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On interval-valued intuitionistic fuzzy modal operators

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Abstract: An survey of the existing interval-valued intuitionistic fuzzy modal operators is given. Eight new operators are introduced that extend the older ones. Some of their basic properties are discussed. Open problems are formulated.
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1 Introduction

An Interval-Valued Intuitionistic Fuzzy Set (IVIFS) A^* (over a basic set E) is an object of the form: $A^* = \{\langle x, M_A(x), N_A(x)\rangle \mid x \in E\}$, where $A \subseteq E$, $M_A(x) \subset [0,1]$ and $N_A(x) \subset [0,1]$ are intervals and for all $x \in E$:

$$\sup M_A(x) + \sup N_A(x) \leq 1.$$

This definition is analogous to the definition of an IFS, that is a partial case of an IVIFS for the case, when $\mu_A(x) = \inf M_A(x) = \sup M_A(x)$, $\mu_A(x) = \inf N_A(x) = \sup N_A(x)$, and

$$\mu_A(x) + \nu_A(x) = \sup M_A(x) + \sup N_A(x) \leq 1.$$

The definition of the IVIFS can be however rewritten to become an analogue of the second definition of the IFS (see [5]) – namely, if M_A and N_A are interpreted as functions. Then, an IVIFS A (over a basic set E) is given by functions

$$M_A : E \to INT([0,1]) \text{ and } N_A : E \to INT([0,1])$$

and the above inequality.

We must note that there is no difference in principle between the two approaches. And what is more, the same exist also in the ordinary fuzzy sets theory. The author originally used the first one influenced by the Kaufmann's book [11]. Perhaps it was this approach that helped him develop the theory of operators over IFS in its present form. The same notation was used in 1987–1988 in the research on IVIFSs, too (see [3, 10]).

Because below we will use only notation A^*, for brevity, the asterisk will be omitted.

IVIFSs have geometrical interpretations similar to, but more complex than these of the IFSs (Fig. 1).

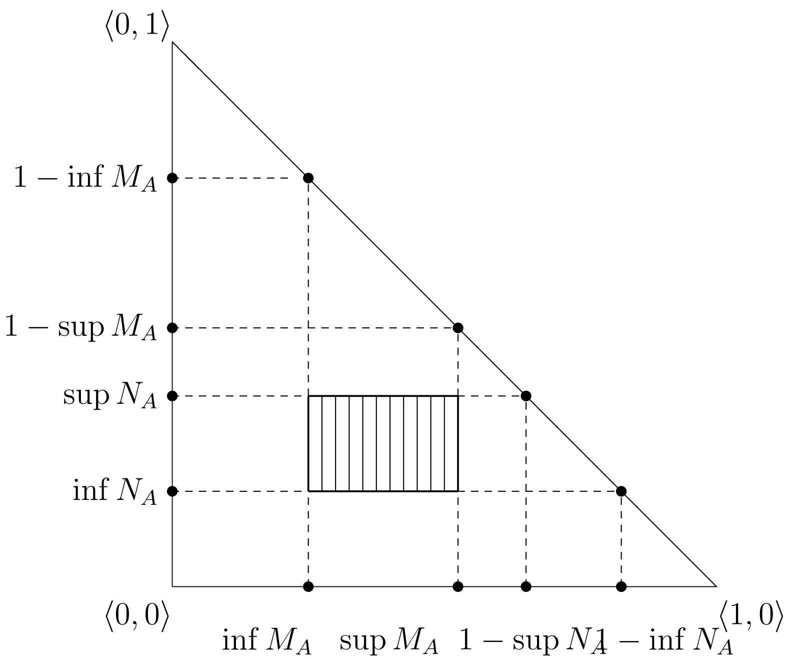

Figure 1.

It is suitable to define $P_A(x) = [0, 1 - \sup M_A(x) - \sup N_A(x)]$.

Therefore, $\inf P_A(x) = 0$ and

$$\sup P_A(x) = 1 - \sup M_A(x) - \sup N_A(x). \qquad (*)$$

First, we define some relations over IVFSs. For every two IVIFSs A and B the following relations hold ("iff" is a abbriviation of "if and only if"):

$$A \subset_{\Box,\inf} B \quad \text{iff}^{1} \quad (\forall x \in E)(\inf M_A(x) \leq \inf M_B(x)),$$

$$A \subset_{\Box,\sup} B \quad \text{iff} \quad (\forall x \in E)(\sup M_A(x) \leq \sup M_B(x)),$$

$$A \subset_{\Diamond,\inf} B \quad \text{iff} \quad (\forall x \in E)(\inf N_A(x) \geq \inf N_B(x)),$$

