NIFS 11 (2005), 1, 16-27

Interval-Valued Intuitionistic Fuzzy Matrices

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Abstract

In this paper, the interval-valued intuitionistic fuzzy matrix (IVIFM) is introduced. The interval-valued intuitionistic fuzzy determinant is also defined. Some fundamental operations are also presented. The need of IVIFM is explain by an example.

Keywords: Intuitionistic fuzzy matrix, interval-valued intuitionistic fuzzy matrix.

1 Introduction

Matrices play important roles in various areas in science and engineering. The classical matrix theory can not solve the problems involving various types of uncertainties. That type of problems are solved by using fuzzy matrix [14]. Later much works have been done by many researchers. Fuzzy matrix deals with only membership values. These matrices can not deal non membership values. Intuitionistic fuzzy matrices (IFMs) introduced first time by Khan, Shyamal and Pal [11]. Several properties on IFMs have been studied in [6]. But, practically it is difficult to measure the membership or non membership value as a point. So, we consider the membership value as an interval and also in the case of non membership values, it is not selected as a point, it can be considered as an interval. Here, we introduce the interval valued intuitionistic fuzzy matrices (IVIFMs) and introduce some basic operators on IVIFMs. The interval-valued intuitionistic fuzzy determinant (IVIFD) is also defined. A real life problem on IVIFM is presented. Interpretation of some of the operators are given with the help of this example.

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2 Definition and Preliminaries

In this section, we first define the intuitionistic fuzzy matrix (IFM) based on the definition of intuitionistic fuzzy sets introduced by Atanassov [2]. The intuitionistic fuzzy matrices are introduced by Pal, Khan and Shyamal [11].

Def. 1 Intuitionistic fuzzy matrix (IFM)[11]: *An intuitionistic fuzzy matrix (IFM) A of order* $m \times n$ *is defined as* $A = [x_{ij}, < a_{ij\mu}, a_{ij\nu} >]_{m\times n}$*, where* $a_{ij\mu}$ *and* $a_{ij\nu}$ *are called membership and non membership values of* x_{ij} *in A, which maintaining the condition* $0 \le a_{ij\mu} + a_{ij\nu} \le 1$*.*
For simplicity, we write $A = [x_{ij}, a_{ij}]_{m\times n}$ *or simply* $[a_{ij}]_{m\times n}$ *where* $a_{ij} =< a_{ij\mu}, a_{ij\nu} >$*.*

Using the concept of intuitionistic fuzzy sets and interval valued fuzzy sets, we define interval-valued intuitionistic fuzzy matrices as follows:

Def. 2 Interval-valued intuitionistic fuzzy matrix (IVIFM): *An interval valued intuitionistic fuzzy matrix (IVIFM) A of order* $m\times n$ *is defined as* $A = [x_{ij}, < a_{ij\mu}, a_{ij\nu} >]_{m\times n}$ *where* $a_{ij\mu}$ *and* $a_{ij\nu}$ *are both the subsets of* $[0, 1]$ *which are denoted by* $a_{ij\mu} = [a_{ij\mu L}, a_{ij\mu U}]$ *and* $a_{ij\nu} = [a_{ij\nu L}, a_{ij\nu U}]$ *which maintaining the condition* $a_{ij\mu U} + a_{ij\nu U} \le 1$ *for* $i = 1, 2, \cdots, m$ *and* $j = 1, 2, \cdots, n$*.*

Def. 3 Interval-valued intuitionistic fuzzy determinant (IVIFD): *An interval valued intuitionistic fuzzy determinant (IVIFD) function* $f : M \to F$ *is a function on the set* M *(of all* $n \times n$ *IVIFMs) to the set* F*, where* F *is the set of elements of the form* $< [a_{\mu L}, a_{\mu U}], [a_{\nu L}, a_{\nu U}] >$*, maintaining the condition* $0 \le a_{\mu U} + a_{\nu U} \le 1$*,* $0 \le a_{\mu L} \le a_{\mu U} \le 1$ *and* $0 \le a_{\nu L} \le a_{\nu U} \le 1$ *and* $0 \le a_{ij\nu L} \le a_{ij\nu U} \le 1$ *such that* $A \subset M$ *then* $f(A)$ *or* $|A|$ *or* $det(A)$ *belongs to* F *and is given by*

$$|A| = \sum_{\sigma\in S_n} \prod_{i=1}^{n} < [a_{i\sigma(i)\mu L}, a_{i\sigma(i)\mu U}], [a_{i\sigma(i)\nu L}, a_{i\sigma(i)\nu U}] >$$

and S_n *denotes the symmetric group of all permutations of the symbols* $\{1, 2, \cdots, n\}$*.*

Def. 4 The adjoint IVIFM of an IVIFM: *The adjoint IVIFM of an IVIFM A of order* $n \times n$*, is denoted by adj.A and is defined by* $adj.A = [A_{ji}]$*, where* A_{ji} *is the determinant of the IVIFM A of order* $(n-1)\times(n-1)$ *formed by suppressing row* j *and column* i *of the IVIFM A. In other words,* A_{ji} *can be written in the form*

$$\sum_{\sigma\in S_{n_i n_j}} \prod_{t\in n_j} < [a_{t\sigma(t)\mu L}, a_{t\sigma(t)\mu U}], [a_{t\sigma(t)\nu L}, a_{t\sigma(t)\nu U} >$$

where, $n_j = \{1, 2, \ldots, n\}\backslash\{j\}$ *and* $S_{n_i n j}$ *is the set of all permutations of set* n_j *over the set* n_i*.*

Depending on the values of diagonal elements, the unit IVIFM are classified into two types: (i) $a-unit\ IVIFM$ and (ii) $r-unit\ IVIFM$.

Def. 5 Acceptance unit IVIFM (a-unit IVIFM): *A square IVIFM is a-unit IVIFM if all diagonal elements are* $< [1,1],[0,0] >$ *and all remaining elements are* $< [0,0],[1,1] >$ *and it is denoted by* $I_{<[0,0],[1,1]>}$.

