NIFS 5 (1999) 1, 9-13

FOUR INTUITIONISTIC FUZZY CATEGORIES REPRESENTED BY FUNCTOR CATEGORIES

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Abstract: By the use of the category **IFuz** of the IFSs, four intuitionistic fuzzy categories such as **IFuz**^{2}, **IFuz**$^{\rightarrow}$, **M-IFuz** and **IFBn** (**I**) are obtained and they can be represented as functor categories from small categories to category **IFuz** of IFSs respectively.

Keywords: IFSs, categories, functors, natural transformations.

1. Preliminary

1.1 *Concepts of IFS*[1]

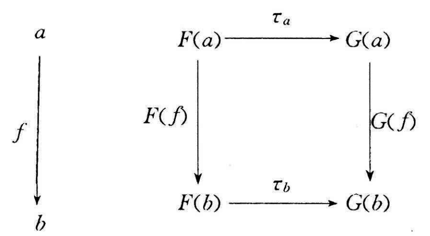
Let a set E be fixed. An IFS A^* in E is an object having the form

$$A^* = \{< x, \mu_A(x), \upsilon_A(x) > | x \in E\}$$

Where the functions $\mu_A : E \rightarrow [0,1]$ and $\upsilon_A : E \rightarrow [0,1]$ define the degree of membership and the degree of nonmembership of the element $x \in E$ to the set A, which is subset of E, and for every $x \in E: 0 \leqslant \mu_A(x) + \upsilon_A(x) \leqslant 1$. Four our convenience, A^* is denoted as (A, μ_A, υ_A).

1.2 *Natural transformation and category* Funct(C, D)

Let **C** and **D** be two categories, F and G be two functors from **C** to **D**. A natural transformation from functor F to functor G is an assignment τ that provides, for each **C**-object a, a **D**-arrow $\tau_a : F(a) \rightarrow G(a)$, such that for any **C**-arrow $f : a \rightarrow b$, the following diagram commutes in **D**, i. e., $\tau_b \circ F(f) = G(f) \circ \tau_a$.



we use the symbolism $\tau: F \dot{\rightarrow} G$, or $F \overset{\tau}{\dot{\rightarrow}} G$, to denote that τ is a natural transformation from F to G. The arrows τ_a are called the components of τ.

By the use of natural transformation, we can form a category **Funct(C,D)**:

——its objects are functors from **C** to **D**;

——an arrow from functor F to functor G is a natural transformation $\tau: F \dot{\rightarrow} G$;

——Let $\tau: F \dot{\rightarrow} G$ and $\sigma: G \dot{\rightarrow} H$ be two natural transformations, the composition of σ and τ satisfies: $(\sigma \circ \tau)_a = \sigma_a \circ \tau_a$, $\forall\ a \in \mathbf{C}$.

1.3 *Small Categories*

Let $\mathbf{2} = \{0,1\}$, then **2** can form a category with only two objects and only two identity arrows, which is called as discrete category.

Let $\mathbf{2} = \{0,1\}$, then **2** can form a category with only two objects and only three arrows: $1_0 = <0,0>: 0 \rightarrow 0$ and $1_1 = <1,1>: 1 \rightarrow 1$ be two identity arrows and arrow $<0,1>: 0 \rightarrow 1$, which is called as poset category.

Let $\mathbf{M} = (M, *, e)$ be a monoid category with only one object M, its arrows are the members m of M, $*$ be a composition, i. e., $m \circ n = m * n$, and e is a identity from M to M.

Let I be a set, then **I** can form a category with only identity arrow and its objects are the members i of **I**, which is called as a discrete category.

In this paper, we construct four intuitionistic fuzzy categories $\mathbf{IFuz}^2$, $\mathbf{IFuz}^{\rightarrow}$, **M-IFuz** and **IFBn(I)** by using category **IFuz** and prove that they are isomorphic with functor categories **Funct(2,IFuz)** (**2** is a discrete category), **Funct(2,IFuz)** (**2** is a poset category), **Funct(M,IFuz)** and **Funct(I,IFuz)** respectively.

