Notes on Intuitionistic Fuzzy Sets
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Inclusion measure for intuitionistic fuzzy sets and embedding of intervals

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Abstract: Comparison of measuring the degree of inclusion for two intuitionistic fuzzy sets (IF-sets) and measuring the degree of embedding of two intervals is considered. Embedding is understood as the classical inclusion of intervals. Inclusion of IF-sets is based on a specific order. In case that the nonmebership function does not exceed the membership function in an IF set, and we replace formally the IF-set by an interval-valued fuzzy set, then the inclusion of IF-sets corresponds to an embedding of interval-valued sets. The embedding measure for interval-valued fuzzy sets was defined previously and we compare the concept of embedding with the inclusion of IF-sets.
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1 Introduction

Intuitionistic fuzzy sets and interval-valued fuzzy sets are two of many generalizations of fuzzy sets, which were introduced by Zadeh in 1965 (see [18]).

Intuitionistic fuzzy sets (IF-sets) were introduced by Atanassov in 1983 [1].

$$IFS(A) = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\},$$

where $\mu_A, \nu_A : X \to [0,1]$ are functions determining the degrees of membership and of non-membership of element $x \in X$ to A and

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

The degree of uncertainty is denoted by π_A and $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$.

Interval-valued fuzzy sets (IVF-sets) were introduced independently by Zadeh [19], Grattan-Guinness [8], Jahn [10], Sambuc [12], in the following way:

$$IVFS(X) = \{\langle x, M_A(x)\rangle : x \in X\},$$

where $M_A(x) \subseteq [0,1]$ is a closed interval and expresses that the degree of membership of an element $x \in X$ is included in that interval. The IVF-sets represent a special case of lattice-valued fuzzy sets. They were introduced by Goguen in 1967 (see [7]) and it is possible to describe them by a lattice $(L^1, \leq_{L^1})$ where $L^1 = \{[\underline{a}, \bar{a}] : [\underline{a}, \bar{a}] \in [0,1]^2$ and $\underline{a} \leq \bar{a}\}$, $[\underline{a}, \bar{a}] \leq_{L^1} [\underline{b}, \bar{b}]$ if and only if $\underline{a} \leq \underline{b}$ and $\bar{a} \leq \bar{b}$, $[\underline{a}, \bar{a}] \wedge [\underline{b}, \bar{b}] = [\min(\underline{a}, \underline{b}), \min(\bar{a}, \bar{b})]$, $[\underline{a}, \bar{a}]^c = [1-\bar{a}, 1-\underline{a}]$, with $0_{L^1} = [0,0]$ and $1_{L^1} = [1,1]$. The order $\leq_{L^1}$ is known as the lattice order.

It is possible to describe IF-sets by the lattice $(L^*, \leq_{L^*})$ where $L^* = \{[\underline{a}, \bar{a}] \in [0,1]^2 : \underline{a} + \bar{a} \leq 1\}$, $[\underline{a}, \bar{a}] \leq_{L^*} [\underline{b}, \bar{b}] \Leftrightarrow \underline{a} \leq \underline{b}$ and $\bar{a} \geq \bar{b}$, $[\underline{a}, \bar{a}] \wedge [\underline{b}, \bar{b}] = [\min(\underline{a}, \underline{b}), \max(\bar{a}, \bar{b})]$, $[\underline{a}, \bar{a}]^c = [\bar{a}, \underline{a}]$, with $0_{L^*} = [0,1]$ and $1_{L^*} = [1,0]$. The IF-sets $\emptyset_{IFS}$ and X_{IFS} are defined as $\emptyset_{IFS} = \{\langle x, 0, 1\rangle : x \in X\}$ and $X_{IFS} = \{\langle x, 1, 0\rangle : x \in X\}$. Every IF-set $\langle x, \mu_A(x), \nu_A(x)\rangle$ can be represented as an IVF-set (see [3, 6]) for every $x \in X$ such as

$$\langle x, [\mu_A(x), 1 - \nu_A(x)]\rangle.$$

In general the inclusion of fuzzy sets A, B is defined as

$$A \subseteq_F B \Leftrightarrow A(x) \leq_F B(x), \text{ for all } x \in X$$

where $\leq_F$ is the corresponding order for fuzzy sets. In our case

$$A \subseteq_{IVF} B \Leftrightarrow A(x) \leq_{L^1} B(x), \text{ for all } x \in X,$$

$$A \subseteq_{IF} B \Leftrightarrow A(x) \leq_{L^*} B(x), \text{ for all } x \in X.$$

