Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–4926, Online ISSN 2367–8283
2026, Volume 2, Number 3, 245–257
DOI: 10.7546/nifs.2026.32.3.245-257

About the sum of intuitionistic fuzzy observables

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Received: 19 June 2026 **Revised:** 30 September 2026
Accepted: 7 October 2026 **Online First:** 6 November 2026

Abstract: In the paper, we show another way of defining the sum of intuitionistic fuzzy observables using max-min connectives. We prove the existence of this sum, and we show its properties.
Keywords: Intuitionistic fuzzy event, Intuitionistic fuzzy observable, Sum of intuitionistic fuzzy observables, Max-min connectives.
2020 Mathematics Subject Classification: 60A86, 60A10.

1 Motivation

The sum of observables defined via max-min connectives has been studied on various structures. In paper [5], A. Dvurečenskij defined the sum of compatible observables $x \oplus y$ in a logic L. This sum is connected with the next system of the elements of a logic L

$$B_{x\oplus y}(t) = \bigvee_{r\in Q} \big(B_x(r) \wedge B_y(t-r)\big) = \bigvee_{r\in Q} \Big(x\big((-\infty, r)\big) \wedge y\big((-\infty, t-r)\big)\Big), t \in R,$$

where Q is the set of all rationals. The logic L is a σ-lattice with the first element 0 and the last element 1 and with an orthocomplementation $\perp : a \to a^{\perp}$, which satisfies the followin g conditions:

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(i) $(a^{\perp})^{\perp} = a$;

(ii) if $a < b$, then $b^{\perp} < a^{\perp}$;

(iii) $a \vee a^{\perp} = 1$;

(iv) if $a < b$, then $b = a \vee (b \wedge a^{\perp})$,

for all $a, b \in L$. As an observable x in logic L he understand a mapping $x : \mathcal{B}(R) \to L$ such that

(i) $x(R) = 1$, $x(\varnothing) = 0$;

(ii) if $A \cap B = \varnothing$, then $x(A) \perp x(B)$;

(iii) if $A \cap B = \varnothing$, $i \neq j$, then $x\Big(\bigcup\limits_{i} A_i\Big) = \bigvee\limits_{i} x(A_i)$,

for all $A, B, A_i \in \mathcal{B}(R)$, see [5].

In paper [7] the authors determined a sum of F-observables x, y in F-quantum space (X, M) with the system $\{B_{x+y}(t) : t \in R\}$ given by

$$B_{x+y}(t) = \bigcup_{r \in Q} \big(B_x(r) \cap B_y(t-r)\big) = \bigcup_{r \in Q} \Big(x\big((-\infty, r)\big) \cap y\big((-\infty, t-r)\big)\Big), t \in R,$$

where Q is the set of all rationals. The F-quantum space (fuzzy quantum space) is a couple (X, M), where X is a nonempty set and $M \subset [0,1]^X$ satisfies the following conditions:

(i) if $1(x) = 1$ for any $x \in X$, then $1 \in M$;

(ii) if $a \in M$, then $a^{\perp} = 1 - a$, $a^{\perp} \in M$;

(iii) if $\frac{1}{2}(x) = \frac{1}{2}$ for any $x \in X$, then $\frac{1}{2} \notin M$;

(iv) $\bigcup\limits_{n=1}^{\infty} a_n = \sup\limits_{n} a_n$, $\bigcup\limits_{n=1}^{\infty} a_n \in M$ for any $\{a_n\}_{n=1}^{\infty} \subset M$.

As an F-observable x on an F-quantum space (X, M) they understand a mapping $x : \mathcal{B}(R) \to M$ such that

(i) $x(A^c) = 1 - x(A)$ for any $A \in \mathcal{B}(R)$;

(ii) if $\{A_n\}_{n=1}^{\infty} \subset \mathcal{B}(R)$, then $x\Big(\bigcup\limits_{n=1}^{\infty} A_n\Big) = \bigcup\limits_{n=1}^{\infty} x(A_n)$,

where $\mathcal{B}(R)$ is the Borel σ-algebra of the real line R and A^c denotes the complement of A in R, see [7].

A very similar situation to that in paper [7], was studied in papers [4, 6]. Here, the authors considered an F-quantum space $(X, \mathcal{M})$ identical to the (X, M) in [7], but there they replaced property (iv) in the definition of the space (X, M) by the following one

(iv) $\bigvee\limits_{n=1}^{\infty} a_n = \sup\limits_{n} a_n$, $\bigvee\limits_{n=1}^{\infty} a_n \in \mathcal{M}$ for any $\{a_n\}_{n=1}^{\infty} \subset \mathcal{M}$.

Naturally, condition (ii) in the definition of F-observable in [7] is replaced by the condition

(ii) if $\{A_n\}_{n=1}^{\infty} \subset \mathcal{B}(R)$, then $x\Big(\bigcup\limits_{n=1}^{\infty} A_n\Big) = \bigvee\limits_{n=1}^{\infty} x(A_n)$,

in papers [4, 6]. In this case, the sum of F-observables x, y in F-quantum space $(X, \mathcal{M})$ is connected with the system $\{B_{x+y}(t) : t \in R\}$ given by

$$B_{x+y}(t) = \bigvee_{r\in Q} \big(B_x(r) \wedge B_y(t-r)\big) = \bigvee_{r\in Q} \Big(x\big((-\infty, r)\big) \wedge y\big((-\infty, t-r)\big)\Big), t \in R,$$

see [4, 6].