Def. 6 Rejection unit IVIFM (r-unit IVIFM): *A square IVIFM is a r-unit IVIFM if all diagonal elements are* $< [0,0],[1,1] >$ *and all remaining elements are* $< [1,1],[0,0] >$ *and it is denoted by* $I_{<[1,1],[0,0]>}$.

Similarly, three types of null IVIFMs are defined on its elements.

Def. 7 Complete null IVIFM (c-null IVIFM): *An IVIFM is a c-null IVIFM if all the elements are* $< [0,0],[0,0] >$.

Def. 8 Acceptance null IVIFM (a-null IVIFM): *An IVIFM is a a-null IVIFM if all the elements are* $< [0,0],[1,1] >$.

Def. 9 Rejection null IVIFM (r-null IVIFM): *An IVIFM is a r-null IVIFM if all the elements are* $< [1,1],[0,0] >$.

2.1 Some operations on IVIFM

Let $A = [< [a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}] >]$ and $B = [< [b_{ij\mu L}, b_{ij\mu U}], [b_{ij\nu L}, b_{ij\nu U}] >]$ be two IVIFMs. Then,

(i) $< [a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}] > + < [b_{ij\mu L}, b_{ij\mu U}], [b_{ij\nu L}, b_{ij\nu U}] >$
$=< [\max(a_{ij\mu L}, b_{ij\mu L}), \max(a_{ij\mu U}, b_{ij\mu U})], [\min(a_{ij\nu L}, b_{ij\nu L}), \min(a_{ij\nu U}, b_{ij\nu U})] >$.

(ii) $< [a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}] > \cdot < [b_{ij\mu L}, b_{ij\mu U}], [b_{ij\nu L}, b_{ij\nu U}] >$
$=< [\min(a_{ij\mu L}, b_{ij\mu L}), \min(a_{ij\mu U}, b_{ij\mu U})], [\max(a_{ij\nu L}, b_{ij\nu L}), \max(a_{ij\nu U}, b_{ij\nu U})] >$.

(iii) $A + B = [< [\max\{a_{ij\mu L}, b_{ij\mu L}\}, \max\{a_{ij\mu U}, b_{ij\mu U}\}], [\min\{a_{ij\nu L}, b_{ij\nu L}\}, \min\{a_{ij\nu U}, b_{ij\nu U}\}] >]$.

(iv) $A \cdot B = [< [\min\{a_{ij\mu L}, b_{ij\mu L}\}, \min\{a_{ij\mu U}, b_{ij\mu U}\}], [\max\{a_{ij\nu L}, b_{ij\nu L}\}, \max\{a_{ij\nu U}, b_{ij\nu U}\}] >]$.

(v) $\bar{A} = [< [a_{ij\nu L}, a_{ij\nu U}], [a_{ij\mu L}, a_{ij\mu U}] >]$. (complement of A)

(vi) $A^T = [< [a_{ji\mu L}, a_{ji\mu U}], [a_{ji\nu L}, a_{ji\nu U}] >]_{n\times m}$. (transpose of A)

(vii) $A \oplus B = [< [a_{ij\mu L} + b_{ij\mu L} - a_{ij\mu L} \cdot b_{ij\mu L}, a_{ij\mu U} + b_{ij\mu U} - a_{ij\mu U} \cdot b_{ij\mu U}],$
$[a_{ij\nu L} + b_{ij\nu L} - a_{ij\nu L} \cdot b_{ij\nu L}, a_{ij\nu U} + b_{ij\nu U} - a_{ij\nu U} \cdot b_{ij\nu U}] >]$.

(viii) $A \odot B = [< [a_{ij\mu L}.b_{ij\mu L}, a_{ij\mu U}.b_{ij\mu U}],$
$[a_{ij\nu L} + b_{ij\nu L} - a_{ij\nu L}.b_{ij\nu L}, a_{ij\nu U} + b_{ij\nu U} - a_{ij\nu U}.b_{ij\nu U}] >]$.

(ix) $A@B = \Big[\Big\langle\Big[\frac{a_{ij\mu L} + b_{ij\mu L}}{2}, \frac{a_{ij\mu U} + b_{ij\mu U}}{2}\Big], \Big[\frac{a_{ij\nu L} + b_{ij\nu L}}{2}, \frac{a_{ij\nu U} + b_{ij\nu U}}{2}\Big]\Big\rangle\Big]$.

(x) $A\$B = \Big[\Big\langle\Big[\sqrt{a_{ij\mu L}.b_{ij\mu L}}, \sqrt{a_{ij\mu U}.b_{ij\mu U}}\Big], \Big[\sqrt{a_{ij\nu L}.b_{ij\nu L}}, \sqrt{a_{ij\nu U}.b_{ij\nu U}}\Big]\Big\rangle\Big]$.

(xi) $A\#B = \Big[\Big\langle\Big[\dfrac{2a_{ij\mu L}.b_{ij\mu L}}{a_{ij\mu L}+b_{ij\mu L}}, \dfrac{2a_{ij\mu U}.b_{ij\mu U}}{a_{ij\mu U}+a_{ij\mu U}}\Big], \Big[\dfrac{2a_{ij\nu L}.b_{ij\nu L}}{a_{ij\nu L}+b_{ij\nu L}}, \dfrac{2a_{ij\nu U}.b_{ij\nu U}}{a_{ij\nu U}+b_{ij\nu U}}\Big]\Big\rangle\Big]$.

(xii) $A * B = \Big[\Big\langle\Big[\dfrac{a_{ij\mu L}+b_{ij\mu L}}{2(a_{ij\mu L}.b_{ij\mu L}+1)}, \dfrac{a_{ij\mu U}+b_{ij\mu U}}{2(a_{ij\mu U}.b_{ij\mu U}+1)}\Big], \Big[\dfrac{a_{ij\nu L}+b_{ij\nu L}}{2(a_{ij\nu L}.b_{ij\nu L}+1)}, \dfrac{a_{ij\nu U}+b_{ij\nu U}}{2(a_{ij\nu U}.b_{ij\nu U}+1)}\Big]\Big\rangle\Big]$.

