On Reassessment of Expert Evaluations in the Case of Intuitionistic Fuzziness

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Abstract

In the present paper an Open Problem proposed by K. Atanassov is completely solved. As a result a possible approach for correction of inconsistent expert estimates is provided. The solution is constructed by selecting a specific p-intuitionistic fuzzy set and then mapping it to the traditional interpretation triangle of the intuitionistic fuzzy sets.

Keywords: Intuitionistic fuzzy set, evaluation, trinomial equation

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1 Introduction

The Fuzzy set theory was first introduced by Zadeh ([1]) as an appropriate mathematical instrument for the description of uncertainty observed in nature (see e.g. [2],[3]). One of the extensions of the notion Fuzzy set is the so-called Intuitionistic Fuzzy set introduced by K. Atanassov (see [4]). We will briefly remind some of the basic definitions and notions.

Let E be a universe set (i.e. the set of all the (*relevant*) elements that will be considered), $A \subset E$, $\mu_A(x) \to [0,1]$ and $\nu_A(x) \to [0,1]$ are mappings reflecting the degree of membership (belongingness) and non-membership (non-belongingness) of the element x from E to the set A, respectively, such that for every x it is fulfilled that

$$\mu_A(x) + \nu_A(x) \leq 1 \tag{1}$$

Definition. *Following [4], we call the set*

$$A^* \stackrel{\text{def}}{=} \{x, \mu_A(x), \nu_A(x) | x \in E\}$$

an intuitionistic fuzzy set (IFS) and the function $\pi_A(x)$, which is given by

$$\pi_A(x) \stackrel{\text{def}}{=} 1 - \mu_A(x) - \nu_A(x), \tag{2}$$

is called degree of uncertainty of the membership of the element x to the set A.

Obviously in the case of

$$(\forall x \in E)\pi_A(x) = 0$$

we have a fuzzy set. The widely used geometric interpretation of IFS is the so-called interpretation triangle (see [4])

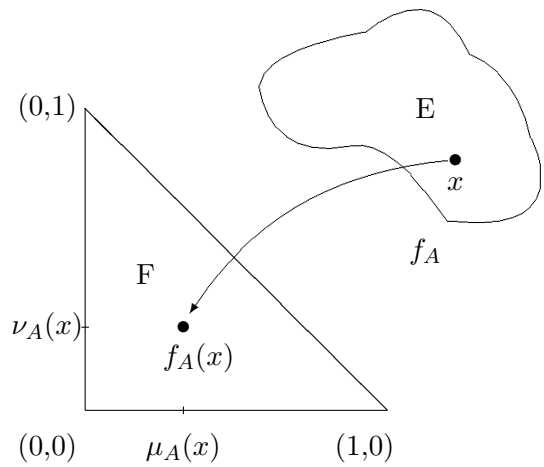

Figure 1.

Let us now return to equation (1). It is possible that when an evaluation task is assigned to a group of experts some of them produce values for $\mu_A(x)$ and $\nu_A(x)$ such that (1) is violated for certain $x \in E$. In this case it is possible to simply disregard the evaluation, or to correct it or, as the current paper suggests, to re-evaluate all of the expert's produced values.

2 Definition of the problem

Let $k \geq 1$ be an integer, $E = \{x_1, \ldots, x_k\}$ be a finite universe set. We consider the case of a group of n experts (numbered from 1 to n). For the membership and non-membership of x_j, let i-th expert has given the following values:

$$\langle \mu_i(x_j), \nu_i(x_j) \rangle.$$

Let $Z \subseteq \{1, \ldots, n\}$ and $T \subseteq \{1, \ldots, k\}$ be such that for all $\langle z, t\rangle \in Z \times T$ (the Cartesian product of Z and T) we have

$$\mu_z(x_t) + \nu_z(x_t) > 1, \tag{3}$$

Then $Z \times T$ is the set of all inconsistent evaluations.

In [6] is proposed the following Open Problem for the case when $Z \times T$ is a non-empty set.

"For a fixed $z \in Z$ find the smallest real $p > 1$ for which

$$A_z = \{\langle x, \mu_z(x), \nu_z(x)\rangle | x \in E\}$$

is a p-IFS (see e.g. [5])[1]*.*

Provide a formula transforming the p-IFS to IFS. "

Here we are faced with the following problems which need to be solved. First and foremost we have to obtain the "smallest" p-IFS which the new evaluations belong to. The second problem is how to map the obtained set to the interpretation triangle.

3 The proposed solution

In [7] the equation

$$a^p + b^p = 1, \tag{4}$$

is solved under the conditions: $a \in (0, 1), b \in (0, 1)$ are real numbers. It is shown that if $a = b$

$$p = -\frac{\ln 2}{\ln a} \tag{5}$$

and when $0 < a < b < 1$ there are seven expressions for the solution p:

$$p = \frac{\ln\left(1 - \frac{\ln b}{\ln a} - \sum\limits_{k=2}^{\infty} \frac{1}{k!}\left(\frac{\ln b}{\ln a}\right)^k E(k, \frac{\ln a}{\ln b})\right)}{\ln b} \tag{6}$$

with

$$E(k, m) = (-1)^{k-1} m^{k-1} \frac{\Gamma(\frac{k+1}{m})}{\Gamma(\frac{k+1}{m} - k + 1)}$$

