q$.

*Then the intuitionistic fuzzy initial value problem*

$$\begin{cases} \langle u, v\rangle = f(t, \langle u, v\rangle), \\ \langle u, v\rangle(t_0) = \langle u, v\rangle_{t_0} \end{cases} \tag{7}$$

*has a unique solution*

$$\langle u, v\rangle \in \mathcal{C}^1\Big([t_0, t_0 + r], B(x_0, q)\Big), \quad \textit{where} \quad r = \min\Big\{p, \frac{q}{M}, \frac{q}{M_1}, d\Big\}$$

*and the successive iterations*

$$\langle u, v\rangle_0(t) = \langle u, v\rangle_{t_0}, \;\; \langle u, v\rangle_{n+1}(t) = \langle u, v\rangle_{t_0} + \int_{t_0}^{t} f\Big(s, \langle u, v\rangle_n(s)\Big)ds \tag{8}$$

*converge to* $\langle u, v\rangle(t)$ *on* $[t_0, t_0 + r]$.

# 4 Variational iteration method

## 4.1 VIM for first-order intuitionistic fuzzy differential equation

Consider the initial value problem (4). To solve this problem we construct the following correction functional

$$[x_{n+1}(t)]_l^+(\alpha) = [x_n(t)]_l^+ + \int_0^t \lambda_1(s)\big([x'_n(s)]_l^+(\alpha) - f_l^+\big(t, [\tilde{x_n}(s)]_l^+(\alpha), [\tilde{x_n}(s)]_r^+(\alpha)\big)\big)ds,$$

$$[x_{n+1}(t)]_r^+(\alpha) = [x_n(t)]_r^+ + \int_0^t \lambda_2(s)\big([x'_n(s)]_r^+(\alpha) - f_r^+\big(t, [\tilde{x_n}(s)]_l^+(\alpha), [\tilde{x_n}(s)]_r^+(\alpha)\big)\big)ds,$$

$$[x_{n+1}(t)]_l^-(\alpha) = [x_n(t)]_l^- + \int_0^t \lambda_3(s)\big([x'_n(s)]_l^-(\alpha) - f_l^-\big(t, [\tilde{x_n}(s)]_l^-(\alpha), [\tilde{x_n}(s)]_r^-(\alpha)\big)\big)ds,$$

$$[x_{n+1}(t)]_r^-(\alpha) = [x_n(t)]_r^- + \int_0^t \lambda_4(s)\big([x'_n(s)]_r^-(\alpha) - f_r^-\big(t, [\tilde{x_n}(s)]_l^-(\alpha), [\tilde{x_n}(s)]_r^-(\alpha)\big)\big)ds,$$

where $\lambda_1(s)$, $\lambda_2(s)$, $\lambda_3(s)$, $\lambda_4(s)$ are general Lagrange multipliers (which can be identified optimally via the variational theory), and $[\tilde{x_n}(t)]_l^+$, $[\tilde{x_n}(t)]_r^+$, $[\tilde{x_n}(t)]_l^-$ and $[\tilde{x_n}(t)]_r^-$ are considered as restricted variations, i.e., $\delta[\tilde{x_n}(t)]_l^+ = \delta[\tilde{x_n}(t)]_r^+ = \delta[\tilde{x_n}(t)]_l^- = \delta[\tilde{x_n}(t)]_r^- = 0$.

$$\delta[x_{n+1}(t)]_l^+(\alpha) = \delta[x_n(t)]_l^+ + \delta\int_0^t \lambda_1(s)\big([x_n'(s)]_l^+(\alpha) - f_l^+\big(t, [\tilde{x_n}(s)]_l^+(\alpha), [\tilde{x_n}(s)]_r^+(\alpha)\big)\big)ds,$$
$$\delta[x_{n+1}(t)]_r^+(\alpha) = \delta[x_n(t)]_r^+ + \delta\int_0^t \lambda_2(s)\big([x_n'(s)]_r^+(\alpha) - f_r^+\big(t, [\tilde{x_n}(s)]_l^+(\alpha), [\tilde{x_n}(s)]_r^+(\alpha)\big)\big)ds,$$
$$\delta[x_{n+1}(t)]_l^-(\alpha) = \delta[x_n(t)]_l^- + \delta\int_0^t \lambda_3(s)\big([x_n'(s)]_l^-(\alpha) - f_l^-\big(t, [\tilde{x_n}(s)]_l^-(\alpha), [\tilde{x_n}(s)]_r^-(\alpha)\big)\big)ds,$$
$$\delta[x_{n+1}(t)]_r^-(\alpha) = \delta[x_n(t)]_r^- + \delta\int_0^t \lambda_4(s)\big([x_n'(t)]_r^-(\alpha) - f_r^-\big(t, [\tilde{x_n}(s)]_l^-(\alpha), [\tilde{x_n}(s)]_r^-(\alpha)\big)\big)ds.$$