$$A \subset_{\Diamond,\sup} B \quad \text{iff} \quad (\forall x \in E)(\sup N_A(x) \geq \sup N_B(x)),$$

$$
\begin{array}{lll}
A \subset_{\Box} B & \text{iff} & A \subset_{\Box,\inf} B \ \&\ A \subset_{\Box,\sup} B, \\
A \subset_{\Diamond} B & \text{iff} & A \subset_{\Diamond,\inf} B \ \&\ A \subset_{\Diamond,\sup} B, \\
A \subset B & \text{iff} & A \subset_{\Box} B \ \&\ B \subset_{\Diamond} A, \\
A \subseteq_{\Box,\inf} B & \text{iff} & (\forall x \in E)(\inf M_A(x) \leq \inf M_B(x)), \\
A \subseteq_{\Box,\sup} B & \text{iff} & (\forall x \in E)(\sup M_A(x) \leq \sup M_B(x)), \\
A \subseteq_{\Diamond,\inf} B & \text{iff} & (\forall x \in E)(\inf N_A(x) \geq \inf N_B(x)), \\
A \subseteq_{\Diamond,\sup} B & \text{iff} & (\forall x \in E)(\sup N_A(x) \geq \sup N_B(x)), \\
A \subseteq_{\Box} B & \text{iff} & A \subseteq_{\Box,\inf} B \ \&\ A \subseteq_{\Box,\sup} B, \\
A \subseteq_{\Diamond} B & \text{iff} & A \subseteq_{\Diamond,\inf} B \ \&\ A \subseteq_{\Diamond,\sup} B, \\
A \subseteq B & \text{iff} & A \subseteq_{\Box} B \ \&\ B \subseteq_{\Diamond} A, \\
A = B & \text{iff} & A \subset B \ \&\ B \subset A,
\end{array}
$$

Second, we describe the basic operations, defined for every two IVIFSs A and B. They are:

$$
\begin{array}{lll}
\neg A & = & \{\langle x, N_A(x), M_A(x)\rangle \mid x \in E\}, \\
A \cap B & = & \{\langle x, [\min(\inf M_A(x), \inf M_B(x)), \min(\sup M_A(x), \sup M_B(x))], \\
& & [\max(\inf N_A(x), \inf N_B(x)), \max(\sup N_A(x), \sup N_B(x))]\rangle \mid x \in E\}, \\
A \cup B & = & \{\langle x, [\max(\inf M_A(x), \inf M_B(x)), \max(\sup M_A(x) \sup M_B(x))], \\
& & [\min(\inf N_A(x), \inf N_B(x)), \min(\sup N_A(x), \sup N_B(x))]\rangle \mid x \in E\} \\
A + B & = & \{\langle x, [\inf M_A(x) + \inf M_B(x) - \inf M_A(x). \inf M_B(x), \\
& & \sup M_A(x) + \sup M_B(x) - \sup M_A(x). \sup M_B(x)], \\
& & [\inf N_A(x). \inf N_B(x), \sup N_A(x). \sup N_B(x)]\rangle \mid x \in E\} \\
A.B & = & \{\langle x, [\inf M_A(x). \inf M_B(x), \sup M_A(x). \sup M_B(x)], \\
& & [\inf N_A(x) + \inf N_B(x) - \inf N_A(x). \inf N_B(x), \\
& & \sup N_A(x) + \sup N_B(x) - \sup N_A(x). \sup N_B(x)]\rangle \mid x \in E\} \\
A@B & = & \{\langle x, [(\inf M_A(x) + \inf M_B(x))/2, (\sup M_A(x) + \sup M_B(x))/2], \\
& & [(\inf N_A(x) + \inf N_B(x))/2, (\sup N_A(x)\rangle + \sup N_B(x))/2] \mid x \in E\}
\end{array}
$$

Now, over IFSs 185 different implications are defined (see, e.g., [6, 7]). On their basis, three types of 185 conjunctions and discunctions can be introduced (see, e.g., [1, 2]. After publishing of [7], 5 new implications were introduced, but for them there are not constructed new conjunctions and discunctions. Now, the following **Open Problems** are interesting: **1.** To construct analogous of all 190 implicationss for the IVIFS-case; **2.** On their basis, to construct the respective triples of conjunctions and discunctions.