[1]We only remind that for p-IFS ($p \in (0, +\infty)$) is real number equation (1) is replaced by

$$\mu_A^p(x) + \nu_A^p(x) \leq 1$$

and Γ denotes Euler's gamma function;

$$p = \frac{\ln\left(1 - \frac{\ln b}{\ln a} + \frac{\ln b}{\ln a}\sum\limits_{k=2}^{\infty}\frac{(-1)^k}{(k-1)k}\left(B(k-1,(k+1)\frac{\ln b}{\ln a} - k + 1)\right)^{-1}\right)}{\ln b} \tag{7}$$

$$p = \frac{\ln\left(1 - \frac{\ln b}{\ln a} + \sum\limits_{k=2}^{\infty}\frac{(-1)^k}{k(k+1)}\left(B(k,(k+1)\frac{\ln b}{\ln a} - k + 1)\right)^{-1}\right)}{\ln b} \tag{8}$$

$$p = \frac{\ln\left(1 - \frac{\ln b}{\ln a} + \frac{\ln a}{\ln b}\sum\limits_{k=2}^{\infty}\frac{(-1)^k(k+1)^{-1}}{k+1+\frac{\ln a}{\ln b}}\left(B(k+1,\frac{(k+1)\ln b}{\ln a} - k + 1)\right)^{-1}\right)}{\ln b} \tag{9}$$

where $B(x,y)$ denotes Euler's beta function;

$$p = \frac{\ln\left(1 - \frac{\ln b}{\ln a} + \sum\limits_{k=2}^{\infty}(-1)^k A_k\right)}{\ln b} \tag{10}$$

with

$$A_k = \frac{(-1)^{k-1}}{k!\left(\frac{\ln a}{\ln b}\right)^k}\prod_{i=1}^{k-1}(i\frac{\ln a}{\ln b} - k - 1);$$

$$p = \frac{\ln\left(1 - \frac{\ln b}{\ln a} + \sum\limits_{k=2}^{\infty}\frac{1}{(k+1)!}\left\langle -(k+1)\frac{\ln b}{\ln a}\right\rangle_k\right)}{\ln b} \tag{11}$$

where $\langle x\rangle_k \stackrel{\text{def}}{=} \prod\limits_{j=0}^{k-1}(x+j)$ denotes the Pochhammer symbol;

$$p = \frac{\ln\left(\sum\limits_{k=0}^{\infty}\frac{(-1)^k}{k+1}\binom{(k+1)\frac{\ln b}{\ln a}}{k}\right)}{\ln b}. \tag{12}$$

with

$$\binom{u}{v} \stackrel{\text{def}}{=} \frac{\Gamma(u+1)}{\Gamma(v+1)\Gamma(u-v+1)}$$

It is clear that further we must consider not all $x \in E$ but only the elements $x_t,\ t \in T$. Putting in one of the above formulas (by choice) instead of a

$$a(t) \stackrel{\text{def}}{=} \min(\mu_z(x_t), \nu_z(x_t))$$

and instead of b

$$b(t) \stackrel{\text{def}}{=} \max(\mu_z(x_t), \nu_z(x_t)),$$

we obtain $p = p(t)$, when $a \neq b$. For the case $a = b$ we put:

$$p = p(t) = p = -\frac{\ln 2}{\ln \mu_z(x_t)}.$$

(see (5)).

Finally we introduce

$$\hat{p} = \max_{t \in T}(p(t)). \tag{13}$$

Below we shall prove that $\hat{p}$ is the minimal $p > 1$ for which

$$A_z = \{\langle x, \mu_z(x), \nu_z(x)\rangle | x \in E\}$$

is a p-IFS.

Indeed, let us assume that p^*, $p^* < \hat{p}$, satisfies the same property. Let $t_0 \in T$ is such that

$$p(t_0) = \hat{p}.$$

Then

$$a^{\hat{p}}(t_0) + b^{\hat{p}}(t_0) = 1.$$

Therefore,

$$a^{p^*}(t_0) + b^{p^*}(t_0) > 1.$$

But the last contrdicts the assumption that

$$A_z = \{\langle x, \mu_z(x), \nu_z(x)\rangle | x \in E\}$$

is a p^*-IFS.

To complete the solution of the Open Problem it remains to give a transformation formula mapping the $\hat{p}$-IFS to IFS.

The formula is the following

$$\text{new}\mu_z = (\text{old}\mu_z)^{\hat{p}}; \text{new}\nu_z = (\text{old}\nu_z)^{\hat{p}}, \tag{14}$$

Of course

$$\text{new}\mu_z(x) < \text{old}\mu_z(x)\ ; \text{new}\nu_z(x) < \text{old}\nu_z(x)$$

for all $x \in E$, since $\hat{p} > 1$. This reflects the fact that the expert has given an inconsistent evaluations and thus the estimates are re-evaluated to diminished value.

Below is shown that (14) generates an IFS

$$A_z = \{\langle x, \text{new}\mu_z(x), \text{new}\nu_z(x)\rangle | x \in E\}.$$

Since $\hat{p} > 1$, for $i \notin T$ we have

$$\text{new}\mu_z(x_i) + \text{new}\nu_z(x_i) < \text{old}\mu_z(x_i) + \text{old}\mu_z(x_i) < 1$$

Also for $i \in T$ we have

$$\text{old}\mu_z(x_i) + \text{old}\nu_z(x_i) > 1$$

but

$$\text{new}\mu_z(x_i) + \text{new}\nu_z(x_i) \leq 1$$

(see (4) and (13)).

4 Conclusion

Using numerical approximations it is possible to find $\hat{p}$ with arbitrary precision but we chose to focus on the analytical form of the solution. The last is more convenient since it provides more flexibility for numerical computation but also states the underlying logic of the solution. In future research we shall discuss the benfits of certain numerical approximations in different cases.

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