Making the above correction functional stationary, we obtain the following stationary condition

$$\begin{aligned}
&\lambda_1'(s) = \lambda_2'(s) = \lambda_3'(s) = \lambda_4'(s) = 0,\\
&1 + \lambda_1(s)|_{s=t} = 1 + \lambda_2(s)|_{s=t} = 1 + \lambda_3(s)|_{s=t} = 1 + \lambda_4(s)|_{s=t} = 0.
\end{aligned}$$

The Lagrange multipliers, therefore, can be identified in the following form:

$$\lambda_1(s) = \lambda_2(s) = \lambda_3(s) = \lambda_4(s) = -1.$$

So we have

$$[x_{n+1}(t)]_l^+(\alpha) = [x_n(t)]_l^+ - \int_0^t \big([x_n'(s)]_l^+(\alpha) - f_l^+\big(t, [x_n(s)]_l^+(\alpha), [x_n(s)]_r^+(\alpha)\big)\big)ds,$$
$$[x_{n+1}(t)]_r^+(\alpha) = [x_n(t)]_r^+ - \int_0^t \big([x_n'(s)]_r^+(\alpha) - f_r^+\big(t, [x_n(s)]_l^+(\alpha), [x_n(s)]_r^+(\alpha)\big)\big)ds,$$
$$[x_{n+1}(t)]_l^-(\alpha) = [x_n(t)]_l^- - \int_0^t \big([x_n'(s)]_l^-(\alpha) - f_l^-\big(t, [x_n(s)]_l^-(\alpha), [x_n(s)]_r^-(\alpha)\big)\big)ds,$$
$$[x_{n+1}(t)]_r^-(\alpha) = [x_n(t)]_r^- - \int_0^t \big([x_n'(s)]_r^-(\alpha) - f_r^-\big(t, [x_n(s)]_l^-(\alpha), [x_n(s)]_r^-(\alpha)\big)\big)ds,$$

where $[x_0(t)]_l^+ = [x(0)]_l^+$, $[x_0(t)]_l^- = [x(0)]_l^-$, $[x_0(t)]_r^+ = [x(0)]_r^+$ and $[x_0(t)]_r^- = [x(0)]_r^-$. Then, the series of approximations $[x(t)]_l^+$, $[x(t)]_r^+$, $[x(t)]_l^-$ and $[x(t)]_r^-$ can be determined. Consequently, the solution is given by

$$\begin{aligned}
\Big[x(t)\Big]_l^+ &= \lim_{n\longrightarrow\infty}\Big[x_n(t)\Big]_l^+, \qquad & \Big[x(t)\Big]_r^+ &= \lim_{n\longrightarrow\infty}\Big[x_n(t)\Big]_r^+,\\
\Big[x(t)\Big]_l^- &= \lim_{n\longrightarrow\infty}\Big[x_n(t)\Big]_l^-, \qquad & \Big[x(t)\Big]_r^- &= \lim_{n\longrightarrow\infty}\Big[x_n(t)\Big]_r^-.
\end{aligned}$$

## 4.2 VIM for $n$-th order intuitionistic fuzzy differential equation

Consider the following $n$-th order intuitionistic fuzzy differential equation

$$\begin{cases} x^n(t) = f\Big(t, x(t), x'(t), \ldots, x^n\Big), & t \in I \\ x^i(t_0) = m_i \in \mathbb{F}_1 \end{cases}, \tag{9}$$

where $i = 0, 1, \ldots, n-1$.

