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Choquet and Sugeno integrals and intuitionistic fuzzy integrals as aggregation operators

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Abstract: In this paper, a comparison of the Choquet and Sugeno integrals is presented and discussed in their fuzzy and intuitionistic fuzzy set forms.
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1 Introduction

In a variety of real applications, when necessary manipulate information, it is common to find obstacles due to the uncertainty which occurs when using inaccurate or imprecise data. Zadeh in 1965 propose the solution to the problem giving the definition of fuzzy set [1], and to complete the solution Michio Sugeno introduced the terms of fuzzy measure and fuzzy integral [2] as the most appropriate way to measure a certain degree of uncertainty, and these values that depend only on human subjectivity.

The particular function of the aggregation operators is for combining information when they can be mathematically formalized. Aggregation operators are aimed at reducing a set of numbers into a unique representative value. It is important to note that any aggregation or fusion process is based on numerical aggregation. The operator considers that the input variables are the information sources to combine and the output is the aggregation results.

We can mention aggregation operators, like the arithmetic mean, geometric mean, weighted mean, OWA, OWA weighted, harmonic mean, Choquet integral [3] and Sugeno integral [2].

The focus of this paper is the aggregation operators that use measures, in particular the Choquet and Sugeno integrals is their fuzzy and intuitionistic fuzzy set forms (for intuitionistic fuzzy sets see, e.g., [4]).

2 Sugeno measures and fuzzy integrals

2.1 Monotonic measures

Monotonic measures are often referred to in the literature as fuzzy measures. This name is somewhat confusing, since no fuzzy sets are involved in the definition of such measures.

To avoid this confusion, the term "fuzzy measures" should be reserved to measures (additive or non-additive) that are defined on families of fuzzy sets. Given a universe of discourse X and a nonempty family C of subsets of X, a monotone measure μ on $\langle X, C\rangle$ is a function of the form μ: $C \to [0, \infty]$. It is assumed that the universal set X is finite and that $C = P(X)$. That is, it is normally assumed that the monotonic measures of concern are sets of functions μ: $P(X) \to [0, 1]$.

Definition 1: A monotonic set measure μ on space X is a mapping μ: $P(X) \to [0, 1]$ such that the following properties hold [5], [6]:

1) $\mu (\phi) = 0$
2) $\mu (X) = 1$
3) For all A, $B \in P(X)$, if $A \subseteq B$, then $\mu(A) \leq \mu(B)$

2.2 Sugeno measures

The Sugeno λ-measures are special types of monotonic measures [7, 8] defined as follows.

Definition 2: Let $X = \{x_1, \ldots, x_n\}$ be any finite set. A discrete fuzzy measure on X is a function μ: $2^x \to [0,1]$ with the following properties:

1) $\mu(\phi) = 0$ and $\mu(X) = 1$;
2) given A, $B \in 2^X$ if $A \subset B$, then $\mu(A) \leq \mu(B)$ (monotonicity property).

The set X is considered to contain the identifiers of sources of information (features, sensors, etc.) For a subset $A \subseteq X$, $\mu(A)$ is considered to be the relevance degree of this subset of information.

Definition 3: Let $X = \{x_1, \ldots, x_n\}$ be any finite set and let $\lambda \in (-1,+\infty)$. A Sugeno λ-measure is a function μ from 2^X to [0, 1] with the following properties:

1) $\mu(X) = 1$;
2) if A, $B \subseteq X$ with $A \cap B = \phi$, then

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\, \mu(A)\mu(B), \tag{1}$$

where $\lambda > -1$.

Equation (1) is usually called the λ-rule. When X is a finite set and the values $\mu(\{x\})$, called fuzzy densities, are given for each $x \in X$, these densities are interpreted as the importance of the individual information sources. The measure of a set A of information sources is interpreted as the importance of that subset of sources towards answering a particular question or problem (such as class membership) [8].

The value of $\mu(A)$ for each $A \subset P(X)$, can be determined by the recurrent application of the λ-rule. This value can be expressed in the following way.

$$\mu(A) = \frac{\left[\prod_{x \in A} (1 + \lambda\mu(\{x\}))\right]}{\lambda} \tag{2}$$

We can observe that given the values of the fuzzy densities $\mu(\{x\})$ for each $x \in X$, the value of λ can be determined by using the constraint $\mu(\{x\}) = 1$. Applying this constraint to (2) results in the following expression:

$$\lambda + 1 = \prod_{i=1}^{n} (1 + \lambda\mu(\{x_i\})) \tag{3}$$

The parameter λ is specific to this class of measures and can be computed from (3) once the densities are known. Sugeno proved that this polynomial has a real root greater than -1 and several researchers have observed that this polynomial equation is easily solved numerically. By property (1), specifying the n different densities, thereby reducing the number of free parameters from $2^n - 2$ to n [7], the value of parameter λ is determined with the help of the following theorem [9].