(xiii) $A \leq B$ iff $a_{ij\mu L} \leq b_{ij\mu L}, a_{ij\mu U} \leq b_{ij\mu U}, a_{ij\nu L} \geq b_{ij\nu L}$ and $a_{ij\nu U} \geq b_{ij\nu U}$.

(xiv) $A = B$ iff $A \leq B$ and $B \leq A$.

In the following section, we consider a daily life problem which can be studied using IVIFMs in better way.

3 Need of IVIFM

We consider a network consisting of six important cities (vertices) in a country. They are interconnected by roads (edges). The network is shown in Figure 1.

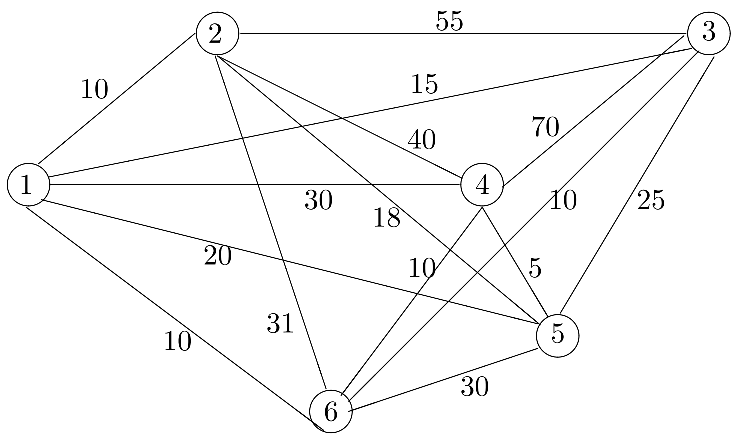

Figure 1: A network.

The number adjacent to an edge represents the distance between the cities (vertices). The above network can be represented with the help of a classical matrix $A = [a_{ij}]$, $i, j = 1, 2, \ldots, n$, where, n is the total number of nodes. The ijth element a_{ij} of A is defined as

$$a_{ij} = \begin{cases} 0, & \text{if } i = j \\ \infty, & \text{the vertices } i \text{ and } j \text{ are not directly connected by an edge} \\ w_{ij}, & w_{ij} \text{ is the distance of the road connecting } i \text{ and } j. \end{cases}$$

Thus the adjacent matrix of the network of Figure 1 is

$$
\begin{array}{c c}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} &
\begin{bmatrix}
0 & 10 & 15 & 30 & 20 & 10 \\
10 & 0 & 55 & 40 & 18 & 30 \\
15 & 55 & 0 & 70 & 25 & 10 \\
30 & 40 & 70 & 0 & 5 & 10 \\
20 & 18 & 25 & 5 & 0 & 30 \\
10 & 31 & 10 & 10 & 30 & 0
\end{bmatrix}
\end{array}
$$

Since the distance between two vertices are known, precisely, so the above matrix is obviously a classical matrix. Generally, the distance between two cities are crisp value, so the corresponding matrix is crisp matrix.

Now, we consider the crowdness of the roads connecting cities. It is clear that the crowdness of a road obviously, is a fuzzy quantity. The amount of crowdness depends on the decision makers mentality, habits, natures, etc. i.e., completely depends on the decision maker. The measurement of crowdness as a point is a difficult task for the decision maker. So, here we consider the amount of crowdness as an interval instead of a point. Similarly, the loneliness is also considered as an interval. The crowdness and loneliness of a network can not be represented as a crisp matrix, it can be represented appropriately by a matrix which we designate by interval-valued intuitionistic fuzzy matrices (IVIFMs).

For illustration, we consider the crowdness and loneliness of the road (i, j) connecting the places i and j as follows:

Roads	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(2,3)	(2,4)	(2,5)
Crowdness	[.1,.3]	[.2,.4]	[.3,.4]	[.2,.4]	[.3,.6]	[.7,.8]	[.3,.5]	[.3,.4]
Loneliness	[.2,.5]	[.1,.5]	[.5,.6]	[.4,.5]	[.2,.3]	[0,.1]	[.4,.5]	[.4,.6]
Roads	(2,6)	(3,4)	(3,5)	(3,6)	(4,5)	(4,6)	(5,6)	
Crowdness	[.2,.3]	[.5,.6]	[.3,.5]	[.3,.6]	[.4,.6]	[.2,.4]	[.3,.5]	
Loneliness	[.4,.5]	[.2,.3]	[.2,.3]	[.2,.3]	[.3,.4]	[.3,.5]	[.2,.4]	

Table 1: The crowdness and loneliness of the network of Figure 1.

The matrix representation of the traffic crowdness and loneliness of the network of Figure 1 is shown in the following IVIFM.

	1	2	3	4	5	6
1	$<[0,0],[1,1]>$	$<[.1,.3],[.2,.5]>$	$<[.2,.4],[.1,.5]>$	$<[.3,.4],[.5,.6]>$	$<[.2,.4],[.4,.5]>$	$<[.3,.6],[.2,.3]>$
2	$<[.1,.3],[.2,.5]>$	$<[0,0],[1,1]>$	$<[.7,.8],[0,.1]>$	$<[.3,.5],[.4,.5]>$	$<[.3,.4],[.4,.6]>$	$<[.2,.3],[.4,.5]>$
3	$<[.2,.4],[.1,.5]>$	$<[.7,.8],[0,.1]>$	$<[0,0],[1,1]>$	$<[.5,.6],[.2,.3]>$	$<[.3,.5],[.2,.3]>$	$<[.3,.6],[.2,.3]>$
4	$<[.3,.4],[.5,.6]>$	$<[.3,.5],[.4,.5]>$	$<[.5,.6],[.2,.3]>$	$<[0,0],[1,1]>$	$<[.4,.6],[.3,.4]>$	$<[.2,.4],[.3,.5]>$
5	$<[.2,.4],[.4,.5]>$	$<[.3,.4],[.4,.6]>$	$<[.3,.5],[.2,.3]>$	$<[.4,.6],[.3,.4]>$	$<[0,0],[1,1]>$	$<[.3,.5],[.2,.4]>$
6	$<[.3,.6],[.2,.3]>$	$<[.2,.3],[.4,.5]>$	$<[.3,.6],[.2,.3]>$	$<[.2,.4],[.3,.5]>$	$<[.3,.5],[.2,.4]>$	$<[0,0],[1,1]>$