Using VIM to solve (9), we obtain

$$\lambda_1(s) = \lambda_2(s) = \lambda_3(s) = \lambda_4(s) = (-1)^n \frac{(s-t)^{n-1}}{(n-1)!}. \tag{10}$$

After identifying the multiplier, we have

$$\begin{aligned}[x_{k+1}(t)]_l^+(\alpha) &= [x_k(t)]_l^+(\alpha) \\ &+ (-1)^n \int_0^t \frac{(s-t)^{n-1}}{(n-1)!} \Big[ [x_k^{(n)}(s)]_l^+(\alpha) - f_l^+\Big(t, [x_k^{(i)}(s)]_l^+(\alpha), [x_k^{(i)}(s)]_r^+(\alpha)\Big)\Big] ds, \end{aligned} \tag{11}$$

$$\begin{aligned}[x_{k+1}(t)]_r^+(\alpha) &= [x_k(t)]_r^+(\alpha) \\ &+ (-1)^n \int_0^t \frac{(s-t)^{n-1}}{(n-1)!} \Big[ [x_k^{(n)}(s)]_r^+(\alpha) - f_r^+\Big(t, [x_k^{(i)}(s)]_l^+(\alpha), [x_k^{(i)}(s)]_r^+(\alpha)\Big)\Big] ds, \end{aligned} \tag{12}$$

$$\begin{aligned}[x_{k+1}(t)]_l^-(\alpha) &= [x_k(t)]_l^-(\alpha) \\ &+ (-1)^n \int_0^t \frac{(s-t)^{n-1}}{(n-1)!} \Big[ [x_k^{(n)}(s)]_l^-(\alpha) - f_l^-\Big(t, [x_k^{(i)}(s)]_l^-(\alpha), [x_k^{(i)}(s)]_r^-(\alpha)\Big)\Big] ds, \end{aligned} \tag{13}$$

$$\begin{aligned}[x_{k+1}(t)]_r^-(\alpha) &= [x_k(t)]_r^-(\alpha) \\ &+ (-1)^n \int_0^t \frac{(s-t)^{n-1}}{(n-1)!} \Big[ [x_k^{(n)}(s)]_r^-(\alpha) - f_r^-\Big(t, [x_k^{(i)}(s)]_l^-(\alpha), [x_k^{(i)}(s)]_r^-(\alpha)\Big)\Big] ds, \end{aligned} \tag{14}$$

where $i = 0, 1, \ldots, n-1$.
For any initial solution $x_0(t)$, we have

$$[x_0(t)]_l^+ = [x(0)]_l^+ + t[x^{(1)}(0)]_l^+ + \frac{1}{2}t^2[x^{(2)}(0)]_l^+ + \cdots + \frac{1}{k!}t^k[x^{(k)}(0)]_l^+, \tag{15}$$

$$[x_0(t)]_r^+ = [x(0)]_r^+ + t[x^{(1)}(0)]_r^+ + \frac{1}{2}t^2[x^{(2)}(0)]_r^+ + \cdots + \frac{1}{k!}t^k[x^{(k)}(0)]_r^+, \tag{16}$$

$$[x_0(t)]_l^- = [x(0)]_l^- + t[x^{(1)}(0)]_l^- + \frac{1}{2}t^2[x^{(2)}(0)]_l^- + \cdots + \frac{1}{k!}t^k[x^{(k)}(0)]_l^-, \tag{17}$$

$$[x_0(t)]_r^- = [x(0)]_r^- + t[x^{(1)}(0)]_r^- + \frac{1}{2}t^2[x^{(2)}(0)]_r^- + \cdots + \frac{1}{k!}t^k[x^{(k)}(0)]_r^-. \tag{18}$$

This leads to a series solution converging to the exact solution.

According to Banach's Fixed Point Theorem, it is easy to obtain the convergence condition for the sequences obtained from (11), (12), (13) and (14).