A different situation is considered in the article [10]. Here, A. Kolesárová and B. Riečan studied the sum of fuzzy observables x and y in the context of Łukasiewicz (Giles) operations $\oplus, \odot$ given by

$$\begin{aligned} x \oplus y &= \min(x + y, 1) = (x + y) \wedge 1, \\ x \odot y &= \max(x + y - 1, 0) = (x + y - 1) \vee 0. \end{aligned}$$

As a fuzzy observable x of a fuzzy space $(\Omega, \mathcal{R})$ they understand a mapping $x : \mathcal{B}(R) \to \mathcal{R}$ satisfying the conditions

(i) $x(A^c) = 1 - x(A)$ for each $A \in \mathcal{B}(R)$;

(ii) if $\{A_n\}_{n=1}^{\infty} \subset \mathcal{B}(R)$, $A_i \cap E_j = \varnothing$, $i \neq j$, then $x\Big(\bigcup\limits_{n=1}^{\infty} A_n\Big) = \bigoplus\limits_{n=1}^{\infty} x(A_n)$,

where $\bigoplus\limits_{n=1}^{\infty} f_n = \lim\limits_{n\to\infty} \bigoplus\limits_{i=1}^{n} f_i$ and $\bigoplus\limits_{i=1}^{n} f_i = \Big(\bigoplus\limits_{i=1}^{n-1} f_i\Big) \oplus f_n$. There $\mathcal{R} \subset [0, 1]^{\Omega}$ is the generated fuzzy σ-algebra on a measurable space $(\Omega, \mathcal{S})$, i.e., the set of all $\mathcal{S}$-measurable functions and $\mathcal{B}(R)$ is the Borel σ-algebra of the real line R, see [10]. In this paper, the authors defined the sum of two fuzzy observables x and y as the fuzzy observable $z = x + y$ given by

$$z\big((-\infty, t)\big) = \bigvee_{r\in Q} \Big[x\big((-\infty, r)\big) \wedge y\big((-\infty, t-r)\big)\Big], t \in R, \tag{1}$$

see [10]. They showed that z is a fuzzy observable and that for any fixed $\omega \in \Omega$, $z\big((-\infty, \cdot)\big)(\omega)$ is a probability distribution on R, i.e., it is a non-decreasing and left-continuous function on R with properties $\lim\limits_{t\to-\infty} z\big((-\infty, \cdot)\big)(\omega) = 0$, $\lim\limits_{t\to\infty} z\big((-\infty, \cdot)\big)(\omega) = 1$.

In [13], we can find the construction of the sum of observables for both structures $(X, \mathcal{M})$, $(\Omega, \mathcal{R})$, which were described in the previously mentioned articles [4, 6, 10].

2 Introduction of elementary notions

In this paper, we will study a sum of two intuitionistic fuzzy observables. As an intuitionistic fuzzy observable, we understand the mapping $x : \mathcal{B}(R) \to \mathcal{F}$, which satisfies the conditions:

(i) $x(\varnothing) = (0_\Omega, 1_\Omega)$, $x(R) = (1_\Omega, 0_\Omega)$;

(ii) if $E \cap F = \varnothing$, then $x(E \cup F) = x(E) \oplus x(F)$ and $x(E) \odot x(F) = (0_\Omega, 1_\Omega)$;

(iii) if $E_i \nearrow E$, then $x(E_i) \nearrow x(E)$,

for each $E, F, E_i \in \mathcal{B}(R)$, $i \in N$, see [12].

Here $\mathcal{B}(R)$ denotes the σ-algebra of Borel sets, i.e., the σ-algebra $\sigma(\mathcal{I})$ of the family $\mathcal{I}$ of all intervals in R in the form $[c,d) = \{x \in R : c \leq x < d\}$. The elements of this σ-algebra are called Borel sets, see [13].

By $\mathcal{F}$ we denote the family of intuitionistic fuzzy events on a measurable space $(\Omega, \mathcal{S})$, i.e., the family of intuitionistic fuzzy sets $\mathbf{E} = (\mu_E, \nu_E)$, $\mu_E + \nu_E \leq 1_\Omega$ such that $\mu_E, \nu_E : \Omega \to [0,1]$ are $\mathcal{S}$-measurable functions. There $\mathcal{S}$ is a σ-algebra of subsets of a nonempty set Ω. The concept of intuitionistic fuzzy sets was proposed by Krassimir T. Atanassov in the works [1–3].

In this paper, we will use the Łukasiewicz binary operations $\oplus, \odot$ defined on $\mathcal{F}$ by

$$\begin{aligned} \mathbf{E} \oplus \mathbf{F} &= ((\mu_E + \mu_F) \wedge 1_\Omega, (\nu_E + \nu_F - 1_\Omega) \vee 0_\Omega)) \\ &= (\mu_E \oplus \mu_F, \nu_E \odot \nu_F), \\ \mathbf{E} \odot \mathbf{F} &= ((\mu_E + \mu_F - 1_\Omega) \vee 0_\Omega, (\nu_E + \nu_F) \wedge 1_\Omega)) \\ &= (\mu_E \odot \mu_F, \nu_E \oplus \nu_F) \end{aligned}$$

for $\mathbf{E} = (\mu_E, \nu_E) \in \mathcal{F}$, $\mathbf{F} = (\mu_F, \nu_F) \in \mathcal{F}$. The functions 0_Ω, 1_Ω represent the constant functions defined by $0_\Omega(x) = 0$, $1_\Omega(x) = 1$ for each $x \in \Omega$.

The partial ordering $\leq$ on family $\mathcal{F}$ is introduced as follows

$$\mathbf{E} \leq \mathbf{F} \text{ if and only if } \mu_E \leq \mu_F, \nu_E \geq \nu_F.$$

There the largest element of $\mathcal{F}$ is $(1_\Omega, 0_\Omega)$ and the smallest element of $\mathcal{F}$ is $(0_\Omega, 1_\Omega)$.