2 Inclusion measures

There are more approaches to measuring the degree of inclusion for fuzzy sets. In the original works by Atanassov (see e.g. [2]) an IF-set $\langle x, \mu_A(x), \nu_A(x)\rangle$ is a subset of $\langle x, \mu_B(x), \nu_B(x)\rangle$ if and only if for any x there is $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$. This relation is obviously crisp. Later some generalizations appeared, alowing it to obtain more degrees than just 0 or 1. The best known axiomatizations were presented by Kitainik [11], Sinha and Dougherty [13], or Young [16].

Cornelis and Kerre [5] extended one of the approaches and proposed the axioms for the inclusion measure Inc what is a mapping $IFS(X) \times IFS(X) \to L^*$.

Definition 1. *The mapping* $Inc : IFS(X) \times IFS(X) \to L^*$ *is an inclusion measure in the sense of Cornelis and Kerre, if the following properties hold:*

(CK1) $Inc(A, B) = Inc(B^c, A^c)$*;*

(CK2) $Inc(A, B \cap C) = \inf(Inc(A, B), Inc(A, C))$*;*

(CK3) $Inc(A, B) = Inc(P(A), P(B))$*, where* P *is a* $IFS(X) \to IFS(X)$ *mapping defined for* $x \in X$ *as* $P(A)(x) = A(p(x))$*, and* p *is a permutation of* X *;*

(CK4a) $Inc(A, B) = [1, 0]$ *if and only if* $A \subseteq_{IFS} B$*;*

(CK4b) $Inc(A, B) = [0, 1]$ *if and only if* $(\exists x \in X : A(x) = [1, 0]$ *and* $B(x) = [0, 1])$*;*

(CK4c) *If* $A, B \in F(X)$*, then* $Inc(A, B) \in D$ *(*$D = \{[a, b] \in L^* : a + b = 1\}$*).*

Despite the intuition that the values of the degree of the inclusion measure should be also uncertain enough, namely the element of L^*, the inclusion measure $Inc : IFS(X) \times IFS(X) \to [0, 1]$ is preferred. The following axiomatization based on axioms of Young was proposed by Vlachos and Sergiadis [14], and Xie, Han and Mi [15].

Definition 2. *The mapping* $Inc : IFS(X) \times IFS(X) \to [0, 1]$ *is an inclusion measure in the sense of Young, if the following properties hold:*

(Y1) $Inc(A, B) = 1$ *if and only if* $A \subseteq_{IFS} B$

(Y2) *If* $A^c \subseteq_{IFS} A$ *(i.e.* $\mu_A(x) \geq \nu_A(x)$ *for all* $x \in X$*), then* $Inc(A, A^c) = 0$ *if and only if* $A = X_{IFS}$ *]]*

(Y3a) *If* $A \subseteq_{IFS} B \subseteq_{IFS} C$*, then* $Inc(C, A) \leq Inc(B, A)$*]*

(Y3b) *if* $A \subseteq_{IFS} B$*, then* $Inc(C, A) \leq Inc(C, B)$*.*

In 2007, Zhang et al. [20] proposed another axiomatization, which is intended for IF-sets and also for IVF-sets. The axioms are formulated for a general case of a lattice L.

Definition 3. *The mapping* $Inc : L \times L \to [0, 1]$ *is a hybrid monotonic inclusion measure on a lattice* L *in the sense of Zhang, if the following properties hold:*

(Z1) $0 \leq Inc(a, b) \leq 1$*, for all* $a, b \in L$

(Z2) $Inc(a, b) = 0$ *if* $a = 1_L$ *and* $b = 0_L$

(Z3) $Inc(a, b) = 1$ *if and only if* $a \leq_L b$*, for all* $a, b \in L$

(Z4) *if* $a \leq_L b$*, then* $Inc(c, a) \leq Inc(c, b)$ *and* $Inc(b, c) \leq Inc(a, c)$ *for all* $c \in L$

In 2008 Yu and Luo [17] proposed another axiomatization.