Third, we give the list of the operators of modal type that are defined over an IVIFS A (see also [4, 5]):

$$\begin{aligned}
\Box A = & \{\langle x, M_A(x), [\inf N_A(x), 1-\sup M_A(x)]\rangle \mid x \in E\},\\
\Diamond A = & \{\langle x, [\inf M_A(x), 1-\sup N_A(x)], N_A(x)\rangle \mid x \in E\},\\
D_\alpha(A) = & \{\langle x, [\inf M_A(x), \sup M_A(x)+\alpha.(1-\sup M_A(x)-\sup N_A(x))],\\
& [\inf N_A(x), \sup N_A(x)+(1-\alpha).(1-\sup M_A(x)-\sup N_A(x))]\rangle \mid x \in E\},\\
F_{\alpha,\beta}(A) = & \{\langle x, [\inf M_A(x), \sup M_A(x)+\alpha.(1-\sup M_A(x)-\sup N_A(x))],\\
& [\inf N_A(x), \sup N_A(x)+\beta.(1-\sup M_A(x)-\sup N_A(x))]\rangle \mid x \in E\}, \alpha+\beta \leq 1,\\
G_{\alpha,\beta}(A) = & \{\langle x, [\alpha.\inf M_A(x), \alpha.\sup M_A(x)], [\beta.\inf N_A(x), \beta.\sup N_A(x)]\rangle \mid x \in E\},\\
H_{\alpha,\beta}(A) = & \{\langle x, [\alpha.\inf M(x), \alpha.\sup M_A(x)], [\inf N_A(x), \sup N_A(x)\\
& +\beta.(1-\sup M_A(x)-\sup N_A(x))]\rangle \mid x \in E\},\\
H^*_{\alpha,\beta}(A) = & \{\langle x, [\alpha.\inf M_A(x), \alpha.\sup M_A(x)], [\inf N_A(x), \sup N_A(x)\\
& +\beta.(1-\alpha.\sup M_A(x)-\sup N_A(x))]\rangle \mid x \in E\},\\
J_{\alpha,\beta}(A) = & \{\langle x, [\inf M_A(x), \sup M_A(x)+\alpha.(1-\sup M_A(x)\\
& -\sup N_A(x))], [\beta.\inf N_A(x), \beta.\sup N_A(x)]\rangle \mid x \in E\},\\
J^*_{\alpha,\beta}(A) = & \{\langle x, [\inf M_A(x), \sup M_A(x)+\alpha.(1-\sup M_A(x)\\
& -\beta.\sup N_A(x))], [\beta.\inf N_A(x), \beta.\sup N_A(x)]\rangle \mid x \in E\},
\end{aligned}$$

where $\alpha, \beta \in [0, 1]$.

Obviously, $\Box A = D_0(A) = F_{0,1}(A)$, $\Diamond A = D_1(A) = F_{1,0}(A)$, $D_\alpha(A) = F_{\alpha,1-\alpha}(A)$ and by this reason, below we will not discuss these three operators.

In [5], the operators $F_{\alpha.\beta}, ..., J^*_{\alpha,\beta}$ are extended to the following operators, where $\alpha, \beta, \gamma, \delta \in [0, 1]$ such that $\alpha \leq \beta$ and $\gamma \leq \delta$:

$$\begin{aligned}
\overline{F}_{\alpha,\beta,\gamma,\delta}(A) = & \{\langle x, [\inf M_A(x)+\alpha.(1-\sup M_A(x)-\sup N_A(x)),\\
& \sup M_A(x)+\beta.(1-\sup M_A(x)-\sup N_A(x))],\\
& [\inf N_A(x)+\gamma.(1-\sup M_A(x)-\sup N_A(x)),\\
& \sup N_A(x)+\delta.(1-\sup M_A(x)-\sup N_A(x))]\rangle \mid x \in E\}, \beta+\delta \leq 1,\\
\overline{G}_{\alpha,\beta,\gamma,\delta}(A) = & \{\langle x, [\alpha.\inf M_A(x), \beta.\sup M_A(x)],\\
& [\gamma.\inf N_A(x), \delta.\sup N_A(x)]\rangle \mid x \in E\},\\
\overline{H}_{\alpha,\beta,\gamma,\delta}(A) = & \{\langle x, [\alpha.\inf M_A(x), \beta.\sup M_A(x)],\\
& [\inf N_A(x)+\gamma.(1-\sup M_A(x)-\sup N_A(x)),\\
& \sup N_A(x)+\delta.(1-\sup M_A(x)-\sup N_A(x))]\rangle \mid x \in E\}\\
\overline{H}^*_{\alpha,\beta,\gamma,\delta}(A) = & \{\langle x, [\alpha.\inf M_A(x), \beta.\sup M_A(x)],\\
& [\inf N_A(x)+\gamma.(1-\beta.\sup M_A(x)-\sup N_A(x)),\\
& \sup N_A(x)+\delta.(1-\beta.\sup M_A(x)-\sup N_A(x))]\rangle \mid x \in E\}
\end{aligned}$$