By $\mathbf{E}_n \nearrow \mathbf{E}$ on $\mathcal{F}$ we mean the convergences $\mu_{E_n} \nearrow \mu_E$, $\nu_{E_n} \searrow \nu_E$ and similarly, for $\mathbf{E}_n \searrow \mathbf{E}$ on $\mathcal{F}$ we have the convergences $\mu_{E_n} \searrow \mu_E$, $\nu_{E_n} \nearrow \nu_E$, where $\mathbf{E}_n = (\mu_{E_n}, \nu_{E_n})$, $\mathbf{E} = (\mu_E, \nu_E) \in \mathcal{F}$, $n \in N$.

In this paper, we define the sum of intuitionistic fuzzy observables with the help of the system of intuitionistic fuzzy (IF) sets and max-min connectives formulated by

$$\begin{aligned} \mathbf{E} \vee \mathbf{F} &= (\mu_E \vee \mu_F, \nu_E \wedge \nu_F), \\ \mathbf{E} \wedge \mathbf{F} &= (\mu_E \wedge \mu_F, \nu_E \vee \nu_F) \end{aligned}$$

for each $\mathbf{E} = (\mu_E, \nu_E), \mathbf{F} = (\mu_F, \nu_F) \in \mathcal{F}$. The De Morgan's laws are given by $(f \vee g)' = f' \wedge g'$, $(f \wedge g)' = f' \vee g'$, where $f' = 1 - f$, see [13].

In the next section, we use the fact that each IF-observable $x : \mathcal{B}(R) \to \mathcal{F}$ can be expressed using two observables $x^\flat, x^\sharp : \mathcal{B}(R) \to \mathcal{T}$ as follows

$$x(C) = \big(x^\flat(C), 1_\Omega - x^\sharp(C)\big)$$

for each $C \in \mathcal{B}(R)$, where $\mathcal{T} = \{f : \Omega \to [0,1]; f \text{ is } \mathcal{S} - \text{measurable}\}$, see [11]. Moreover, the observables $x^\flat$ and $x^\sharp$ coincide. We prove this in the following theorem.

Theorem 1. *Let $x : \mathcal{B}(R) \to \mathcal{F}$ be an IF-observable. For any $C \in \mathcal{B}(R)$ denote*

$$x(C) = \big(x^\flat(C), 1_\Omega - x^\sharp(C)\big).$$

Then $x^\flat, x^\sharp : \mathcal{B}(R) \to \mathcal{T}$ are observables and $x^\flat(C) = x^\sharp(C)$ for each $C \in \mathcal{B}(R)$, where $\mathcal{T} = \{f : \Omega \to [0,1]; f$ is $\mathcal{S}$-measurable$\}$.

Proof. Let $x : \mathcal{B}(R) \to \mathcal{F}$ be an IF-observable and denote $x(C) = \big(x^\flat(C), 1_\Omega - x^\sharp(C)\big)$ for any $C \in \mathcal{B}(R)$. Really from properties $x(\varnothing) = (0_\Omega, 1_\Omega)$, $x(R) = (1_\Omega, 0_\Omega)$ we have

$$x^\flat(\varnothing) = 0_\Omega, \qquad x^\sharp(\varnothing) = 0_\Omega,$$
$$x^\flat(R) = 1_\Omega, \qquad x^\sharp(R) = 1_\Omega.$$

Also if $E \cap F = \varnothing$, then $x(E \cup F) = x(E) \oplus x(F)$ and $x(E) \odot x(F) = (0_\Omega, 1_\Omega)$. Hence

$$x^\flat(E) \odot x^\flat(F) = 0_\Omega \ , \ x^\sharp(E) \odot x^\sharp(F) = 0_\Omega$$

and

$$\begin{aligned} x^\flat(E \cup F) &= x^\flat(E) + x^\flat(F) = x^\flat(E) \oplus x^\flat(F), \\ x^\sharp(E \cup F) &= x^\sharp(E) + x^\sharp(F) = x^\sharp(E) \oplus x^\sharp(F), \end{aligned}$$

because $x^\flat(E) + x^\flat(F) \le 1_\Omega$, $x^\sharp(E) + x^\sharp(F) \le 1_\Omega$.

Finally, if $E_i \nearrow E$, then $x(E_i) \nearrow x(E)$, and hence

$$x^\flat(E_i) \nearrow x^\flat(E) \ , \ x^\sharp(E_i) \nearrow x^\sharp(E),$$

for each $E, F, E_i \in \mathcal{B}(R)$, $i \in N$. Thus, $x^\flat$ and $x^\sharp$ satisfy the three properties of the observables mentioned in [10, 13], where $\oplus$, $\odot$ are the Łukasiewicz operations.

Since any $A \in \mathcal{B}(R)$ and its complement $A^c \in \mathcal{B}(R)$ are disjoint sets (i.e. $A \cap A^c = \varnothing$) and $A \cup A^c = R$, then using (ii) property of IF-observable x and the De Morgan's laws we have

$$(1_\Omega, 0_\Omega) = x(R) = x(A \cup A^c) = x(A) \oplus x(A^c) = \big(x^\flat(A) \oplus x^\flat(A^c), 1_\Omega - x^\sharp(A) \oplus x^\sharp(A^c)\big)$$

and

$$(0_\Omega, 1_\Omega) = x(A) \odot x(A^c) = \big(x^\flat(A) \odot x^\flat(A^c), 1_\Omega - x^\sharp(A) \odot x^\sharp(A^c)\big).$$

Therefore

$$1_\Omega = x^\flat(A) \oplus x^\flat(A^c) = x^\flat(A) + x^\flat(A^c), \quad \text{i.e.} \quad x^\flat(A^c) = 1_\Omega - x^\flat(A);$$
$$1_\Omega = x^\sharp(A) \oplus x^\sharp(A^c) = x^\sharp(A) + x^\sharp(A^c), \quad \text{i.e.} \quad x^\sharp(A^c) = 1_\Omega - x^\sharp(A);$$

because $x^\flat(A) + x^\flat(A^c) \le 1_\Omega$, $x^\sharp(A) + x^\sharp(A^c) \le 1_\Omega$. But $x(A), x(A^c) \in \mathcal{F}$, hence

$$x^\flat(A) + (1 - x^\sharp(A)) \le 1_\Omega, \quad \text{i.e.} \quad x^\flat(A) \le x^\sharp(A);$$
$$x^\flat(A^c) + (1 - x^\sharp(A^c)) \le 1_\Omega, \quad \text{i.e.} \quad x^\flat(A^c) \le x^\sharp(A^c).$$