2. Category $\mathbf{IFuz}^2$, $\mathbf{IFuz}^{\rightarrow}$, M-IFuz and IFBn(I)

Let **IFuz** be a category. Its objects are IFs(A, μ_A, υ_A) satisfying: $\mu_A(a) > 0$, $\forall\ a \in A$; its arrow from (A, μ_A, υ_A) to (B, μ_B, υ_B) is a mapping $f: A \rightarrow B$ satisfying: $\mu_B(f(a)) \geqslant \mu_A(a)$, $\upsilon_B(f(a)) \leqslant \upsilon_A(a)$, $\forall\ a \in A$. by the use of category **IFuz**, we can obtain the following categories:

(1)Category $\mathbf{IFuz}^2$

Its objects are $<(A, \mu_A, \upsilon_A), (B, \mu_B, \upsilon_B)>$, where (A, μ_A, υ_A) and (B, μ_B, υ_B) are **IFuz**-objects.

An arrow from $<(A, \mu_A, \upsilon_A), (B, \mu_B, \upsilon_B)>$ to $<(C, \mu_C, \upsilon_C), (D, \mu_D, \upsilon_D)>$ is a pair of mappings (f, g), where $f: A \rightarrow C$ and $g: B \rightarrow D$ satisfying:

(i)$\mu_C(f(a)) \geqslant \mu_A(a)$, $\upsilon_C(f(a)) \leqslant \upsilon_A(a)$, $\forall\ a \in A$;

(ii)$\mu_D(g(b)) \geqslant \mu_B(b)$, $\upsilon_D(g(b)) \leqslant \upsilon_B(b)$, $\forall\ b \in B$.

Composition of (f, g) and (f', g') satisfies: $(f, g) \circ (f', g') = (f \circ f', g \circ g')$.

(2)Category $\mathbf{IFuz}^{\rightarrow}$

Its objcts are $[(A, \mu_A) \xrightarrow{f} (B, \mu_B, \upsilon_B)]$, where (A, μ_A, υ_A) and (B, μ_B, υ_B) are **IFuz**-objects and f is a **IFuz**-arrow.

An arrow from $[(A,\mu_A,\upsilon_A)\xrightarrow{f}(B,\mu_B,\upsilon_B)]$ to $[(C,\mu_C,\upsilon_C)\xrightarrow{g}(D,\mu_D,\upsilon_D)]$ is a pair of mappins (h,k), where $h:A\to C$ and $k:B\to D$ satisfying:

(i) $\mu_C(h(a))\geqslant\mu_A(a),\upsilon_C(h(a))\leqslant\upsilon_A(a),\forall\ a\in A$;

(ii) $\mu_D(k(b))\geqslant\mu_B(b),\upsilon_D(k(b))\leqslant\upsilon_B(b),\forall\ b\in B$;

(iii) $k\circ f=g\circ h$.

Composition of (h,k) and (h',k') satisfies: $(h,k)\circ(h',k')=(h\circ h',k\circ k')$.

(3)Category **M-IFuz**

Let $(M,*,e)$ be a monoid with identity e. objects of category **M-IFuz** are (A,μ_A,υ_A,ξ) where (A,μ_A,υ_A) is **IFuz**-objects. mapping $\xi:M\times A\to A$ satisfies

(i) $\xi(e,a)=a,\forall\ a\in A$;

(ii) $\xi(m,\xi(n,a))=\xi(m*n,a),\forall\ a\in A,m,n\in M$;

(iii) $\mu_A(\xi(m,a))\geqslant\mu_A(a),\upsilon_A(\xi(m,a))\leqslant\upsilon_A(a),\forall\ a\in A$.

An arrow from (A,μ_A,υ_A,ξ) to $(B,\mu_B,\upsilon_B,\eta)$ is a mapping $f:A\to B$ *Satisfying*

(a) $f(\xi(m,a))=\eta(m,f(a)),\forall\ a\in A$; (b) $\mu_B(f(a))\geqslant\mu_A(a),\upsilon_A(f(a))\leqslant\upsilon_A(a),\forall\ a\in A$.

Composition of arrows is composition of mappings.

(4)Category **IFBn(I)**

Let I be a set. Objects of category **IFBn(I)** are (A,μ_A,υ_A,f), where (A,μ_A,υ_A) is **IFuz**-objects, $f:A\to I$ is a mapping.

An arrow from (A,μ_A,υ_A,f) to (B,μ_B,υ_B,g) is a mapping $k:A\to B$ such that

(i) $g\circ k=f$; (ii) $\mu_B(k(a))\geqslant\mu_A(a),\upsilon_B(k(a))\leqslant\upsilon_A(a)$.