$$\overline{J}_{\alpha,\beta,\gamma,\delta}(A) = \{\langle x, [\inf M_A(x) + \alpha.(1 - \sup M_A(x) - \sup N_A(x)), \sup M_A(x) + \beta.(1 - \sup M_A(x) - \sup N_A(x))], [\gamma. \inf N_A(x), \delta. \sup N_A(x)]\rangle \mid x \in E\}$$

$$\overline{J}^*_{\alpha,\beta,\gamma,\delta}(A) = \{\langle x, [\inf M_A(x) + \alpha.(1 - \delta. \sup M_A(x) - \sup N_A(x)), \sup M_A(x) + \beta.(1 - \sup M_A(x) - \delta. \sup N_A(x))], [\gamma. \inf N_A(x), \delta. \sup N_A(x)]\rangle \mid x \in E\}.$$

Obviosly, the new operators are extensions of the previous ones.

In [6, 8], the idea for changing of real number parameters α, β in the operators defined over IFS A with whole IFS B is discussed. Here, we discuss wimilar idea for the case, when set A is an IVIFS.

2 Main results

Here, following [8], we describe step by step 8 intuitionistic fuzzy extended modal operators and discuss some of their propertiers. We give the proof of only the first two properties, formulated below and the rest properties are proved by the same manner.

2.1 Operator F_B

Let B be an IVIFS. Then for each IVIFS A:

$$F_B(A) = \{\langle x, [\inf M_A(x) + \inf M_B(x) \sup P_A(x), \sup M_A(x) + \sup M_B(x) \sup P_A(x)],$$

$$[\inf N_A(x) + \inf N_B(x) \sup P_A(x), \sup N_A(x) + \sup N_B(x) \sup P_A(x)]\rangle | x \in E\}.$$

Theorem 1. For every six IFSs A, B, C, D, P, Q:

$$\neg F_{\neg B}(\neg A) = F_B(A),$$

$$F_B(F_C(A)) = F_{F_B(C)}(A),$$

$$F_B(A \cap D) \subseteq F_B(A) \cap F_B(D),$$

$$F_B(A \cup D) \supseteq F_B(A) \cup F_B(D),$$

$$F_B(A@D) = F_B(A)@F_B(D),$$

$$F_B(A) \subseteq F_C(A), \text{ where } B \subseteq C.$$

Proof: Let the IVIFSs A and B be given. Then, for the first equality we obtain that

$$\neg F_{\neg B}(\neg A) = \neg F_{\neg\{\langle x, M_B(x), N_B(x)\rangle | x \in E\}}(\neg A)$$

$$= \neg F_{\{\langle x, N_B(x), M_B(x)\rangle | x \in E\}}(\{\langle x, N_A(x), M_A(x)\rangle | x \in E\})$$

$$= \neg\{\langle x, [\inf N_A(x) + \inf N_B(x) \sup P_A(x), \sup N_A(x) + \sup N_B(x) \sup P_A(x)],$$

$$[\inf M_A(x) + \inf M_B(x) \sup P_A(x), \sup M_A(x) + \sup M_B(x) \sup P_A(x)]\rangle | x \in E\}$$

$$= \{\langle x, [\inf M_A(x) + \inf M_B(x) \sup P_A(x), \sup M_A(x) + \sup M_B(x) \sup P_A(x)],$$
$$[\inf N_A(x) + \inf N_B(x) \sup P_A(x), \sup N_A(x) + \sup N_B(x) \sup P_A(x)]\rangle | x \in E\}$$
$$= F_B(A).$$