Definition 4. *The mapping* $Inc : IFS(X) \times IFS(X) \to [0,1]$ *is an inclusion measure in the sense of Yu and Luo, if the following properties hold:*

(YL1) *If* $A \subseteq_{IFS} B$*, then* $Inc(A,B) = 1$*;*

(YL2) $Inc(X_{IFS}, \emptyset_{IFS}) = 0$*;*

(YL3) *If* $A \subseteq_{IFS} B \subseteq_{IFS} C$*, then* $Inc(C,A) = \min(Inc(B,A), Inc(C,B))$.

There exist also other approaches on how to measure inclusion of IF-sets. The one based on possible and necessary inclusion is given by Grzegorzewski in [9].

In 2023 Bouchet et al. [4] defined the measure of embedding for interval-valued fuzzy sets. Embedding of intervals is understood as classical inclusion of intervals, so

$$A \subseteq B \Leftrightarrow A(x) \subseteq B(x), \text{ for all } x \in X.$$

Definition 5. *The mapping* $Emb : IVFS(X) \times IVFS(X) \to [0,1]$ *is an embedding measure for interval-valued fuzzy sets, if the following properties hold:*

(E1) $Emb(A,B) = 1$ *if and only if* $A \subseteq B$

(E2) *If* $A(x) \cap B(x) = \emptyset$ *for all* $x \in X$*, then* $Emb(A,B) = 0$

(E3) *If* $Emb(B,C) = 1$*, then* $Emb(A,B) \leq Emb(A,C)$ *for every* $A \in IVFS(X)$

(E4) *If* $Emb(A,B) = Emb(B,C) = 1$*, then* $Emb(C,A) \leq Emb(B,A)$

One kind of construction of embedding, proposed in [4], is based on a mapping $E : L([0,1]) \times L([0,1]) \to [0,1]$ where $L([0,1])$ is the set of all closed subintervals of $[0,1]$. The axioms for the measure E are analogous to the axioms for a measure Emb. The trivial embedding measure of intervals $E_0 : L([0,1]) \times L([0,1]) \to [0,1]$ is defined as follows

$$E_0(a,b) = \begin{cases} 1, & \text{if } a \subseteq b, \\ 0, & \text{otherwise.} \end{cases}$$

The followings mappings are also embedding measures of intervals

$$E_w(a,b) = \begin{cases} 1, & \text{if } w(a) = 0, a \cap b \neq \emptyset, \\ 0, & \text{if } w(a) = 0, a \cap b = \emptyset, \\ \dfrac{w(a \cap b)}{w(a)}, & \text{if } w(a) \neq 0, \end{cases}$$

$$E_\lambda(a,b) = \begin{cases} 1, & \text{if } a \subseteq b, \\ 0, & \text{if } a \cap b = \emptyset, \\ \lambda, & \text{otherwise,} \end{cases}$$

where $w(a)$ is the width of the interval a and $\lambda \in [0,1)$ is arbitrary.

The construction for $Emb^E_{\mathcal{M}} : IVFS(X) \times IVFS(X) \to [0,1]$ is defined as follows

$$Emb^E_{\mathcal{M}}(A,B) = \mathcal{M}_{x\in X} E(A(x), B(x)),$$

where $\mathcal{M} : [0,1]^n \to [0,1]$ is a one-strict aggregation function, i.e., an increasing mapping in each argument with $\mathcal{M}(0,0,\ldots,0) = 0$ and $\mathcal{M}(1,1,\ldots,1) = 1$, where $\mathcal{M}(x_1, x_2, \ldots, x_n) = 1$ if and only if $x_i = 1$ for all $i = 1,2,\ldots,n$. Examples of one-strict aggregation functions are the arithmetic mean, the geometric mean and the harmonic mean or any t-norm generated by their aggregation functions (for example the minimum t-norm or the product t-norm).