**Theorem 5.** *Define a nonlinear mapping*

$$\begin{aligned}T_1\Big[x(t)\Big]_l^+(\alpha) &= \Big[x(t)\Big]_l^+(\alpha) \\ &+ (-1)^n \int_0^t \frac{(s-t)^{n-1}}{(n-1)!} \Big[ [x^{(n)}(s)]_l^+(\alpha) - f_l^+\Big(t, [x^{(i)}(s)]_l^+(\alpha), [x^{(i)}(s)]_r^+(\alpha)\Big)\Big] ds \end{aligned} \tag{19}$$

$$T_2\Big[x(t)\Big]_r^+(\alpha) = \Big[x(t)\Big]_r^+(\alpha)$$
$$+(-1)^n \int_0^t \frac{(s-t)^{n-1}}{(n-1)!}\Bigg[[x^{(n)}(s)]_r^+(\alpha) - f_r^+\Big(t, [x^{(i)}(s)]_l^+(\alpha), [x^{(i)}(s)]_r^+(\alpha)\Big)\Bigg] ds \quad (20)$$

$$T_3\Big[x(t)\Big]_l^-(\alpha) = \Big[x(t)\Big]_l^-(\alpha)$$
$$+(-1)^n \int_0^t \frac{(s-t)^{n-1}}{(n-1)!}\Bigg[[x^{(n)}(s)]_l^-(\alpha) - f_l^-\Big(t, [x^{(i)}(s)]_l^-(\alpha), [x^{(i)}(s)]_r^-(\alpha)\Big)\Bigg] ds \quad (21)$$

$$T_4\Big[x(t)\Big]_r^-(\alpha) = \Big[x(t)\Big]_r^-(\alpha)$$
$$+(-1)^n \int_0^t \frac{(s-t)^{n-1}}{(n-1)!}\Bigg[[x^{(n)}(s)]_r^-(\alpha) - f_r^-\Big(t, [x^{(i)}(s)]_l^-(\alpha), [x^{(i)}(s)]_r^-(\alpha)\Big)\Bigg] ds \quad (22)$$

*however, the strict contraction of functions $T_1$, $T_2$, $T_3$ and $T_4$ implies the convergence of the iterative sequences $\Big[x_n(t)\Big]_l^+(\alpha)$, $\Big[x_n(t)\Big]_r^+(\alpha)$, $\Big[x_n(t)\Big]_l^-(\alpha)$ and $\Big[x_n(t)\Big]_r^-(\alpha)$. So the limit is a solution of (9).*

**Example 1** Consider the following second-order intuitionistic fuzzy differential equation

$$\begin{cases} y'' + y = -t, \quad 0 \le t \le 1 \\ y(0) = (0.1r - 0.1, 0.1 - 0.1r, -0.2r, 0.2r) \\ y'(0) = (0.088 + 0.1r, 0.288 - 0.1r, 0.188 + 0.178r, 0.188 + 0.288r) \end{cases} . \quad (23)$$

The exact intuitionistic fuzzy solution is:

$$\Big[y(t,r)\Big]_l^+ = (0.1r - 0.1)\cos t + (1.088 + 0.1r)\sin t - t,$$
$$\Big[y(t,r)\Big]_r^+ = (0.1 - 0.1r)\cos t + (1.288 - 0.1r)\sin t - t,$$
$$\Big[y(t,r)\Big]_l^- = -0.2r\cos t + (1.188 + 0.178r)\sin t - t,$$
$$\Big[y(t,r)\Big]_r^- = 0.2r\cos t + (1.188 + 0.288r)\sin t - t.$$

Using the VIM, we obtain

$$[y_{n+1}(t,r)]_l^+ = [y_n(t,r)]_l^+ + \int_0^t \sin(s-t)\{\frac{d^2}{ds^2}[y_n(t,r)]_l^+ + [y_n(t,r)]_l^+ + s\},$$

$$[y_{n+1}(t,r)]_r^+ = [y_n(t,r)]_r^+ + \int_0^t \sin(s-t)\{\frac{d^2}{ds^2}[y_n(t,r)]_r^+ + [y_n(t,r)]_r^+ + s\},$$

$$[y_{n+1}(t,r)]_l^- = [y_n(t,r)]_l^- + \int_0^t \sin(s-t)\{\frac{d^2}{ds^2}[y_n(t,r)]_l^- + [y_n(t,r)]_l^- + s\},$$

$$[y_{n+1}(t,r)]_r^- = [y_n(t,r)]_r^- + \int_0^t \sin(s-t)\{\frac{d^2}{ds^2}[y_n(t,r)]_r^- + [y_n(t,r)]_r^- + s\}.$$