For the second equality we obtain

$$F_B(F_C(A))$$
$$= F_B(\{\langle x, [\inf M_A(x) + \inf M_C(x) \sup P_A(x), \sup M_A(x) + \sup M_C(x) \sup P_A(x)],$$
$$[\inf N_A(x) + \inf N_C(x) \sup P_A(x), \sup N_A(x) + \sup N_C(x) \sup P_A(x)]\rangle | x \in E\})$$
$$= \{\langle x, [\inf M_A(x) + \inf M_C(x) \sup P_A(x) + \inf M_B(x)(1 - \sup M_A(x)$$
$$- \sup M_C(x) \sup P_A(x) - \sup N_A(x) - \sup N_C(x) \sup P_A(x)),$$
$$\sup M_A(x) + \sup M_C(x) \sup P_A(x) + \sup M_B(x)(1 - \sup M_A(x)$$
$$- \sup M_C(x) \sup P_A(x) - \sup N_A(x) - \sup N_C(x) \sup P_A(x))],$$
$$[\inf N_A(x) + \inf N_C(x) \sup P_A(x) + \inf N_B(x)(1 - \sup M_A(x)$$
$$- \sup M_C(x) \sup P_A(x) - \sup N_A(x) - \sup N_C(x) \sup P_A(x)),$$
$$\sup N_A(x) + \sup N_C(x) \sup P_A(x) + \sup N_B(x)(1 - \sup M_A(x)$$
$$- \sup M_C(x) \sup P_A(x) - \sup N_A(x) - \sup N_C(x) \sup P_A(x))]\rangle | x \in E\}$$

$$= \{\langle x, [\inf M_A(x) + \inf M_C(x) \sup P_A(x) + \inf M_B(x) - \inf M_B(x) \sup M_A(x)$$
$$- \inf M_B(x) \sup M_C(x) \sup P_A(x) - \inf M_B(x) \sup N_A(x) - \inf M_B(x) \sup N_C(x) \sup P_A(x)),$$
$$\sup M_A(x) + \sup M_C(x) \sup P_A(x) + \sup M_B(x) - \sup M_B(x) \sup M_A(x)$$
$$- \sup M_B(x) \sup M_C(x) \sup P_A(x) - \sup M_B(x) \sup N_A(x) - \sup M_B(x) \sup N_C(x) \sup P_A(x))],$$
$$[\inf N_A(x) + \inf N_C(x) \sup P_A(x) + \inf N_B(x) - \inf N_B(x) \sup M_A(x)$$
$$- \inf N_B(x) \sup M_C(x) \sup P_A(x) - \inf N_B(x) \sup N_A(x) - \inf N_B(x) \sup N_C(x) \sup P_A(x)),$$
$$\sup N_A(x) + \sup N_C(x) \sup P_A(x) + \sup N_B(x) - \sup N_B(x) \sup M_A(x)$$
$$- \sup N_B(x) \sup M_C(x) \sup P_A(x) - \sup N_B(x) \sup N_A(x)$$
$$- \sup N_B(x) \sup N_C(x) \sup P_A(x))]\rangle | x \in E\}$$
$$= \{\langle x, [\inf M_A(x) + \inf M_B(x)(1 - \sup M_A(x) - \sup N_A(x))$$
$$+ (\inf M_C(x) - \inf M_B(x) \sup M_C(x) - \inf M_B(x) \sup N_C(x)) \sup P_A(x)),$$
$$\sup M_A(x) + \sup M_B(x)(1 - \sup M_A(x) - \sup N_A(x))$$
$$+ (\sup M_C(x) - \sup M_B(x) \sup M_C(x) - \sup M_B(x) \sup N_C(x)) \sup P_A(x))],$$
$$[\inf N_A(x) + \inf N_B(x)(1 - \sup M_A(x) - \sup N_A(x))$$

$$+(\inf N_C(x) - \inf N_B(x) \sup M_C(x) - \inf N_B(x) \sup N_C(x)) \sup P_A(x)),$$

$$\sup N_A(x) + \sup N_B(x)(1 - \sup M_A(x) - \sup N_A(x))$$

$$+(\sup N_C(x) - \sup N_B(x) \sup M_C(x) - \sup N_B(x) \sup N_C(x)) \sup P_A(x))]\rangle | x \in E\}$$

(from (*))

$$= \{\langle x, [\inf M_A(x) + \inf M_B(x) \sup P_A(x)$$

$$+(\inf M_C(x) - \inf M_B(x) \sup M_C(x) - \inf M_B(x) \sup N_C(x)) \sup P_A(x)),$$

$$\sup M_A(x) + \sup M_B(x) \sup P_A(x)$$

$$+(\sup M_C(x) - \sup M_B(x) \sup M_C(x) - \sup M_B(x) \sup N_C(x)) \sup P_A(x))],$$

$$[\inf N_A(x) + \inf N_B(x) \sup P_A(x)$$

$$+(\inf N_C(x) - \inf N_B(x) \sup M_C(x) - \inf N_B(x) \sup N_C(x)) \sup P_A(x)),$$

$$\sup N_A(x) + \sup N_B(x) \sup P_A(x)$$

$$+(\sup N_C(x) - \sup N_B(x) \sup M_C(x) - \sup N_B(x) \sup N_C(x)) \sup P_A(x))]\rangle | x \in E\}$$