Similarly, we define the inclusion measure of intervals $I : L^* \times L^* \to [0,1]$, where the axioms of I will be analogous to the corresponding axioms of the inclusion measure $Inc : IFS(X) \times IFS(X) \to [0,1]$. Assume the following mapping $I_0 : L^* \times L^* \to [0,1]$

$$I_0(a,b) = \begin{cases} 1 & \text{if } a \subseteq_{L^*} b, \\ 0 & \text{otherwise,} \end{cases}$$

which is a candidate for trivial inclusion measure of intervals.

It is clear that $I_0(a,b) = 1$ if and only if $a \subseteq_{L^*} b$, so the axioms (Y1) and (YL1) hold. Obviously $I_0([1,0],[0,1]) = 0$, so the axiom (YL2) is satisfied.

But I_0 is not an inclusion measure in the sense of Young. For example we can take $a^c = [0.6, 0.4]^c = [0.4, 0.6]$, $[0.4, 0.6] \subseteq_{L^*} [0.6, 0.4]$ and $I_0([0.6, 0.4], [0.4, 0.6]) = 0$ despite $[0.6, 0.4] \neq [1, 0]$, so the axiom (Y2) does not hold.

Assume $a, b, c \in L^*$ where $a \subseteq_{L^*} b \subseteq_{L^*} c$ and $a = [\underline{a}, \bar{a}], b = [\underline{b}, \bar{b}], c = [\underline{c}, \bar{c}]$. So $\underline{a} \le \underline{b} \le \underline{c}$ and $\bar{a} \ge \bar{b} \ge \bar{c}$. The case $c \subseteq_{L^*} a$, i.e. $\underline{c} \le \underline{a}$ and $\bar{c} \ge \bar{a}$, can occur when $\underline{c} = \underline{a} = \underline{b}$ and $\bar{c} = \bar{a} = \bar{b}$, so $b \subseteq_{L^*} a$ and $c \subseteq_{L^*} b$. The following holds

$$I_0(c,a) = 1 \Leftrightarrow I_0(b,a) = 1 \text{ and } I_0(c,b) = 1,$$

so $I_0(c,a) = \min(I_0(b,a), I_0(c,b))$. If $c \not\subseteq_{L^*} a$, then $I_0(c,a) = 0$ and $\underline{c} > \underline{a}$ or $\bar{c} < \bar{a}$. Assume the case $\underline{c} > \underline{a}$. Then there are cases

$$\left.\begin{array}{l} \underline{a} < \underline{b} < \underline{c} \\ \underline{a} < \underline{b} = \underline{c} \end{array}\right\} \Rightarrow I_0(b,a) = 0$$

$$\underline{a} = \underline{b} < \underline{c} \ \ \Rightarrow I_0(c,b) = 0$$

and $I_0(c,a) = \min(I_0(b,a), I_0(c,b))$. Analogic result we can get for upper bounds. So the axiom (YL3) holds. We showed that I_0 is the inclusion measure for intervals in the sense of Yu and Luo.

The consequence of the proof in the previous paragraph is that if $a \subseteq_{L^*} b \subseteq_{L^*} c$ for $a, b, c \in L^*$, then $I(c,a) \le I(b,a)$, then the axiom (Y3a) holds. Suppose that $a \subseteq_{L^*} b$ for $a, b \in L^*$, so $\underline{a} \le \underline{b}$ and $\bar{a} \ge \bar{b}$. Assume $c \in L^*$ such as $I(c,a) = 0$, then the inequality $I(c,a) \le I(c,b)$ always holds. Assume $c \in L^*$ such as $I(c,a) = 1$, so $\underline{c} \le \underline{a}$ and $\bar{c} \ge \bar{a}$. Because $I(A,B) = 1$, then $\underline{c} \le \underline{a} \le \underline{b}$ and $\bar{c} \ge \bar{a} \ge \bar{b}$ and $I(C,B) = 1$, so the inequality of the axiom (Y3b) holds.