Using $x^\flat(A^c) = 1_\Omega - x^\flat(A)$, $x^\sharp(A^c) = 1_\Omega - x^\sharp(A)$ and $x^\flat(A^c) \le x^\sharp(A^c)$ we obtain $x^\flat(A) \ge x^\sharp(A)$, and together with $x^\flat(A) \le x^\sharp(A)$ we have that $x^\flat(A) = x^\sharp(A)$ for every $A \in \mathcal{B}(R)$. □

Remark 1. *It follows from Theorem 1 that an IF-observable $x : \mathcal{B}(R) \to \mathcal{F}$ can be expressed in the form $x(C) = \big(x^\flat(C), 1_\Omega - x^\flat(C)\big)$ for each $C \in \mathcal{B}(R)$, where $x^\flat : \mathcal{B}(R) \to \mathcal{T}$ is observable and $\mathcal{T} = \{f : \Omega \to [0, 1]; f$ is $\mathcal{S}$-measurable$\}$.*

3 Sum of IF-observables induced by system of IF-sets

In this section, we will present the main results. We will define the sum of IF-observables using max-min connectives and the system of IF-sets.

We are inspired by papers mentioned in Section 1 and by the formulation of the sum of two random variables $\xi, \eta : \Omega \to R$ in classical probability space. There for each $t \in R$ is $\xi + \eta < t$ if and only if $\xi < t - \eta$. In this case, we can choose $r \in Q$ such that $\xi < r < t - \eta$, i.e., $\xi < r$ and $\eta < t - r$. Hence

$$\begin{aligned}\{u \in \Omega : (\xi + \eta)(u) < t\} &= \bigcup_{r \in Q} \big(\{u \in \Omega : \xi(u) < r\} \cap \{u \in \Omega : \eta(u) < t - r\}\big) \\ &= \bigcup_{r \in Q} \Big[\xi^{-1}\big((-\infty, r)\big) \cap \eta^{-1}\big((-\infty, t - r)\big)\Big].\end{aligned}$$

In the IF-space $(\Omega, \mathcal{F})$ to the set $\xi^{-1}\big((-\infty, t)\big)$ corresponds the IF-set $x\big((-\infty, t)\big)$. Denote it by $\mathbf{B}_x(t)$ for each $t \in R$. In the next theorems, we show some properties of the system $\{\mathbf{B}_x(t) : t \in R\}$ of IF-sets from $\mathcal{F}$.

Theorem 2. *Let $x : \mathcal{B}(R) \to \mathcal{F}$ be an IF-observable. Denote $\mathbf{B}_x(t) = x\big((-\infty, t)\big)$ for each $t \in R$. Then the system of IF-sets $\{\mathbf{B}_x(t) : t \in R\}$ satisfies the conditions:*

(i) *if $s \leq t$, then $\mathbf{B}_x(s) \leq \mathbf{B}_x(t)$;*

(ii) $\bigvee\limits_{t \in R} \mathbf{B}_x(t) = (1_\Omega, 0_\Omega)$, $\bigwedge\limits_{t \in R} \mathbf{B}_x(t) = (0_\Omega, 1_\Omega)$;

(iii) $\bigvee\limits_{t < s} \mathbf{B}_x(t) = \mathbf{B}_x(s)$.

Proof. (i) It is trivial, because for $s \leq t$ is $(-\infty, s) \subseteq (-\infty, t)$ and from the property of IF-observable x we obtain $\mathbf{B}_x(s) = x\big((-\infty, s)\big) \leq x\big((-\infty, t)\big) = \mathbf{B}_x(t)$.

(ii) For each $t \in R$ we have $(-\infty, t) \subseteq R$, so $\mathbf{B}_x(t) = x\big((-\infty, t)\big) \leq x(R) = (1_\Omega, 0_\Omega)$. Therefore $\bigvee\limits_{t \in R} \mathbf{B}_x(t) \leq (1_\Omega, 0_\Omega)$. For each integer n we have

$$\mathbf{B}_x(n) = x\big((-\infty, n)\big) \leq x(R) = (1_\Omega, 0_\Omega)$$

and

$$(1_\Omega, 0_\Omega) = x(R) = x\Big(\bigcup_{n=1}^{\infty} (-\infty, n)\Big) = \bigvee_{n=1}^{\infty} x\big((-\infty, n)\big) = \bigvee_{n=1}^{\infty} \mathbf{B}_x(n).$$

Therefore $\bigvee\limits_{t \in R} \mathbf{B}_x(t) = (1_\Omega, 0_\Omega)$.

Similarly for each $t \in R$ we have $\varnothing \subseteq (-\infty, t)$, so $(0_\Omega, 1_\Omega) = x(\varnothing) \leq x\big((-\infty, t)\big) = \mathbf{B}_x(t)$. Therefore $(0_\Omega, 1_\Omega) \leq \bigwedge\limits_{t \in R} \mathbf{B}_x(t)$. But for each integer n we have

$$(0_\Omega, 1_\Omega) = x(\varnothing) \leq x\big((-\infty, -n)\big) = \mathbf{B}_x(-n)$$

and

$$(0_\Omega, 1_\Omega) = x(\varnothing) = x\Big(\bigcap_{n=1}^{\infty}(-\infty, -n)\Big) = \bigwedge_{n=1}^{\infty} x\big((-\infty, -n)\big) = \bigwedge_{n=1}^{\infty} \mathbf{B}_x(-n).$$

Therefore $\bigwedge\limits_{t\in R} \mathbf{B}_x(t) = (0_\Omega, 1_\Omega)$.