$$= \{\langle x, [\inf M_A(x) + (\inf M_B(x) + \inf M_C(x) - \inf M_B(x) \sup M_C(x)$$

$$- \inf M_B(x) \sup N_C(x)) \sup P_A(x)),$$

$$\sup M_A(x) + (\sup M_B(x) + \sup M_C(x) - \sup M_B(x) \sup M_C(x)$$

$$- \sup M_B(x) \sup N_C(x)) \sup P_A(x))],$$

$$[\inf N_A(x) + (\inf N_B(x) + \inf N_C(x) - \inf N_B(x) \sup M_C(x)$$

$$- \inf N_B(x) \sup N_C(x)) \sup P_A(x)),$$

$$\sup N_A(x) + (\sup N_B(x) + \sup N_C(x) - \sup N_B(x) \sup M_C(x)$$

$$- \sup N_B(x) \sup N_C(x)) \sup P_A(x))]\rangle | x \in E\}$$

(let us denote conditionaly this set as)

$$= F_X(A),$$

where

$$X = \{\langle x, M_X(x), N_X(x)\rangle \mid x \in E\},$$

$$\inf M_X(x) = \inf M_B(x) + \inf M_C(x) - \inf M_B(x) \sup M_C(x) - \inf M_B(x) \sup N_C(x),$$

$$\sup M_X(x) = \sup M_B(x) + \sup M_C(x) - \sup M_B(x) \sup M_C(x) - \sup M_B(x) \sup N_C(x),$$

$$\inf N_X(x) = \inf N_B(x) + \inf N_C(x) - \inf N_B(x) \sup M_C(x) - \inf N_B(x) \sup N_C(x),$$

$$\sup N_X(x) = \sup N_B(x) + \sup N_C(x) - \sup N_B(x) \sup M_C(x) - \sup N_B(x) \sup N_C(x).$$

Now, we must prove that set X is an IVIFS.

First, we see diretly, that

$$0 \leq \inf M_B(x) \sup P_C(x) + \inf M_C(x)$$

$$= \inf M_B(x) + \inf M_C(x) - \inf M_B(x) \sup M_C(x) - \inf M_B(x) \sup N_C(x)$$

$$= \inf M_X(x) \le \sup M_X(x)$$

$$= \sup M_B(x) + \sup M_C(x) - \sup M_B(x) \sup M_C(x) - \sup M_B(x) \sup N_C(x)$$

$$= \sup M_B(x)(1 - \sup M_C(x) - \sup N_C(x)) + \sup M_C(x)$$

$$= \sup M_B(x) \sup P_C(x) + \sup M_C(x)$$

$$\le \sup P_C(x) + \sup M_C(x) \le 1.$$

Second, analogously, we see that

$$0 \le \inf N_X(x) \le \sup N_X(x) \le 1.$$

Third, we check validity of condition (*) for set X.

$$\sup M_X(x) + \sup N_X(x)$$

$$= \sup M_B(x) + \sup M_C(x) - \sup M_B(x) \sup M_C(x) - \sup M_B(x) \sup N_C(x)$$

$$+ \sup N_B(x) + \sup N_C(x) - \sup N_B(x) \sup M_C(x) - \sup N_B(x) \sup N_C(x)$$

$$= \sup M_B(x)(1 - \sup M_C(x) - \sup N_C(x)) + \sup M_C(x)$$

$$+ \sup N_B(x)(1 - \sup M_C(x) - \sup N_C(x)) + \sup N_C(x)$$

(from (*))

$$= \sup M_B(x) \sup P_C(x) + \sup N_B(x) \sup P_C(x)$$

$$= (\sup M_B(x) + \sup N_B(x)) \sup P_C(x) \le \sup P_C(x) \le 1.$$

Therefore, set X is an IVIFS.

Finally, se wee that

$$\inf M_X(x) = \inf M_C(x) + \inf M_B(x)(1 - \sup M_C(x) - \sup N_C(x))$$

$$= \inf M_C(x) + \inf M_B(x) \sup P_C(x),$$

$$\sup M_X(x) = \sup M_C(x) + \sup M_B(x)(1 - \sup M_C(x) - \sup N_C(x))$$

$$= \sup M_C(x) + \sup M_B(x) \sup P_C(x),$$

$$\inf N_X(x) = \inf N_C(x) + \inf N_B(x)(1 - \sup M_C(x) - \sup N_C(x))$$

$$= \inf N_C(x) + \inf N_B(x) \sup P_C(x),$$

$$\sup N_X(x) = \sup N_C(x) + \sup N_B(x)(1 - \sup M_C(x) - \sup N_C(x))$$

$$= \sup N_C(x) + \sup N_B(x) \sup M_C(x).$$

Therefore, $X = F_B(C)$. □

2.2 Operator $G_{B,C}$

Let B and C be IVIFSs. Then for each IVIFS A:

$$G_{B,C}(A) = \{\langle x, [\inf M_B(x) \inf M_A(x), \sup M_B(x) \sup M_A(x)],$$

$$[\inf N_C(x) \inf N_A(x), \sup N_C(x) \sup N_A(x)]\rangle | x \in E\}.$$

Theorem 2. For every six IFSs A, B, C, D, P, Q:

$$G_{\Box B, \Diamond C}(A) = G_{B,C}(A).$$

$$\neg G_{\neg C, \neg B}(\neg A) = G_{B,C}(A),$$

$$G_{B,C}(G_{P,Q}(A)) = G_{P,Q}(G_{B,C}(A)),$$

$$G_{B,C}(A \cap D) = G_{B,C}(A) \cap G_{B,C}(D),$$

$$G_{B,C}(A \cup D) = G_{B,C}(A) \cup G_{B,C}(D),$$

$$G_{B,C}(A) \subseteq G_{P,Q}(A), \ \text{ where } B \subseteq P, C \subseteq Q.$$

2.3 Operator $H_{B,C}$

It is defined for every three IVIFSs A, B and C by:

$$H_{B,C}(A) = \{\langle x, [\inf M_B(x) \inf M_A(x), \sup M_B(x) \sup M_A(x)],$$

$$[\inf N_A(x) + \inf N_C(x) \sup P_A(x), \sup N_A(x) + \sup N_C(x) \sup P_A(x)]\rangle | x \in E\}.$$

Theorem 3. For every six IFSs A, B, C, D, P, Q, so that $B \subseteq P, C \subseteq Q$:

$$H_{\Box B, \Diamond C}(A) = H_{B,C}(A),$$

$$\neg H_{\neg C, \neg B}(\neg A) = J_{B,C}(A),$$

(see Subsection 2.6),

$$H_{B,C}(A \cap D) \subseteq H_{B,C}(A) \cap H_{B,C}(D),$$

$$H_{B,C}(A \cup D) \supseteq H_{B,C}(A) \cup H_{B,C}(D),$$

$$H_{B,C}(A) \subseteq H_{P,Q}(A), \ \text{ where } B \subseteq P, C \subseteq Q.$$

2.4 Operator $H^*_{B,C}$

Let B and C be IVIFSs. Then for each IVIFS A:

$$H^*_{B,C}(A) = \{\langle x, [\inf M_B(x) \inf M_A(x), \sup M_B(x) \sup M_A(x)],$$

$$[\inf N_A(x) + \inf N_C(x)(1 - \sup M_B(x) \sup M_A(x) - \sup N_A(x)),$$

$$\sup N_A(x) + \sup N_C(x)(1 - \sup M_B(x) \sup M_A(x) - \sup N_A(x))]\rangle | x \in E\}.$$

Theorem 4. For every three IFSs A, B, C:

$$H^*_{\square B, \diamond C}(A) = H^*_{B,C}(A),$$

$$\neg H^*_{\neg C, \neg B}(\neg A) = J^*_{B,C}(A)$$

(see Subsection 2.7),

$$H^*_{B,C}(A \cap D) \subseteq H^*_{B,C}(A) \cap H_{B,C}(D),$$

$$H^*_{B,C}(A \cup D) \supseteq H^*_{B,C}(A) \cup H_{B,C}(D).$$

2.5 Operator $\overline{H}_{B,C}$

It is defined for every three IVIFSs A, B and C, so that

$$\sup M_B(x) + \sup N_C(x) \leq 1$$

for each $x \in E$, by:

$$\overline{H}_{B,C}(A) = \{\langle x, [\inf M_B(x) \inf M_A(x), \sup M_B(x) \sup M_A(x)],$$

$$[\inf N_A(x) + \inf N_C(x) - \inf N_A(x) \inf N_C(x),$$

$$\sup N_A(x) + \sup N_C(x) - \sup N_A(x) \sup N_C(x)]\rangle | x \in E\}.$$

This operator is a modification of the operator $\overline{H}_{\alpha.\beta}$, defined in [9].