To explain the meaning of the operators defined earlier we consider two IVIFMs A and B. Let A and B represent respectively the crowdness and the loneliness of the network at two time instances t and t'. Now, the IVIFM $A+B$ represents the maximum amount of traffic crowdness and minimum amount of loneliness of the network between the time instances t and t'. $A.B$ represents the minimum amount of traffic crowdness and maximum amount of loneliness of the network. $\bar{A}$ matrix represents the loneliness and crowdness of the network. $A@B, A\$B$ and $A\#B$ reveals the arithmetic mean, geometric mean and harmonic mean of the crowdness and loneliness in between the two time instances t and t' of the network .

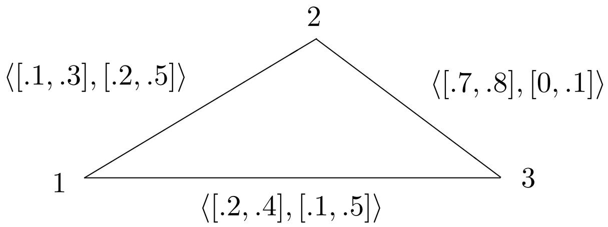

Figure 2:

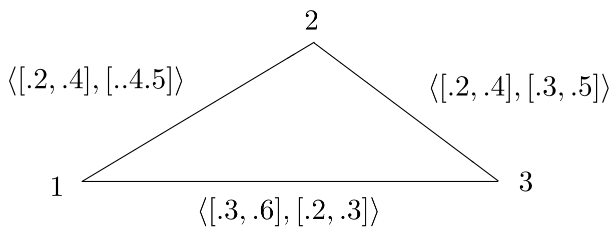

Figure 3:

To illustrate the operators $A.B$, $A + B$ and $|A|$, we consider a network consisting three vertices and three edges. The crowdness and loneliness of the network are observed at two different time instances t and t'. The matrices A_t and $A_{t'}$ represent the status of the network at t (Figure 2) and at t' (Figure 3). The number adjacent to the sides represents the crowdness and loneliness of the roads at two different instances of the same network. A_t and $A_{t'}$ be the matrix representation of crowdness and loneliness at time t and t' respectively,

$$\text{Let } A_t = \begin{bmatrix} <[0,0],[1,1]> & <[.1,.3],[.2,.5]> & <[.2,.4],[.1,.5]> \\ <[.1,.3],[.2,.5]> & <[0,0],[1,1]> & <[.7,.8],[0,.1]> \\ <[.2,.4],[.1,.5]> & <[.7,.8],[0,.1]> & <[0,0],[1,1]> \end{bmatrix}$$

$$\text{and } A_{t'} = \begin{bmatrix} <[0,0],[1,1]> & <[.2,.4],[.4,.5]> & <[.3,.6],[.2,.3]> \\ <[.2,.4],[.1,.5]> & <[0,0],[1,1]> & <[.2,.4],[.3,.5]> \\ <[.3,.6],[.2,.3]> & <[.2,.4],[.3,.5]> & <[0,0],[1,1]> \end{bmatrix}.$$

$$\text{So, } A_t.A_{t'} = \begin{bmatrix} <[0,0],[1,1]> & <[.1,.3],[.4,.5]> & <[.2,.4],[.2,.5]> \\ <[.1,.3],[.4,.5]> & <[0,0],[1,1]> & <[.2,.4],[.3,.5]> \\ <[.2,.4],[.2,.5]> & <[.2,.4],[.3,.5]> & <[0,0],[1,1]> \end{bmatrix}$$

$$\text{and, } A_t + A_{t'} = \begin{bmatrix} <[0,0],[1,1]> & <[.2,.4],[.2,.5]> & <[.3,.6],[.1,.3]> \\ <[.2,.4],[.2,.5]> & <[0,0],[1,1]> & <[.7,.8],[0,.1]> \\ <[.3,.6],[.1,.3]> & <[.7,.8],[0,.1]> & <[0,0],[1,1]> \end{bmatrix}.$$

$$\begin{aligned}
|A_t| = &<[0,0],[1,1]> \{<[0,0],[1,1]><[0,0],[1,1]> + <[.7,.8],[.0,.1]><[.7,.8],[0,1]>\} \\
&+ <[.1,.3],[.2,.5]> \{<[.7,.8],[0,.1]><[.2,.4],[.1,.5]> + <.1,.3],[.2,.5]><[0,0],[1,1]>\} \\
&+ <[.2,.4],[.1,.5]> \{<[.1,.3],[.2,.5]><.7,.8],[0,.1]> + <[0,0],[1,1]><[.2,.4],[.1,.5]>\} \\
= &<[0,0],[1,1]> \{<[0,0],[1,1]> + <[.7,.8],[0,.1]>\} \\
&+ <[.1,.3],[.2,.5]> \{<[.2,.4],[.1,.5]> + <[0,0],[1,1]>\} \\
&+ <[.2,.4],[.1,.5]> \{<[.1,.3],[.2,.5]> + <[0,0],[1,1]>\} \\
= &<[0,0],[1,1]><[.7,.8],[0,.1]> \\
&+ <[.1,.3],[.2,.5]><[.2,.4],[.1,.5]> \\
&+ <[.2,.4],[.1,.5]><[.1,.3],[.2,.5]> \\
= &<[0,0],[1,1]> + <[.1,.3],[.2,.5]> + <[.1,.3],[.2,.5]> \\
= &<[.1,.3],[.2,.5]>
\end{aligned}$$

It may be noted that if the ij-th element of the IVIFM A_t is $<[0,0],[1,1]>$ then it indicates that the road (i,j) is fully lonely (not crowd), but, if it is $<[1,1],[0,0]>$ then the road (i,j) is fully crowd or blocked.