Composition of arrows is composition of mappins.

Category $\mathbf{IFuz}^2$, $\mathbf{IFuz}^{\to}$, **M-IFuz** and **IFBn(I)** have intimate connections with category **IFuz**. We shall prove that they can be represented as category **Funct (C, IFuz)** of functors from small category **C** to category **IFuz**.

Theorem 1. Let $\mathbf{2}=\{0,1\}$ be a discrete category, then category $\mathbf{IFuz}^2$ is isomorphic with category **Funct(2, IFuz)**.

Proof. Let $F:\mathbf{2}\to\mathbf{IFuz}$ be a functor, then $(F(0),F(1))$ is $\mathbf{IFuz}^2$-object. Since **2** is a discrete category with only identity arrows $1_0=<0,0>:0\to 0$ and $1_1=<1,1>:1\to 1$, so $F(1_0)$ and $F(1_1)$ are identity arrows from $F(0)$ to $F(0)$, $F(1)$ to $F(1)$ respectively. Hence F can be seen as $<F(0),F(1)>$.

Let F and G be two functors from **2** to **IFuz** and $\tau:F\dot{\to}G$ be a natural transformation, then τ has two components $\tau_0:F(0)\to G(0)$ and $\tau_1:F(1)\to G(1)$ and consequently τ can be seen as (τ_0,τ_1). Let

$$\mathscr{F}:\mathbf{Funct(2,IFuz)}\to\mathbf{IFuz}^2$$
$$F\mapsto<F(0),F(1)>$$
$$F\xrightarrow[\cdot]{\tau}G\mapsto(\tau_0,\tau_1),$$

then $\mathscr{F}$ is a functor. Clearly, $\mathscr{F}$ is a isomrphism. Hence $\mathbf{IFuz}^2$ is isomorphic with category **Funct(2, IFuz)**.

Theorem 2. Let $\mathbf{2}=\{0,1\}$ be a poset category, then $\mathbf{IFuz}^{\rightarrow}$ is isomorphic with category **Funct(2, IFuz)**.

Proof. Let $F:\mathbf{2}\rightarrow\mathbf{IFuz}$ be a functor and $F(1_0)=1_{F(0)}, F(1_1)=1_{F(1)}, F(<0,1>)=f$, then F can be seen as $[F(0)\xrightarrow{f}F(1)]$.

Let $G:\mathbf{2}\rightarrow\mathbf{IFuz}$ be a functor and $\tau: F\rightarrow G$ be a natural transformation with component (τ_0,τ_1), then (τ_0,τ_1) is a arrow from $[F(0)\xrightarrow{f}F(1)]$ to $[G(0)\xrightarrow{g}G(1)]$ (where $g=G(<0,1>)$). Let

$\mathscr{P}$: **Funct(2, IFuz)** $\rightarrow \mathbf{IFuz}^{\rightarrow}$

$$F\mapsto[F(0)\xrightarrow{f}F(1)]$$

$$F\xrightarrow{\tau}G\mapsto(\tau_0,\tau_1)$$

$\mathscr{F}$: $\mathbf{IFuz}^{\rightarrow}\longrightarrow$ **Funct(2, IFuz)**

$$[(A,\mu_A,\upsilon_A)\xrightarrow{f}(B,\mu_B,\upsilon_B)]\mapsto F$$

$$[(A,\mu_A,\upsilon_A)\xrightarrow{f}(B,\mu_B,\upsilon_B)]\xrightarrow{(h,k)}[(C,\mu_C,\upsilon_C)\xrightarrow{g}(D,\mu_D,\upsilon_D)]\mapsto\tau$$

where $F(0)=(A,\mu_A,\upsilon_A), F(1)=(B,\mu_B,\upsilon_B), F(1_0)=Id_A, F(1_1)=Id_B, F(<0,1>)=f, \tau_0=h$ and $\tau_1=k$.

Then $\mathscr{P}$ and $\mathscr{F}$ are functors and $\mathscr{F}$ is an inverse of $\mathscr{P}$. Hence, category $\mathbf{IFuz}^{\rightarrow}$ is isomorphic with **Funct(2, IFuz)**.