Theorem 5. For every five IFSs A, B, C, P, Q, so that $B \subseteq P, C \subseteq Q$:

$$\overline{H}_{\square B, \diamond C}(A) = \overline{H}_{B,C}(A),$$

$$\neg \overline{H}_{\neg C, \neg B}(\neg A) = \overline{J}_{B,C}(A),$$

(see Subsection 2.8),

$$\overline{H}_{B,C}(A \cap D) \subseteq \overline{H}_{B,C}(A) \cap \overline{H}_{B,C}(D),$$

$$\overline{H}_{B,C}(A \cup D) \supseteq \overline{H}_{B,C}(A) \cup \overline{H}_{B,C}(D),$$

$$\overline{H}_{B,C}(A) \subseteq \overline{H}_{P,Q}(A).$$

2.6 Operator $J_{B,C}$

Let B and C be IVIFSs. Then for each IVIFS A the operator $J_{B,C}$ is defined by:

$$J_{B,C}(A) = \{\langle x, [\inf M_A(x) + \inf M_B(x) \sup P_A(x), \sup M_A(x) + \sup M_B(x) \sup P_A(x)],$$

$$[\inf N_C(x) \inf N_A(x), \sup N_C(x) \sup N_A(x)]\rangle | x \in E\}.$$

Theorem 6. For every five IFSs A, B, C, P, Q, so that $B \subseteq P, C \subseteq Q$:

$$J_{\square B, \diamond C}(A) = J_{B,C}(A),$$

$$\neg J_{\neg C, \neg B}(\neg A) = H_{B,C}(A),$$

$$J_{B,C}(A \cap D) \supseteq J_{B,C}(A) \cap J_{B,C}(D),$$

$$J_{B,C}(A \cup D) \subseteq J_{B,C}(A) \cup J_{B,C}(D),$$

$$J_{B,C}(A) \subseteq J_{P,Q}(A).$$

2.7 Operator $J^*_{B,C}$

Let B and C be IVIFSs. Then for each IVIFS A:

$$J^*_{B,C}(A) = \{\langle x, [\inf M_A(x) + \inf M_B(x)(1 - \sup M_A(x) - \sup N_C(x) \sup N_A(x)),$$
$$\sup M_A(x) + \sup M_B(x)(1 - \sup M_A(x) - \sup N_C(x) \sup N_A(x))],$$
$$[\inf N_C(x) \inf N_A(x), \sup N_C(x) \sup N_A(x)]\rangle | x \in E\}.$$

Theorem 7. For every three IFSs A, B, C:

$$J^*_{\square B, \diamond C}(A) = J^*_{B,C}(A),$$
$$\neg J^*_{\neg C, \neg B}(\neg A) = H^*_{B,C}(A),$$
$$J^*_{B,C}(A \cap D) \supseteq J^*_{B,C}(A) \cap J_{B,C}(D),$$
$$J^*_{B,C}(A \cup D) \subseteq J^*_{B,C}(A) \cup J_{B,C}(D).$$

2.8 Operator $\overline{J}_{B,C}$

It is defined for every three IVIFSs A, B and C, so that $\sup M_B(x) + \sup N_C(x) \leq 1$ for each $x \in E$, by:

$$\overline{J}_{B,C}(A) = \{\langle x, [\inf M_A(x) + \inf M_B(x) - \inf M_A(x) \inf M_B(x),$$
$$\sup M_A(x) + \sup M_B(x) - \sup M_A(x) \sup M_B(x)],$$
$$[\inf N_C(x) \inf N_A(x), \sup N_C(x) \sup N_A(x)]\rangle | x \in E\}.$$

This operator is a modification of the operator $\overline{J}_{\alpha.\beta}$, defined in [9].

Theorem 8. For every five IFSs A, B, C, P, Q, so that $B \subseteq P, C \subseteq Q$:

$$\overline{J}_{\square B, \diamond C}(A) = \overline{J}_{B,C}(A),$$
$$\neg \overline{J}_{\neg C, \neg B}(\neg A) = \overline{H}_{B,C}(A),$$
$$\overline{J}_{B,C}(A \cap D) \supseteq \overline{J}_{B,C}(A) \cap \overline{J}_{B,C}(D),$$
$$\overline{J}_{B,C}(A \cup D) \subseteq \overline{J}_{B,C}(A) \cup \overline{J}_{B,C}(D),$$
$$\overline{J}_{B,C}(A) \subseteq \overline{J}_{P,Q}(A).$$

3 Conclusion

In a next research, other properties of the newly defined operators will be studied. All they can be used in a lot of areas of informatics and especially, in the Artificial Intelligence for modification of intuitionistic fuzzy evaluations.

Also, they can be used in Intercriteria Analysis procedures for changing of evaluations of the compared criteria.

An interesting **Open Problem** is: Can these operators be extended additionally?

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