(iii) Evidently $(-\infty, s) = \bigcup\limits_{n=1}^{\infty}\big(-\infty, s - \frac{1}{n}\big)$. Denote $t_n = s - \frac{1}{n}$ for $n \in N$. As $t_n \nearrow s$, so $(-\infty, t_n) \nearrow (-\infty, s)$. But by condition (i) we have $\mathbf{B}_x(t) \le \mathbf{B}_x(s)$ for $t \le s$ and hence

$$\mathbf{B}_x(s) = x\big((-\infty, s)\big) = x\Big(\bigcup_{n=1}^{\infty}\Big(-\infty, s - \frac{1}{n}\Big)\Big) = \bigvee_{n=1}^{\infty} x\Big(\Big(-\infty, s - \frac{1}{n}\Big)\Big)$$
$$= \bigvee_{n=1}^{\infty} \mathbf{B}_x(t_n) = \bigvee_{t<s} \mathbf{B}_x(t). \qquad \square$$

Now define the sum of IF-observables x, y by the system of IF-sets.

Definition 1. *Let $x, y : \mathcal{B}(R) \to \mathcal{F}$ be the IF-observables. We say that an IF-observable $z : \mathcal{B}(R) \to \mathcal{F}$ is the sum of IF-observables x and y and we write $z = x + y$, if the IF-observable z is determined by*

$$z\big((-\infty, t)\big) = \bigvee_{r\in Q}\Big[x\big((-\infty, r)\big) \wedge y\big((-\infty, t - r)\big)\Big], t \in R,$$

where Q is the set of all rationals.

Theorem 3. *For any two IF-observables $x, y : \mathcal{B}(R) \to \mathcal{F}$ their sum $z = x + y$ exists and*

$$z\big((-\infty, t)\big) = \Big(z^\flat\big((-\infty, t)\big), 1_\Omega - z^\flat\big((-\infty, t)\big)\Big), t \in R,$$

where $z^\flat = x^\flat + y^\flat$ is the sums of observables $x^\flat, y^\flat : \mathcal{B}(R) \to \mathcal{T}$ given by

$$z^\flat\big((-\infty, t)\big) = \bigvee_{r\in Q}\Big[x^\flat\big((-\infty, r)\big) \wedge y^\flat\big((-\infty, t - r)\big)\Big], t \in R,$$

and Q is the set of all rationals. There $x(C) = \big(x^\flat(C), 1_\Omega - x^\flat(C)\big)$, $y(C) = \big(y^\flat(C), 1_\Omega - y^\flat(C)\big)$ for each $C \in \mathcal{B}(R)$ and $\mathcal{T} = \{f : \Omega \to [0, 1]; f$ is $\mathcal{S}$-measurable$\}$.

Proof. Let $x, y : \mathcal{B}(R) \to \mathcal{F}$ be the IF-observables. By Theorem 1, then there exist the observables $x^\flat, y^\flat : \mathcal{B}(R) \to \mathcal{T}$ such that $x(C) = \big(x^\flat(C), 1_\Omega - x^\flat(C)\big)$, $y(C) = \big(y^\flat(C), 1_\Omega - y^\flat(C)\big)$ for each $C \in \mathcal{B}(R)$ and $\mathcal{T} = \{f : \Omega \to [0, 1]; f$ is $\mathcal{S} - \text{measurable}\}$.

Using Proposition 4 in [10] we have that there exists an observable $z^\flat : \mathcal{B}(R) \to \mathcal{T}$ given by

$$z^\flat\big((-\infty, t)\big) = \bigvee_{r\in Q}\Big[x^\flat\big((-\infty, r)\big) \wedge y^\flat\big((-\infty, t - r)\big)\Big], t \in R, \tag{2}$$

and Q is the set of all rationals. There, $z^\flat = x^\flat + y^\flat$ is a sum of the previously mentioned observables. Using Definition 1, De Morgan's laws and (2), we have for the sum $z = x + y$ of IF-observables x and y

$$\begin{aligned} z\big((-\infty,t)\big) &= \bigvee_{r\in Q}\Big[x\big((-\infty,r)\big) \wedge y\big((-\infty,t-r)\big)\Big] \\ &= \Big(\bigvee_{r\in Q}\Big[x^\flat\big((-\infty,r)\big) \wedge y^\flat\big((-\infty,t-r)\big)\Big], 1_\Omega - \bigvee_{r\in Q}\Big[x^\flat\big((-\infty,r)\big) \wedge y^\flat\big((-\infty,t-r)\big)\Big]\Big) \\ &= \Big(z^\flat\big((-\infty,t)\big), 1_\Omega - z^\flat\big((-\infty,t)\big)\Big), \end{aligned}$$

where $t \in R$ and Q is the set of all rationals.

Since Proposition 4 of [10] guarantees the existence of the observable $z^\flat : \mathcal{B}(R) \to \mathcal{T}$, then for any $C \in \mathcal{B}(\mathbb{R})$ we have that

$$z(C) = \big(z^\flat(C), 1_\Omega - z^\flat(C)\big) \in \mathcal{F}$$

and hence $z : \mathcal{B}(R) \to \mathcal{F}$ is an IF-observable, see Theorem 1. □

In [10], A. Kolesárová and B. Riečan used a concept of fuzzy-valued functions introduced by E. P. Klement in [8, 9]. By the extended real line $\overline{R}(I)$ he understand the set of all functions $u : \overline{R} \to I$ such that

(i) $u(-\infty) = 0$, $u(+\infty) = 1$

(ii) $u(r) = \sup\{u(s); s < r, s \in R\}$ for each $r \in R$.