It is easy to show that the axioms (Z1)–(Z4) also hold. In the following section, we will compare the measures E_0 and I_0.

3 Main results

We study the relationship between $I_0(a,b) = 1$ and E_0 for $a, b \in L^*$. Since the value of $E_0(a,b)$ is defined only for intervals a, b, i.e., $a, b \in L([0,1])$, we assume the following transformation

$$a^T = [\underline{a}, \bar{a}]^T = [\underline{a}, 1 - \bar{a}],$$

for the case when a is not an interval. It is based on the bijection between IF-sets and IVF-sets, see [3, 6]. We assume also a complement in the sense of IF-sets

$$a^c = [\underline{a}, \bar{a}]^c = [\bar{a}, \underline{a}].$$

It is clear that $L([0,1]) \subseteq L^*$, so $I_0(a,b)$ is defined also for $a, b \in L([0,1])$.

Proposition 1. *If $a, b \in L([0,1])$, then $I_0(a,b) = 1$ if and only if $E_0(b,a) = 1$.*

Proof. Assume $a, b \in L([0,1])$, so $\underline{a} \le \bar{a}$ and $\underline{b} \le \bar{b}$. If $I_0(a,b) = 1$, then $\underline{a} \le \underline{b}$ and $\bar{a} \ge \bar{b}$. It is

$$\underline{a} \le \underline{b} \le \bar{b} \le \bar{a}$$

and $E_0(b,a) = 1$. □

Proposition 2. *If $a, b \notin L([0,1])$, then $I_0(a,b) = 1$ if and only if $E(a^c, b^c) = 1$.*

Proof. Assume $a, b \notin L([0,1])$, so $\underline{a} > \bar{a}$ and $\underline{b} > \bar{b}$. If $I_0(a,b) = 1$, then $\underline{a} \le \underline{b}$ and $\bar{a} \ge \bar{b}$. It is

$$\bar{b} \le \bar{a} < \underline{a} \le \underline{b}$$

so $[\bar{a}, \underline{a}] \subseteq [\bar{b}, \underline{b}]$ and $E_0(a^c, b^c) = 1$. □

The elements $a \notin L([0,1])$ and $b \in L([0,1])$ with $I_0(a,b) = 1$ do not exist. Let $a = [\underline{a}, \bar{a}]$ and $b = [\underline{b}, \bar{b}]$ with $\underline{a} > \bar{a}$ and $\underline{b} \le \bar{b}$. If $a \subseteq_{L^*} b$, then $\underline{a} \le \underline{b} \le \underline{b} \le \bar{a}$, which is a contradiction with $\bar{a} < \underline{a}$.

Now assume $a \in L([0,1])$ and $b \notin L([0,1])$ with $a \subseteq_{L^*} b$, so $\underline{a} \le \underline{b}$, $\bar{a} \ge \bar{b}$, $\underline{a} \le \bar{a}$ and $\underline{b} > \bar{b}$. There are several cases for the appropriate ordering of $\underline{a}, \bar{a}, \underline{b}, \bar{b}$, where $\underline{b} \ne \bar{b}$:

C1 $\underline{a} \le \bar{b} < \underline{b} \le \bar{a}$

C2 $\underline{a} \le \bar{b} \le \bar{a} \le \underline{b}$

C3 $\bar{b} \le \underline{a} \le \underline{b} \le \bar{a}$

C4 $\bar{b} \le \underline{a} \le \bar{a} \le \underline{b}$

For the cases C1 and C4 there is a relationship with E_0.

Proposition 3. *Let $a \in L([0,1])$ and $b \notin L([0,1])$. If $\underline{a} \le \bar{b} < \underline{b} \le \bar{a}$, then $I_0(a,b) = 1$ and $E_0(b^c, a) = 1$. If $\bar{b} \le \underline{a} \le \bar{a} \le \underline{b}$, then $I_0(a,b) = 1$ and $E_0(a, b^c) = 1$.*

Proof. Straightforward. □

For every case (C1)–(C4) there are some subcases, some of them coinciding. For example the subcase $\underline{a} = \bar{b} < \underline{b} = \bar{a}$ is included in every case (C1)–(C4). We summarize these subcases in the Table 1. It is obvious that for these subcases it holds that $E_0(b^c, a) = 0$ or $E_0(a, b^c) = 0$.