4 Properties of IVIFMs

In this section some properties of IVIFMs are presented.
IVIFMs satisfy the commutative and associative properties over the operators $+, ., \oplus$, and $\odot$. The operator '.' is distributed over '+' in left and right but the left and right distribution laws do not

hold for the operators $\oplus$ and $\odot$.

(1) $A+B=B+A$

(2) $A+(B+C)=(A+B)+C$

(3) $A.B=B.A$

(4) $A.(B.C)=(A.B).C$

(5) (i)$A.(B+C)=A.B+A.C$

(ii) $(B+C).A=B.A+C.A$

(6) $A\oplus B=B\oplus A$

(7) $A\oplus(B\oplus C)=(A\oplus B)\oplus C$

(8) $A\odot B=B\odot A$

(9) $A\odot(B\odot C)=(A\odot B)\odot C$

(10)(i) $A\odot(B\oplus C)\neq(A\odot B)\oplus(A\odot C)$

(ii) $(B\oplus C)\odot A\neq(B\odot A)\oplus(C\odot A)$

Proof of (i): Let $A=[<[a_{ij\mu L},a_{ij\mu U}],[a_{ij\nu L},a_{ij\nu U}]>]$,
$B=[<[b_{ij\mu L},b_{ij\mu U}],[b_{ij\nu L},b_{ij\nu U}]>]$
and $C=[<[c_{ij\mu L},c_{ij\mu U}],[c_{ij\nu L},c_{ij\nu U}]>]$.
So, $B\oplus C=[<[b_{ij\mu L}+c_{ij\mu L}-b_{ij\mu L}.c_{ij\mu L},b_{ij\mu U}+c_{ij\mu U}-b_{ij\mu U}.c_{ij\mu U}],[b_{ij\nu L}.c_{ij\nu L},b_{ij\nu U}.c_{ij\nu U}]>]$
and $A\odot(B\oplus C)=[<[a_{ij\mu L}.(b_{ij\mu L}+c_{ij\mu L}-b_{ij\mu L}.c_{ij\mu L}),a_{ij\mu U}(b_{ij\mu U}+c_{ij\mu U}-b_{ij\mu U}.c_{ij\mu U}],[a_{ij\nu L}+b_{ij\nu L}.c_{ij\nu L}-a_{ij\nu L}.b_{ij\nu L}.c_{ij\nu L},a_{ij\mu U}+b_{ij\mu U}.c_{ij\mu U}-a_{ij\mu U}.b_{ij\mu U}.c_{ij\mu U}]>]$.
$A\odot B=[<[a_{ij\mu L}.b_{ij\mu L},a_{ij\mu U}.b_{ij\mu U}],[a_{ij\nu L}+b_{ij\nu L}-a_{ij\nu L}.b_{ij\nu L},a_{ij\nu U}+b_{ij\nu U}-a_{ij\nu U}.b_{ij\nu U}]>]$,
$A\odot C=[<[a_{ij\mu L}.c_{ij\mu L},a_{ij\mu U}.c_{ij\mu U}],[a_{ij\nu L}+c_{ij\nu L}-a_{ij\nu L}.c_{ij\nu L},a_{ij\nu U}+c_{ij\nu U}-a_{ij\nu U}.c_{ij\nu U}]>]$.
Now, $(A\odot B)\oplus(A\odot C)=[<[a_{ij\mu L}(b_{ij\mu L}+c_{ij\mu L})-a^2_{ij\mu L}.b_{ij\mu L}.c_{ij\mu L},a_{ij\mu U}(b_{ij\mu U}+c_{ij\mu U})-a^2_{ij\mu U}.b_{ij\mu U}.c_{ij\mu U}],[(a_{ij\nu L}+b_{ij\nu L}-a_{ij\nu L}.b_{ij\nu L}).(a_{ij\nu L}+c_{ij\nu L}-a_{ij\nu L}.c_{ij\nu L}),(a_{ij\nu U}+b_{ij\nu U}-a_{ij\nu U}.b_{ij\nu U}).(a_{ij\nu U}+c_{ij\nu U}-a_{ij\nu U}.c_{ij\nu U})]>]$.
So, $A\odot(B\oplus C)\neq(A\odot B)\oplus(A\odot C)$.

Property 1 *Let A be an IVIFM of any order then, $A+A=A$.*

Proof: Let $A=[<[a_{ij\mu L},a_{ij\mu U}],[a_{ij\nu L},a_{ij\nu U}]>]$
Then $A+A=[<[\max(a_{ij\mu L},a_{ij\mu L}),\max(a_{ij\mu U},a_{ij\mu U})],[\min(a_{ij\nu L},a_{ij\nu L}),\min(a_{ij\nu U},a_{ij\nu U})]>]$
$=[<[a_{ij\mu L},a_{ij\mu U}],[a_{ij\nu L},a_{ij\nu U}]>]$
$=A$.

Property 2 *If A be an IVIFM of any order then, $A+I_{<[0,0],[0,0]>}\geq A$ where, $I_{<[0,0],[0,0]>}$ is the null IVIFM of same order.*

Proof: Let $A=[<[a_{ij\mu L},a_{ij\mu U}],[a_{ij\nu L},a_{ij\nu U}]>]$
and $I_{<[0,0],[0,0]>}=<[0,0],[0,0]>$.