There $\overline{R} = R \cup \{-\infty, +\infty\}$ and $I = [0, 1]$. It is easy to see that the fuzzy real number $u \in \overline{R}(I)$ is the cumulative distribution function on $\overline{R}$. A fuzzy number u is said to be finite, if $\inf\{u(r); r \in R\} = 0$ and $\sup\{u(r); r \in R\} = 1$, see [10].

Moreover $\overline{R}(I)$ can be considered as a subspace of $[0, 1]^R$. Therefore we can consider the measurable functions $X : \Omega \to R(I)$ called fuzzy-valued random variables. The measurability of a function $X : \Omega \to R(I)$ is equivalent to the existence of a Markov kernel $\mathcal{K}$ from $(\Omega, \mathcal{S})$ to $\big(\overline{R}, \mathcal{B}(\overline{R})\big)$ such that for all $(\omega, t) \in \Omega \times \overline{R}$, $X(\omega)(t) = \mathcal{K}\big(\omega, [-\infty, t)\big)$, see Proposition 2.1 in [8]. A Markov kernel $\mathcal{K}$ from $(\Omega, \mathcal{S})$ to $\big(\overline{R}, \mathcal{B}(\overline{R})\big)$ is a function $\mathcal{K} : \Omega \times \mathcal{B}(\overline{R}) \to [0, 1]$, which satisfies the conditions:

(i) $\mathcal{K}(\cdot, A) : \Omega \to [0, 1]$ is a measurable function for each $A \in \mathcal{B}(\overline{R})$

(ii) $\mathcal{K}(\omega, \cdot) : \mathcal{B}(\overline{R}) \to [0, 1]$ is a probability distribution for each fixed $\omega \in \Omega$,

see [8].

In [10] A. Kolesárová and B. Riečan proved in Theorem 1 that a mapping $x : \mathcal{B}(R) \to \mathcal{R}$ is a fuzzy observable of $(\Omega, \mathcal{R})$ if and only if $x\big((-\infty, t)\big)(\omega) = X(\omega)(t)$, $t \in R$ defines a finite fuzzy-valued random variable X on $(\Omega, \mathcal{S})$.

Using these results, we will try to show that there exists an IF-observable generated by a system of IF-sets.

Theorem 4. *If a system $\{\mathbf{B}(t) : t \in R\}$ of IF-sets in $\mathcal{F}$ has the following three properties*

(i) *if $s \leq t$, then $\mathbf{B}(s) \leq \mathbf{B}(t)$;*

(ii) $\bigvee\limits_{t\in R} \mathbf{B}(t) = (1_\Omega, 0_\Omega)$, $\bigwedge\limits_{t\in R} \mathbf{B}(t) = (0_\Omega, 1_\Omega)$;

(iii) $\bigvee\limits_{t<s} \mathbf{B}(t) = \mathbf{B}(s)$,

then there exists an IF-observable $x : \mathcal{B}(R) \to \mathcal{F}$ such that $x\big((-\infty, t)\big) = \mathbf{B}(t)$ for each $t \in R$.

Proof. Let $\{\mathbf{B}(t) : t \in R\}$ be a system of IF-sets in $\mathcal{F}$ with properties (i)–(iii). Then for each $t \in R$ there exist two functions $\mu_{B(t)}, \nu_{B(t)} : \Omega \to [0, 1]$ such that $\mu_{B(t)} + \nu_{B(t)} \leq 1_\Omega$ and $\mathbf{B}(t) = \big(\mu_{B(t)}, \nu_{B(t)}\big)$. Denote $B^\flat(t) = \mu_{B(t)}$, $B^\sharp(t) = 1_\Omega - \nu_{B(t)}$ for each $t \in R$, then

$$\mathbf{B}(t) = \big(\mu_{B(t)}, \nu_{B(t)}\big) = \big(B^\flat(t), 1_\Omega - B^\sharp(t)\big). \tag{3}$$

If $s \leq t$, then by (i) we have

$$\begin{aligned} \mathbf{B}(s) &\leq \mathbf{B}(t), \\ \big(\mu_{B(s)}, \nu_{B(s)}\big) &\leq \big(\mu_{B(t)}, \nu_{B(t)}\big), \\ \mu_{B(s)} &\leq \mu_{B(t)}, \\ \nu_{B(s)} &\geq \nu_{B(t)}. \end{aligned}$$

Therefore

$$B^\flat(s) = \mu_{B(s)} \leq \mu_{B(t)} = B^\flat(t), \tag{4}$$

$$B^\sharp(s) = 1_\Omega - \nu_{B(s)} \leq 1_\Omega - \nu_{B(t)} = B^\sharp(t). \tag{5}$$

From (ii) and (4), with the help of De Morgan's laws, we obtain

$$(1_\Omega, 0_\Omega) = \bigvee_{t\in R} \mathbf{B}(t) = \bigvee_{t\in R} \big(B^\flat(t), 1_\Omega - B^\sharp(t)\big) = \Big(\bigvee_{t\in R} B^\flat(t), 1_\Omega - \bigvee_{t\in R} B^\sharp(t)\Big),$$

$$(0_\Omega, 1_\Omega) = \bigwedge_{t\in R} \mathbf{B}(t) = \bigwedge_{t\in R} \big(B^\flat(t), 1_\Omega - B^\sharp(t)\big) = \Big(\bigwedge_{t\in R} B^\flat(t), 1_\Omega - \bigwedge_{t\in R} B^\sharp(t)\Big),$$

i.e.,

$$\bigvee_{t\in R} B^\flat(t) = 1_\Omega, \bigwedge_{t\in R} B^\flat(t) = 0_\Omega, \tag{6}$$

$$\bigvee_{t\in R} B^\sharp(t) = 1_\Omega, \bigwedge_{t\in R} B^\sharp(t) = 0_\Omega. \tag{7}$$