Subcase	C1	C2	C3	C4
$\underline{a} = \bar{b} < \underline{b} = \bar{a}$	✓	✓	✓	✓
$\bar{b} = \underline{a} < \underline{b} < \bar{a}$	✓	✓		
$\bar{b} < \underline{a} < \underline{b} = \bar{a}$		✓		✓
$\bar{b} < \underline{a} = \underline{b} = \bar{a}$		✓		✓
$\underline{a} = \bar{b} < \bar{a} < \underline{b}$			✓	✓
$\underline{a} < \bar{b} < \bar{a} = \underline{b}$	✓		✓	
$\underline{a} = \bar{b} = \bar{a} < \underline{b}$			✓	✓

Table 1. The subcases included in several cases

Few subcases are included only in (C2) or (C3):

(i) $\bar{b} < \underline{a} < \underline{b} < \bar{a}$ in (C2)

(ii) $\bar{b} < \underline{a} = \underline{b} < \bar{a}$ in (C2)

(I) $\underline{a} < \bar{b} < \bar{a} < \underline{b}$ in (C3)

(II) $\underline{a} < \bar{b} = \bar{a} < \underline{b}$ in (C3)

We provide some values of E_0 for some pairs of elements of L^* belonging to the subcases mentioned above in the following Tables 2 and 3. In the columns with pairs there are corresponding values of E for some embedding measure of intervals. In the case where $E = E_0$, then $? = \{.\} = 0$. In the other case, 0 means that the intersection of intervals is empty, $\{.\}$ means that the intervals have one common point and we assume some specific value of E for this situation, and the symbol $?$ means, that the value is not specific and depends on the chosen E. It can happen that $\{.\} = 0$ or $? = 0$.

	$a, b \in L^*, a \subseteq_{L^*} b$	(a, b^T)	(b^T, a)	(a, b^c)	(b^c, a)	(a^T, b^T)	(b^T, a^T)	(a^T, b^c)	(b^c, a^T)
(i)	$[0.3, 0.5], [0.4, 0.1]$	?	?	?	?	?	?	?	?
	$[0.3, 0.6], [0.5, 0.1]$	?	?	?	?	0	0	1	?
	$[0.3, 0.6], [0.4, 0.1]$	?	?	?	?	$\{.\}$	$\{.\}$	1	?
	$[0.2, 0.4], [0.3, 0.1]$	?	?	?	?	?	?	?	?
	$[0.2, 0.7], [0.6, 0.1]$	?	?	?	?	0	0	1	?
(ii)	$[0.3, 0.5], [0.3, 0.2]$	1	?	$\{.\}$	$\{.\}$	1	?	$\{.\}$	$\{.\}$
	$[0.3, 0.6], [0.3, 0.2]$	1	?	$\{.\}$	$\{.\}$	1	?	$\{.\}$	$\{.\}$
	$[0.3, 0.7], [0.3, 0.2]$	1	?	$\{.\}$	$\{.\}$	1	?	?	1
	$[0.3, 0.4], [0.3, 0.2]$	1	?	$\{.\}$	$\{.\}$	1	?	$\{.\}$	$\{.\}$

Table 2. The subcases for (C2)

	$a,b \in L^*, a \subseteq_{L^*} b$	(a,b^T)	(b^T,a)	(a,b^c)	(b^c,a)	(a^T,b^T)	(b^T,a^T)	(a^T,b^c)	(b^c,a^T)
(I)	$[0.1,0.3],[0.4,0.2]$	0	0	?	?	?	?	?	1
	$[0.1,0.5],[0.6,0.3]$	0	0	?	?	0	0	?	?
	$[0.1,0.5],[0.6,0.4]$	0	0	?	?	0	0	?	?
	$[0.1,0.4],[0.6,0.3]$	0	0	?	?	$\{.\}$	$\{.\}$	?	1
	$[0.1,0.3],[0.6,0.2]$	0	0	?	?	?	?	?	1
(II)	$[0.2,0.3],[0.4,0.3]$	0	0	$\{.\}$	$\{.\}$	?	1	?	1
	$[0.2,0.3],[0.6,0.3]$	0	0	$\{.\}$	$\{.\}$	?	1	?	1
	$[0.2,0.3],[0.7,0.3]$	0	0	$\{.\}$	$\{.\}$	?	1	?	1