Then, $A+I_{<[0,0],[0,0]>} = [< [\max(a_{ij\mu L},0), \max(a_{ij\mu U},0)], [\min(a_{ij\nu L},0), \min(a_{ij\nu U},0)] >]$
$= [< [a_{ij\mu L}, a_{ij\mu U}], [0,0] >]$

Therefore, $A+I_{<[0,0],[0,0]>} \geq A$.

Some more properties on determinant and adjoint of IVIFM are presented below.

Property 3 *Like classical matrices the determinant value of an IVIFM and its transpose are equal. If A be a square IVIFM then $|A| = |A^T|$.*

Proof: Let $A = [< [a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}] >]$.
Then $A^T = B = [< [b_{ij\mu L}, b_{ij\mu U}], [b_{ij\nu L}, b_{ij\nu U}] >]$
$= [< [a_{ji\mu L}, a_{ji\mu U}], [a_{ji\nu L}, a_{ji\nu U}] >]$.

Now,

$$|B| = \sum_{\sigma \in S_n} < [b_{1\sigma(1)\mu L}, b_{1\sigma(1)\mu U}], [b_{1\sigma(1)\nu L}, b_{1\sigma(1)\nu U}] >< [b_{2\sigma(2)\mu L}, b_{2\sigma(2)\mu U}], [b_{2\sigma(2)\nu L}, b_{2\sigma(2)\nu U}] > \dots < [b_{n\sigma(n)\mu L}, b_{n\sigma(n)\mu U}], [b_{n\sigma(n)\nu L}, b_{n\sigma(n)\nu U}] >$$

$$= \sum_{\sigma \in S_n} < [a_{\sigma(1)1\mu L}, a_{\sigma(1)1\mu U}], [a_{\sigma(1)1\nu L}, a_{\sigma(1)1\nu U}] >< [a_{\sigma(2)2\mu L}, a_{\sigma(2)2\mu U}], [a_{\sigma(2)2\nu L}, a_{\sigma(2)2\nu U}] > \dots < [a_{\sigma(n)n\mu L}, a_{\sigma(n)n\mu U}], [a_{\sigma(n)n\nu L}, a_{\sigma(n)n\nu U}] > .$$

Let ψ be the permutation of $\{1,2\dots n\}$ such that $\psi\sigma = I$, the identity permutation. Then $\psi = \sigma^{-1}$. As σ runs over the whole set of permutations, so does ψ.
Let $\sigma(i) = j,\ i = \sigma^{-1}(j) = \psi(j)$.

Therefore, $a_{\sigma(i)i\mu L} = a_{j\psi(j)\mu L},\ a_{\sigma(i)i\mu U} = a_{j\psi(j)\mu U},\ a_{\sigma(i)i\nu L} = a_{j\psi(j)\nu L},\ a_{\sigma(i)i\nu U} = a_{j\psi(j)\nu U}$ for all i,j.
As i runs over the set $\{1,2,\dots,n\}$, j does so.
Now, $< [a_{\sigma(1)1\mu L}, a_{\sigma(1)1\mu U}], [a_{\sigma(1)1\nu L}, a_{\sigma(1)1\nu U}] >< [a_{\sigma(2)2\mu L}, a_{\sigma(2)2\mu U}], [a_{\sigma(2)2\nu L}, a_{\sigma(2)2\nu U}] > \dots$
$< [a_{\sigma(n)n\mu L}, a_{\sigma(n)n\mu U}], [a_{\sigma(n)n\nu L}, a_{\sigma(n)n\nu U}] >$
$=< [a_{1\psi(1)\mu L}, a_{1\psi(1)\mu U}], [a_{1\psi(1)\nu L}, a_{1\psi(1)\nu U}] >< [a_{2\psi(2)\mu L}, a_{2\psi(2)\mu U}], [a_{2\psi(2)\nu L}, a_{2\psi(2)\nu U}] > \dots$
$< [a_{n\psi(n)\mu L}, a_{n\psi(n)\mu U}], [a_{n\psi(n)\nu L}, a_{n\psi(n)\nu U}] > .$

Therefore,

$$|B| = \sum_{\sigma \in S_n} < [a_{\sigma(1)1\mu L}, a_{\sigma(1)1\mu U}], [a_{\sigma(1)1\nu L}, a_{\sigma(1)1\nu U}] >< [a_{\sigma(2)2\mu L}, a_{\sigma(2)2\mu U}], [a_{\sigma(2)2\nu L}, a_{\sigma(2)2\nu U}] > \dots < [a_{\sigma(n)n\mu L}, a_{\sigma(n)n\mu U}], [a_{\sigma(n)n\nu L}, a_{\sigma(n)n\nu U}] >$$

$$= \sum_{\psi \in S_n} < [a_{1\psi(1)\mu L}, a_{1\psi(1)\mu U}], [a_{1\psi(1)\nu L}, a_{1\psi(1)\nu U}] >< [a_{2\psi(2)\mu L}, a_{2\psi(2)\mu U}], [a_{2\psi(2)\nu L}, a_{2\psi(2)\nu U}] > \dots < [a_{n\psi(n)\mu L}, a_{n\psi(n)\mu U}], [a_{n\psi(n)\nu L}, a_{n\psi(n)\nu U}] >$$

$= |A|$.