Using (iii), (4) and De Morgan's laws, we have

$$\begin{aligned} \big(B^\flat(s), 1_\Omega - B^\sharp(s)\big) &= \mathbf{B}(s) = \bigvee_{t<s} \mathbf{B}(t) = \bigvee_{t<s} \big(B^\flat(t), 1_\Omega - B^\sharp(t)\big) \\ &= \Big(\bigvee_{t<s} B^\flat(t), 1_\Omega - \bigvee_{t<s} B^\sharp(t)\Big), \end{aligned}$$

i.e.,

$$B^\flat(s) = \bigvee_{t<s} B^\flat(t),\ B^\sharp(s) = \bigvee_{t<s} B^\sharp(t). \tag{8}$$

Denote

$$x^{\flat}\big((-\infty,t)\big)=B^{\flat}(t),\ x^{\sharp}\big((-\infty,t)\big)=B^{\sharp}(t)$$

for each $t\in R$. Evidently using (5), (6), (7), (8) and (9) for any fixed $\omega\in\Omega$ the functions $x^{\flat}\big((-\infty,\cdot)\big)(\omega)$, $x^{\sharp}\big((-\infty,\cdot)\big)(\omega)$ are the probability distributions on R and hence they are the finite fuzzy numbers, see [10]. Therefore the functions $X^{\flat},X^{\sharp}:\Omega\to R(I)$,

$$X^{\flat}(\omega)(t)=x^{\flat}\big((-\infty,t)\big)(\omega),\ X^{\sharp}(\omega)(t)=x^{\sharp}\big((-\infty,t)\big)(\omega),\ t\in R$$

are the fuzzy-valued random variables and using Theorem 1 from [10] we have that $x^{\flat},x^{\sharp}:\mathcal{B}(R)\to\mathcal{T}$ are observables. For that reason there exists the IF-observable $x:\mathcal{B}(R)\to\mathcal{F}$ given by $x(C)=\big(x^{\flat}(C),1_{\Omega}-x^{\sharp}(C)\big)$ for each $C\in\mathcal{B}(R)$ such that

$$x\big((-\infty,t)\big)=\big(x^{\flat}\big((-\infty,t)\big),1_{\Omega}-x^{\sharp}\big((-\infty,t)\big)\big)=\big(B^{\flat}(t),1_{\Omega}-B^{\sharp}(t)\big)=\mathbf{B}(t)$$

for each $t\in R$. □

4 Properties of the sum of IF-observables

In this section, we will show some properties of the sum of IF-observables.

Lemma 1. *Let $x,y:\mathcal{B}(R)\to\mathcal{F}$ be two IF-observables, and let S be a countable dense set in R. Denote*

$$(x+y)^{S}\big((-\infty,t)\big)=\bigvee_{s\in S}\Big[x\big((-\infty,s)\big)\wedge y\big((-\infty,t-s)\big)\Big],$$

then $(x+y)^{S}\big((-\infty,t)\big)=(x+y)\big((-\infty,t)\big)$ for each $t\in R$.

Proof. Let $t_n\nearrow t$, $t_n\in S$, $t\in R$. Evidently

$$\begin{aligned}\bigvee_n(x+y)^{S}\big((-\infty,t_n)\big)&=\bigvee_n\bigvee_{s\in S}\Big[x\big((-\infty,s)\big)\wedge y\big((-\infty,t_n-s)\big)\Big]\\&=\bigvee_{s\in S}\Big[x\big((-\infty,s)\big)\wedge\bigvee_n y\big((-\infty,t_n-s)\big)\Big]\\&=\bigvee_{s\in S}\Big[x\big((-\infty,s)\big)\wedge y\big((-\infty,t-s)\big)\Big]\\&=(x+y)^{S}\big((-\infty,t)\big).\end{aligned}$$

Let n be any integer. Then for each $s\in S$ there exists $r\in Q$ such that $s<r<s+\frac{1}{n}$. For this reason

$$\begin{aligned}x\big((-\infty,s)\big)\wedge y\Big(\Big(-\infty,t-\Big(s+\frac{1}{n}\Big)\Big)\Big)&\le x\big((-\infty,r)\big)\wedge y\big((-\infty,t-r)\big)\\x\big((-\infty,s)\big)\wedge y\Big(\Big(-\infty,t-\frac{1}{n}-s\Big)\Big)&\le x\big((-\infty,r)\big)\wedge y\big((-\infty,t-r)\big)\\\bigvee_{s\in S}\Big[x\big((-\infty,s)\big)\wedge y\Big(\Big(-\infty,t-\frac{1}{n}-s\Big)\Big)\Big]&\le\bigvee_{r\in Q}\Big[x\big((-\infty,r)\big)\wedge y\big((-\infty,t-r)\big)\Big]\\(x+y)^{S}\Big(\Big(-\infty,t-\frac{1}{n}\Big)\Big)&\le(x+y)\big((-\infty,t)\big).\end{aligned}$$

Then

$$\begin{aligned}(x+y)^S\big((-\infty,t)\big) &= \bigvee_n (x+y)^S\big((-\infty,t_n)\big) = \bigvee_n (x+y)^S\Big(\Big(-\infty,t-\frac{1}{n}\Big)\Big)\\ &\leq (x+y)\big((-\infty,t)\big).\end{aligned}$$

Similarly, we can show that $(x+y)\big((-\infty,t)\big) \leq (x+y)^S\big((-\infty,t)\big)$. Therefore

$$(x+y)^S\big((-\infty,t)\big) = (x+y)\big((-\infty,t)\big)$$

for each $t \in R$. □

Theorem 5. *For any IF-observables $x, y, z : \mathcal{B}(R) \to \mathcal{F}$ the following properties hold*

(i) $x + y = y + x$;

(ii) $(x + y) + z = x + (y + z)$;

(iii) $\alpha(x + y) = \alpha x + \alpha y$, $\alpha > 0$, $\alpha \in R$;

(iv) $x + 0_{\mathcal{F}} = x$,

where $\alpha x(A) = x\big(\{t : \alpha t \in A\}\big)$ for each $A \in \mathcal{B}(R)$ and zero IF-observable $0_{\mathcal{F}}$ is given by

$$0_{\mathcal{F}}(A) = \begin{cases} (1_\Omega, 0_\Omega), & \text{if } 0 \in A \\ (0_\Omega, 1_\Omega), & \text{if } 0 \notin A \end{cases}$$

for each $A \in \mathcal{B}(R)$.