Table 3. The subcases for (C3)

We can notice that if $E(x,y) = 1$ and $E(y,x) =?$, then $x \subset y$. For the subcase (ii) it is obvious that $a \subset b^T$ and $a^T \subset b^T$. For the subcase (II) there is $b^T \subset a^T$ and $b^c \subset a^T$. We can notice that in the subcases (I) and (II) there is $a \cap b^T = \emptyset$. However, it is not possible to characterize cases (C1)–(C4) in one way.

Similarly, we obtain the relationships between $I_0(a,b) = 0$ and E_0.

Proposition 4. *If $a,b \in L([0,1])$, then $I_0(a,b) = 0$ if and only if $E_0(b,a) = 0$.*

Proof. Assume $a,b \in L([0,1])$, so $\underline{a} \leq \bar{a}$ and $\underline{b} \leq \bar{b}$. If $I_0(a,b) = 0$, then one of the following cases holds:

- $\underline{a} > \underline{b}, \bar{a} \geq \bar{b}$, specifically $\underline{b} < \underline{a} \leq \bar{b} \leq \bar{a}$ or $\underline{b} \leq \bar{b} < \underline{a} \leq \bar{a}$
- $\underline{a} \leq \underline{b}, \bar{a} < \bar{b}$, specifically $\underline{a} \leq \underline{b} \leq \bar{a} < \bar{b}$ or $\underline{a} \leq \bar{a} < \underline{b} \leq \bar{b}$
- $\underline{a} > \underline{b}, \bar{a} < \bar{b}$, specifically $\underline{b} < \underline{a} \leq \bar{a} < \bar{b}$

In all cases $E_0(b,a) = 0$. □

Proposition 5. *If $a,b \notin L([0,1])$, then $I_0(a,b) = 0$ if and only if $E_0(a^c,b^c) = 0$.*

Proof. Analogous to the Proof of Proposition 4. □

Proposition 6. *If $a \notin L([0,1])$ and $b \in L([0,1])$, then $I_0(a,b) = 0$.*

Proof. We have shown that there are no elements $a \notin L([0,1])$ and $b \in L([0,1])$ such as $I_0(a,b) = 1$. □

Proposition 7. *Let $a \in L([0,1])$ and $b \notin L([0,1])$. Then $I_0(a,b) = 0$ if and only if $E_0(a,b^c) = E_0(b^c,a) = 0$.*

Proof. Assume $a \in L([0,1])$ and $b \notin L([0,1])$, so $\underline{a} \leq \bar{a}$ and $\underline{b} > \bar{b}$. If $a \not\subseteq_{L^*} b$, then one of the following cases holds:

- $\underline{a} \leq \underline{b},\ \bar{a} < \bar{b}$, then $\underline{a} \leq \bar{a} < \bar{b} < \underline{b}$ and $E_0(a,b^c) = E_0(b^c,a) = 0$
- $\underline{a} > \underline{b},\ \bar{a} \geq \bar{b}$, then $\bar{b} < \underline{b} < \underline{a} \leq \bar{a}$ and $E_0(a,b^c) = E_0(b^c,a) = 0$
- $\underline{a} > \underline{b},\ \bar{a} < \bar{b}$, then $\underline{b} < \underline{a} \leq \bar{a} < \bar{b}$ and it is a contradiction with $\underline{b} > \bar{b}$. □

4 Conclusions

We have presented an overview of inclusion measures for intuitionistic fuzzy sets and studied some properties of these measures. There are two general attitudes to this topic – either an inclusion or an embedding. Each of these can be used in different circumstances and reflects a different aspect. This is usually dependent on the context in which the fuzzy set is used and so the proper embedding also depends on this context.

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