Property 4 *If A and B be two square IVIFMs and $A \leq B$, then, adj. $A \leq$ adj. B.*

Proof: Let, $C = [< [c_{ij\mu L}, c_{ij\mu U}], [c_{ij\nu L}, c_{ij\nu U}] >] = adj.\, A$,
$D = [< [d_{ij\mu L}, d_{ij\mu U}], [d_{ij\nu L}, c_{ij\nu U}] >] = adj.\, B$
where, $< [c_{ij\mu L}, c_{ij\mu U}], [c_{ij\nu L}, c_{ij\nu U}] > = \sum_{\sigma \in S_{n_i n_j}} \prod_{t \in n_j} < [a_{t\sigma(t)\mu L}, a_{t\sigma(t)\mu U}], [a_{t\sigma(t)\nu L}, a_{t\sigma(t)\nu U} >$ and $< [d_{ij\mu L}, d_{ij\mu U}], [d_{ij\nu L}, c_{ij\nu U}] > = \sum_{\sigma \in S_{n_i n_j}} \prod_{t \in n_j} < [b_{t\sigma(t)\mu L}, b_{t\sigma(t)\mu U}], [b_{t\sigma(t)\nu L}, b_{t\sigma(t)\nu U} >$. It is clear that $< [c_{ij\mu L}, c_{ij\mu U}], [c_{ij\nu L}, c_{ij\nu U}] > \leq < [d_{ij\mu L}, d_{ij\mu U}], [d_{ij\nu L}, c_{ij\nu U}] >$.
Since, $a_{t\sigma(t)\mu L} \leq b_{t\sigma(t)\mu L}$, $a_{t\sigma(t)\mu U} \leq b_{t\sigma(t)\mu U}$, $a_{t\sigma(t)\nu L} \geq b_{t\sigma(t)\nu L}$, and $a_{t\sigma(t)\nu U} \geq b_{t\sigma(t)\nu U}$ for all $t \neq j$, $\sigma(t) \neq \sigma(j)$.
Therefore $C \leq D$, i.e., $adj.\, A \leq adj.\, B$.

Property 5 *For a square IVIFM A, $adj.\,(A^T) = (adj.\, A)^T$.*

Proof: Let $B = adj.\, A$, $C = adj.\, A^T$.
Therefore, $< [b_{ij\mu L}, b_{ij\mu U}], [b_{ij\nu L}, b_{ij\nu U}] > = \sum_{\sigma \in S_{n_j n_i}} \prod_{t \in n_i} < [a_{t\sigma(t)\mu L}, a_{t\sigma(t)\mu U}], [a_{t\sigma(t)\nu L}, a_{t\sigma(t)\nu U} >$ and $< [c_{ij\mu L}, c_{ij\mu U}], [c_{ij\nu L}, c_{ij\nu U}] > = \sum_{\sigma \in S_{n_i n_j}} \prod_{t \in n_j} < [a_{t\sigma(t)\mu L}, a_{t\sigma(t)\mu U}], [a_{t\sigma(t)\nu L}, a_{t\sigma(t)\nu U} >$ $=< [b_{ij\mu L}, b_{ij\mu U}], [b_{ij\nu L}, b_{ij\nu U}] >$.
Therefore, $adj.\,(A^T) = (adj.\, A)^T$.

The following result is not valid for classical matrices, though it is true for IVIFM.

Property 6 *For an IVIFM A, $|A| = |adj.\, A|$.*

Proof: $adj.\, A = [< [A_{ij\mu L}, A_{ij\mu U}], [A_{ij\nu L}, A_{ij\nu U}] >]$.
where, $< [A_{ij\mu L}, A_{ij\mu U}], [A_{ij\nu L}, A_{ij\nu U}] >$ is the cofactor of the element $< [a_{ij\mu L}, a_{ij\mu U}], [a_{ij\nu L}, a_{ij\nu U}] >$ in the IVIFM A.

$$\begin{aligned}\text{Therefore, } |adj.\, A| = \sum_{\sigma \in S_n} & < [A_{1\sigma(1)\mu L}, A_{1\sigma(1)\mu U}], [A_{1\sigma(1)\nu L}, A_{1\sigma(1)\nu U}] > \\ & < [A_{2\sigma(2)\mu L}, A_{2\sigma(2)\mu U}], [A_{2\sigma(2)\nu L}, A_{2\sigma(2)\nu U}] > \\ & \ldots < [A_{n\sigma(n)\mu L}, A_{n\sigma(n)\mu U}], [A_{n\sigma(n)\nu L}, A_{n\sigma(n)\nu U}] > \end{aligned}$$

$$= \sum_{\sigma \in S_n} \prod_{i=1}^{n} < [A_{i\sigma(i)\mu L}, A_{i\sigma(i)\mu U}], [A_{i\sigma(i)\nu L}, A_{i\sigma(i)\nu U}] >$$

$$= \sum_{\sigma \in S_n} \left[\prod_{i=1}^{n} \left(\sum_{\theta \in S_{n_i n_{\sigma(i)}}} \prod_{t \in n_i} < [a_{t\theta(t)\mu L}, a_{t\theta(t)\mu U}], [a_{t\theta(t)\nu L}, a_{t\theta(t)\nu U}] > \right) \right]$$

$$\begin{aligned} = \sum_{\sigma \in S_n} \Bigg[& \left(\prod_{t \in n_1} < [a_{t\theta_1(t)\mu L}, a_{t\theta_1(t)\mu U}], [a_{t\theta_1(t)\nu L}, a_{t\theta_1(t)\nu U}] > \right) \Bigg(\prod_{t \in n_2} < [a_{t\theta_2(t)\mu L}, a_{t\theta_2(t)\mu U}], \\ & [a_{t\theta_2(t)\nu L}, a_{t\theta_2(t)\nu U}] > \Bigg) \ldots \left(\prod_{t \in n_n} < [a_{t\theta_n(t)\mu L}, a_{t\theta_n(t)\mu U}], [a_{t\theta_n(t)\nu L}, a_{t\theta_n(t)\nu U}] > \right) \Bigg] \end{aligned}$$

$\left(\text{For some } \theta_1 \in S_{n_1 n_{\sigma(1)}}, \theta_2 \in S_{n_2 n_{\sigma(2)}}, \ldots, \theta_n \in S_{n_1 n_{\sigma(n)}}\right)$