Proof. (i) Let $t \in R$ and denote $S_t = \{t - r : r \in Q\}$. Then using Lemma 1 we obtain

$$\begin{aligned}(x+y)\big((-\infty,t)\big) &= \bigvee_{r\in Q}\Big[x\big((-\infty,r)\big) \wedge y\big((-\infty,t-r)\big)\Big]\\ &= \bigvee_{s\in S_t}\Big[x\big((-\infty,t-s)\big) \wedge y\big((-\infty,s)\big)\Big]\\ &= \bigvee_{s\in S_t}\Big[y\big((-\infty,s)\big) \wedge x\big((-\infty,t-s)\big)\Big]\\ &= (y+x)^{S_t}\big((-\infty,t)\big) = (y+x)\big((-\infty,t)\big).\end{aligned}$$

(ii) Calculate

$$\begin{aligned}\big((x+y)+z\big)\big((-\infty,t)\big) &= \bigvee_{r\in Q}\Big[(x+y)\big((-\infty,r)\big) \wedge z\big((-\infty,t-r)\big)\Big]\\ &= \bigvee_{r\in Q}\Big[\bigvee_{s\in Q}\Big[x\big((-\infty,s)\big) \wedge y\big((-\infty,r-s)\big)\Big] \wedge z\big((-\infty,t-r)\big)\Big]\\ &= \bigvee_{s\in Q} x\big((-\infty,s)\big) \wedge \Big[\bigvee_{r\in Q}\Big[y\big((-\infty,r-s)\big) \wedge z\big((-\infty,t-r)\big)\Big]\Big].\end{aligned}$$

But $t-r=t-s-(r-s)$ and if we denote $C_r=\{r-s : s\in Q\}$, then by Lemma 1 we obtain

$$\begin{aligned}
&\bigvee_{s\in Q} x\big((-\infty,s)\big)\wedge\Big[\bigvee_{r\in Q}\Big[y\big((-\infty,r-s)\big)\wedge z\Big(\big(-\infty,t-s-(r-s)\big)\Big)\Big]\Big]\\
&=\bigvee_{s\in Q} x\big((-\infty,s)\big)\wedge\Big[\bigvee_{q\in C_r}\Big[y\big((-\infty,q)\big)\wedge z\Big(\big(-\infty,t-s-q\big)\Big)\Big]\Big]\\
&=\bigvee_{s\in Q}\Big[x\big((-\infty,s)\big)\wedge(y+z)^{C_r}\big((-\infty,t-s)\big)\Big]\\
&=\bigvee_{s\in Q}\Big[x\big((-\infty,s)\big)\wedge(y+z)\big((-\infty,t-s)\big)\Big]=\big(x+(y+z)\big)\big((-\infty,t)\big).
\end{aligned}$$

(iii) Since $\alpha>0$ and $\alpha x(A)=x\big(\{t:\alpha t\in A\}\big)$ for each $A\in\mathcal{B}(R)$, then

$$\begin{aligned}
(\alpha x+\alpha y)\big((-\infty,t)\big) &= \bigvee_{r\in Q}\Big[(\alpha x)\big((-\infty,r)\big)\wedge(\alpha y)\big((-\infty,t-r)\big)\Big]\\
&= \bigvee_{r\in Q}\left[x\Big(\Big(-\infty,\frac{r}{\alpha}\Big)\Big)\wedge y\Big(\Big(-\infty,\frac{t}{\alpha}-\frac{r}{\alpha}\Big)\Big)\right]\\
&= (x+y)\Big(\Big(-\infty,\frac{t}{\alpha}\Big)\Big)=\big(\alpha(x+y)\big)\big((-\infty,t)\big).
\end{aligned}$$

(iv) Since

$$0_{\mathcal{F}}\big((-\infty,t-r)\big)=\begin{cases}(1_\Omega,0_\Omega), & \text{if } t\geq r\\ (0_\Omega,1_\Omega), & \text{if } t<r,\end{cases}$$

then

$$\begin{aligned}
(x+0_{\mathcal{F}})\big((-\infty,t)\big)&=\bigvee_{r\in Q}\Big[x\big((-\infty,r)\big)\wedge 0_{\mathcal{F}}\big((-\infty,t-r)\big)\Big]\\
&=\bigvee_{t<r}\Big[x\big((-\infty,r)\big)\wedge(0_\Omega,1_\Omega)\Big]\vee\bigvee_{r\leq t}\Big[x\big((-\infty,r)\big)\wedge(1_\Omega,0_\Omega)\Big]\\
&=\bigvee_{r\leq t}x\big((-\infty,r)\big)=x\big((-\infty,t)\big).
\end{aligned}$$

□

Remark 2. *Using properties of the sum of IF-observables, we can define the subtraction of two IF-observables x and y as $x-y=x+(-y)$, where $(-y)(A)=y\big(\{t:-t\in A\}\big)$ for each $A\in\mathcal{B}(R)$.*

5 Conclusion

In the paper, we defined the sum of IF-observables using a system of IF sets and max-min connectives. We proved some properties of this sum, too. Such IF-sets are the generalisation of fuzzy sets; we can say that the results are an extension of work [10].

Acknowledgements

This publication was supported by grant VEGA 2/0123/26.

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