Theorem 1: Let $\mu(\{x\}) < 1$ for each $x \in X$ and let $\mu(\{x\}) > 0$ for at least two elements of X. Then (3) determines a unique parameter λ in the following way.

- If $\Sigma_{x \in X}\, \mu(\{x\}) < 1$, then λ is equal to a unique root of the equation in the interval $(0, \infty)$.
- If $\Sigma_{x \in X}\, \mu(\{x\}) = 0$, then $\lambda = 0$; that is the unique root of the equation.
- If $\Sigma_{x \in X}\, \mu(\{x\}) > 1$, then λ is equal to a unique root of the equation in the interval $(-1, 0)$.

According to Theorem 1, three situations can be distinguished:

- If $\Sigma_{x \in X}\, \mu(\{x\}) < 1$, which means that μ qualifies as a lower probability, $\lambda > 0$.
- If $\Sigma_{x \in X}\, \mu(\{x\}) = 1$, which means that μ is a classical probability, $\lambda = 0$.
- If $\Sigma_{x \in X}\, \mu(\{x\}) > 1$, which means that μ qualifies as an upper probability, $\lambda < 0$.

When μ is a λ-fuzzy measure, the values of $\mu(A_i)$ can be computed by means of (2), or recursively, after a descendent reordering of the sets X and $\mu(\{x\})$, with respect to the values of the elements of set X [10].

2.3 Sugeno integral

Using the concept of fuzzy measures, Sugeno proposed the concept of fuzzy integral as nonlinear functions defined with respect to fuzzy measures as λ-fuzzy measure. One can interpret the fuzzy integral as finding the maximum degree of similarity between the target and the expected value as shown in (4).

The Sugeno integral generalizes "max-min" operators.

$$Sugeno_{\mu}(x_1, x_2, \ldots, x_n) = \max_{i=1,\ldots,n} \left(\min\left(f(x_{\sigma(i)}), \mu(A_{\sigma(i)}) \right) \right) \tag{4}$$

with $A_0 = 0$, where $x_{\sigma(i)}$ indicates the indices that must be permuted as shown in (5), and where $A_{\sigma(i)} = \{x_{\sigma(i)}, \ldots, x_{\sigma(n)}\}$.

$$0 \leq f(x_{\sigma(1)}) \leq f(x_{\sigma(2)}) \leq \ldots \leq f(x_{\sigma(n)}) \leq 1 \tag{4}$$

The Sugeno integral can be applied to solve several problems, which consider a finite set of n elements $X = \{x_1, \ldots, x_n\}$.

2.4 Intuitionistic fuzzy Sugeno operator

$$IFSu_{(\mu_A,\nu_A)}(x_1, x_2, \ldots, x_n) = \left(\max_{i=1,\ldots,n} \left(\min \left(f(x_{\sigma(i)}), \mu_A(A_{\sigma(i)}) \right) \right), \max_{i=1,\ldots,n} \left(\min \left(f(x_{\sigma(i)}), \nu_A(A_{\sigma(i)}) \right) \right) \right)$$

2.5 Choquet integral

The Choquet integral generalizes "additive‖ operators, and it was created by the French mathematician Gustave Choquet in 1953 [3]. The Choquet integral with respect to a fuzzy measure is a very popular data aggregation approach. The generalized Choquet integral with respect to a signed fuzzy measure can act as an aggregation tool, which is especially useful in many applications. The Choquet integral can be calculated using

$$(C)\int f d\mu = \sum_{i=1}^{n} \left[f\left(x_{\sigma(i)}\right) - f\left(x_{\sigma(i-1)}\right) \right] \bullet \mu\left(A_{\sigma(i)}\right) \tag{6}$$

where $f(x_{\sigma(0)}) = 0$ and $x_{\sigma(i)}$ indicates the indices that must be permuted as shown in (7), and where $A_{\sigma(i)} = \left\{x_{\sigma(i)}, \ldots, x_{\sigma(n)}\right\}$.

$$0 \leq f(x_{\sigma(1)}) \leq f(x_{\sigma(2)}) \leq \ldots \leq f(x_{\sigma(n)}) \leq 1 \tag{7}$$

2.6 Intuitionistic fuzzy Choquet integral

The intuitionistic fuzzy Choquet integral has the form

$$(IFC)\int fd(\mu_A \nu_A) = \left\langle (C)\int fd(\mu_A), (C)\int fd(\nu_A) \right\rangle$$

(cf. [11]).

3 Conclusions

Future research consists in fusing other databases and the assignation automatic the fuzzy densities through a fuzzy system. We will research the aggregation operators intuitionistic fuzzy Sugeno and intuitionistic fuzzy Choquet integral.

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