If we start by

$$\Big[y_0(t,r)\Big]_l^+ = (0.1r-0.1)+t(0.088+0.1r)$$
$$\Big[y_0(t,r)\Big]_r^+ = (0.1-0.1r)+t(0.288-0.1r)$$

and

$$\Big[y_0(t,r)\Big]_l^- = (-0.2r)+t(00.188+0.178r)$$
$$\Big[y_0(t,r)\Big]_r^- = (0.2r)+t(0.188+0.288r),$$

then we obtain

$$\Big[y(t,r)\Big]_l^+ = (0.1r-0.1)\cos t+(1.088+0.1r)\sin t-t$$
$$\Big[y(t,r)\Big]_r^+ = (0.1-0.1r)\cos t+(1.288-0.1r)\sin t-t$$
$$\Big[y(t,r)\Big]_l^- = -0.2r\cos t+(1.188+0.178r)\sin t-t$$
$$\Big[y(t,r)\Big]_r^- = 0.2r\cos t+(1.188+0.288r)\sin t-t$$

which is the exact explicit solution.

**Example 2.** Consider the following fourth-order intuitionistic fuzzy differential equation

$$\begin{cases} y^{(4)}-y=0 \\ y(0)=(r-1,1-r,-2r,2r) \\ y'(0)=(r-1,1-r,-2r,2r) \\ y''(0)=(r-1,1-r,-2r,2r) \\ y'''(0)=(r-1,1-r,-2r,2r) \end{cases} \tag{24}$$

The exact intuitionistic fuzzy solution is

$$\Big[y(t,r)\Big]_l^+ = (r-1)\exp(t), \quad \Big[y(t,r)\Big]_r^+ = (1-r)\exp(t),$$
$$\Big[y(t,r)\Big]_l^- = -2r\exp(t), \quad \Big[y(t,r)\Big]_r^- = 2r\exp(t).$$

Using the VIM, we obtain

$$[y_{n+1}(t,r)]_l^+ = [y_n(t,r)]_l^+ + \int_0^t \frac{1}{2}(\sinh(s-t)-\sin(s-t))\{\frac{d^4}{ds^4}[y_n(s,r)]_l^+ - [y_n(s,r)]_l^+\}ds,$$

$$[y_{n+1}(t,r)]_r^+ = [y_n(t,r)]_r^+ + \int_0^t \frac{1}{2}(\sinh(s-t)-\sin(s-t))\{\frac{d^4}{ds^4}[y_n(s,r)]_r^+ - [y_n(s,r)]_r^+\}ds,$$

$$[y_{n+1}(t,r)]_l^- = [y_n(t,r)]_l^- + \int_0^t \frac{1}{2}(\sinh(s-t)-\sin(s-t))\{\frac{d^4}{ds^4}[y_n(s,r)]_l^- - [y_n(s,r)]_l^-\}ds,$$

$$[y_{n+1}(t,r)]_r^- = [y_n(t,r)]_r^- + \int_0^t \frac{1}{2}(\sinh(s-t) - \sin(s-t))\{\frac{d^4}{ds^4}[y_n(s,r)]_r^- - [y_n(s,r)]_r^-\}ds.$$

If we begin with

$$\Big[y_0(t,r)\Big]_l^+ = (r-1)\Big(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}\Big),$$
$$\Big[y_0(t,r)\Big]_r^+ = (1-r)\Big(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}\Big)$$

and

$$\Big[y_0(t,r)\Big]_l^- = -2r\Big(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}\Big)$$
$$\Big[y_0(t,r)\Big]_r^- = 2r\Big(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}\Big)$$

then we obtain

$$\Big[y(t,r)\Big]_l^+ = (r-1)\exp(t), \quad \Big[y(t,r)\Big]_r^+ = (1-r)\exp(t),$$
$$\Big[y(t,r)\Big]_l^- = -2r\exp(t), \quad \Big[y(t,r)\Big]_r^- = 2r\exp(t),$$

which is the exact solution.

# 5 Conclusion

In this paper, an iterative method has been presented to solve $n$-th order intuitionistic fuzzy differential equations. It is clear that the method gives rapidly convergent successive approximations by determining the Lagrange multipliers. He's variational iteration method gives several successive approximations by using the iteration of the correction functional. Numerical examples, which show efficiency and applicability of the proposed method have been presented. In future work, intuitionistic fuzzy systems can be studied using VIM.

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