Theorem 3. Let $\mathbf{M}=(M,*,e)$ be a monoid category, then category **M-IFuz** is isomorphic with **Funct (M, IFuz)**.

Proof. Let $F:\mathbf{M}\rightarrow\mathbf{IFuz}$ be a functor and $F(M)=(A,\mu_A,\upsilon_A)$, $\xi_m=F(m)$. Let $\xi(m,a)=\xi_m(a), \forall a\in A, m\in M$, then we have **M-IFuz**-object (A,μ_A,υ_A,ξ).

Let $G:\mathbf{M}\rightarrow\mathbf{IFuz}$ be another functor and $\tau: F\rightarrow G$ be a natural transformation with component $\tau_M=f$. Let $G(m)=(B,\mu_B,\upsilon_B)$, $\eta_m=G(m)$, then f is a arrow from (A,μ_A,υ_A,ξ) to $(B,\mu_B,\upsilon_B,\eta)$. Let

$\mathscr{F}$: **Funct(M, IFuz)** $\rightarrow$ **M-IFuz**

$$F\mapsto(A,\mu_A,\upsilon_A,\xi)$$

$$F\xrightarrow{\tau}G\mapsto f$$

then $\mathscr{F}$ is a functor. Clearly, $\mathscr{F}$ is a isomorphism. Hence, category **M-IFuz** is isomorphic with category **Funct(M, IFuz)**.

Theorem 4.. Let set **I** be a discrete category, then category **IFBn(I)** is isomorphic with category **Funct(I, IFuz)**.

Proof. Let $F:\mathbf{I}\rightarrow\mathbf{IFuz}$ be a functor and $F(i)=(A_i,\mu_{A_i},\upsilon_{A_i})$. Let $A=\bigcup_{i\in I}(A_i\times\{i\})$;

$f: A \to I\ (a,i) \mapsto i$; $\mu_A: A \to (0,1]$, $(a,i) \mapsto \mu_{A_i}(a)$; $\upsilon_A: A \to [0,1]\ (a,i) \mapsto \upsilon_{A_i}(a)$, then we have **IFBn(I)**-object $(A, \mu_A, \upsilon_A, f)$.

Let G: **I**→**IFuz** be another functor and $\tau: F \overset{\cdot}{\to} G$ be a natural transformation with component $\tau_i: F(i) \to G(i)$. Let

$$k: A \to B \quad (a,i) \mapsto (\tau_i(a), i),$$

then k is a arrow from $(A, \mu_A, \upsilon_A, f)$ to $(B, \mu_B, \upsilon_B, g)$ (where $(B, \mu_B, \upsilon_B, g)$ is a object corresponding functor G).

Let

$$\mathscr{F}: \mathbf{Funct(I, IFuz)} \to \mathbf{IFBn(I)}$$
$$F \mapsto (A, \mu_A, \upsilon_A, f)$$
$$F \xrightarrow[\cdot]{\tau} G \mapsto k$$

then $\mathscr{F}$ is a functor.

Conversely, let $(A, \mu_A, \upsilon_A, f)$ be an **IFBn(I)**-object and $A_i = f^{-1}(i)$, $\mu_{A_i} = \mu_A|_{A_i}$, $\upsilon_{A_i} = \upsilon_A|_{A_i}$. Let $F(i) = (A_i, \mu_{A_i}, \upsilon_{A_i})$, then F is a functor from category **I** to **IFuz**.

Let $k: (A, \mu_A, \upsilon_A, f) \to (B, \mu_B, \upsilon_B, g)$ be a arrow and $k_i = k|_{A_i}$, then we have natural transformation $F \xrightarrow[\cdot]{\tau} G$ with component $\tau_i = k_i$ (where G is a functor corresponding $(B, \mu_B, \upsilon_B, g)$). Let

$$\mathscr{D}: \mathbf{IFBn(I)} \to \mathbf{Funct(I, IFuz)}$$
$$(A, \mu_A, \upsilon_A, f) \mapsto F$$
$$(A, \mu_A, \upsilon_A, f) \xrightarrow{k} (B, \mu_B, \upsilon_B, g) \mapsto F \xrightarrow[\cdot]{\tau} G$$

then $\mathscr{D}$ is a functor.

Clearly, $\mathscr{D}$ is a inverse of $\mathscr{F}$. Hence category **IFBn(I)** is isomorphic with category **Funct(I, IFuz)**.

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