$= \sum_{\sigma \in S_n} [(< [a_{2\theta_1(2)\mu L}, a_{2\theta_1(2)\mu U}], [a_{2\theta_1(2)\nu L}, a_{2\theta_1(2)\nu U}] >< [a_{3\theta_1(3)\mu L}, a_{3\theta_1(3)\mu U}], [a_{3\theta_1(3)\nu L}, a_{3\theta_1(3)\nu U}] > \ldots < [a_{n\theta_1(n)\mu L}, a_{n\theta_1(n)\mu U}], [a_{n\theta_1(n)\nu L}, a_{n\theta_1(n)\nu U}] >)(< [a_{1\theta_2(1)\mu L}, a_{1\theta_2(1)\mu U}], [a_{1\theta_2(1)\nu L}, a_{1\theta_2(1)\nu U}] > < [a_{3\theta_2(3)\mu L}, a_{3\theta_2(3)\mu U}], [a_{3\theta_2(3)\nu L}, a_{3\theta_2(3)\nu U}] > \ldots < [a_{n\theta_2(n)\mu L}, a_{n\theta_2(n)\mu U}], [a_{n\theta_2(n)\nu L}, a_{n\theta_2(n)\nu U}] >)$
$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$(< [a_{1\theta_n(1)\mu L}, a_{1\theta_n(1)\mu U}], [a_{1\theta_n(1)\nu L}, a_{1\theta_n(1)\nu U}] >< [a_{2\theta_n(2)\mu L}, a_{2\theta_n(2)\mu U}], [a_{2\theta_n(2)\nu L}, a_{2\theta_n(2)\nu U}] > \dots$
$< [a_{(n-1)\theta_n(n-1)\mu L}, a_{(n-1)\theta_n(n-1)\mu U}], [a_{(n-1)\theta_n(n-1)\nu L}, a_{(n-1)\theta_n(n-1)\nu U}] >)]$
$= \sum_{\sigma\in S_n}[(< [a_{1\theta_2(1)\mu L}, a_{1\theta_2(1)\mu U}], [a_{1\theta_2(1)\nu L}, a_{1\theta_2(1)\nu U}] >< [a_{1\theta_3(1)\mu L}, a_{1\theta_3(1)\mu U}], [a_{1\theta_3(1)\nu L}, a_{1\theta_3(1)\nu U}] >$
$\dots < [a_{1\theta_n(1)\mu L}, a_{1\theta_n(1)\mu U}], [a_{1\theta_n(1)\nu L}, a_{1\theta_n(1)\nu U}] >)(< [a_{2\theta_1(2)\mu L}, a_{2\theta_1(2)\mu U}], [a_{2\theta_1(2)\nu L}, a_{2\theta_1(2)\nu U}] >$
$< [a_{2\theta_3(2)\mu L}, a_{2\theta_3(2)\mu U}], [a_{2\theta_3(2)\nu L}, a_{2\theta_3(2)\nu U}] > \dots < [a_{2\theta_n(2)\mu L}, a_{2\theta_n(2)\mu U}], [a_{2\theta_n(2)\nu L}, a_{2\theta_n(2)\nu U}] >)$
$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$
$(< [a_{n\theta_1(n)\mu L}, a_{n\theta_1(n)\mu U}], [a_{n\theta_1(n)\nu L}, a_{n\theta_1(n)\nu U}] >< [a_{n\theta_2(n)\mu L}, a_{n\theta_2(n)\mu U}], [a_{n\theta_2(n)\nu L}, a_{n\theta_2(n)\nu U}] >$
$\dots < [a_{n\theta_{(n-1)}(n)\mu L}, a_{n\theta_{(n-1)}(n)\mu U}], [a_{n\theta_{(n-1)}(n)\nu L}, a_{n\theta_{(n-1)}(n)\nu U}] >)]$
$= \sum_{\sigma\in S_n}[< [a_{1\theta_{f_1}(1)\mu L}, a_{1\theta_{f_1}(1)\mu U}], [a_{1\theta_{f_1}(1)\nu L}, a_{1\theta_{f_1}(1)\nu U}] >$
$< [a_{2\theta_{f_2}(2)\mu L}, a_{2\theta_{f_2}(2)\mu U}], [a_{2\theta_{f_2}(2)\nu L}, a_{2\theta_{f_2}(2)\nu U}] >$
$\dots < [a_{n\theta_{f_n}(n)\mu L}, a_{n\theta_{f_n}(n)\mu U}], [a_{n\theta_{f_n}(n)\nu L}, a_{n\theta_{f_n}(n)\nu U}] >]$
where, $f_{\hat{\theta}} \in \{1, 2, \dots, n\}\backslash\{\hat{\theta}\}, \hat{\theta} = 1, 2, \dots, n.$

But since, $< [a_{\hat{\theta}\theta_{f_{\hat{\theta}}}(\hat{\theta})\mu L}, a_{\hat{\theta}\theta_{f_{\hat{\theta}}}(\hat{\theta})\mu U}], [a_{\hat{\theta}\theta_{f_{\hat{\theta}}}(\hat{\theta})\nu L}, a_{\hat{\theta}\theta_{f_{\hat{\theta}}}(\hat{\theta})\nu U}] >$
$=< [a_{n\sigma(n)\mu L}, a_{n\sigma(n)\mu U}], [a_{n\sigma(n)\nu L}, a_{n\sigma(n)\nu U}] >.$

Therefore, $|adj.\ A| = \sum_{\sigma\in S_n} < [a_{1\sigma(1)\mu L}, a_{1\sigma(1)\mu U}], [a_{1\sigma(1)\nu L}, a_{1\sigma(1)\nu U}] >$
$< [a_{2\sigma(2)\mu L}, a_{2\sigma(2)\mu U}], [a_{2\sigma(2)\nu L}, a_{2\sigma(2)\nu U}] > \dots$
$< [a_{n\sigma(n)\mu L}, a_{n\sigma(n)\mu U}], [a_{n\sigma(n)\nu L}, a_{n\sigma(n)\nu U}] >$
$= |A|.$

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