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Modal forms of Fodor's type of intuitionistic fuzzy implication

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Abstract: Herewith proposed are four new intuitionistic fuzzy implications, based on the Fodor's type of intuitionistic fuzzy implication, introduced earlier by the authors. These four implications are modifications of the first implication, but in modal forms. Some of their properties are discussed.
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1 Introduction

On the basis of Janos Fodor's fuzzy implication [6], that for $a, c \in [0, 1]$ is defined by

$$a \to c = \begin{cases} 1, & \text{if } a \leq c \\ \max(1-a, c), & \text{otherwise} \end{cases},$$

in [5] the authors defined its intuitionistic fuzzy version. Here, the intuitionistic fuzzy counterpart of Janos Fodor's fuzzy implication will be modified to four new implications of modal type.

In intuitionistic fuzzy propositional calculus, if x is a variable then its truth-value is represented by the ordered couple

$$V(x) = \langle a, b \rangle,$$

so that $a, b, a + b \in [0, 1]$, where a and b are degrees of validity and of non-validity of x. In [4], we called this couple an *"intuitionistic fuzzy pair" (IFP).*

Below we assume that for the two variables x and y the equalities: $V(x) = \langle a, b \rangle$ and $V(y) = \langle c, d \rangle$ $(a, b, c, d, a + b, c + d \in [0, 1])$ hold.

For the needs of the discussion below we shall define the notion of Intuitionistic Fuzzy Tautology (IFT, see [1, 2]) by:

$$x \text{ is an IFT if and only if for } V(x) = \langle a, b \rangle \text{ holds: } a \geq b,$$

while x will be a tautology iff $a = 1$ and $b = 0$. As in the case of ordinary logics, x is a tautology, if $V(x) = \langle 1, 0 \rangle$.

The Fodor's Type of an intuitionistic fuzzy implication from [5] is defined by

$$V(x \to y) = \langle a, b \rangle \to \langle c, d \rangle = \langle \overline{\text{sg}}(a - c) + \text{sg}(a - c) \max(b, c), \text{sg}(a - c) \min(a, d) \rangle,$$

where we use functions sg and $\overline{\text{sg}}$ defined by,

$$\text{sg}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}, \quad \overline{\text{sg}}(x) = \begin{cases} 0, & \text{if } x > 0 \\ 1, & \text{if } x \leq 0 \end{cases}.$$

In [2] a list of 138 intuitionistic fuzzy implications is given. In [3] it is extended, so that the implication from [5] is numbered as $\to_{176}$. Below, we keep this numeration.

2 Main results

First, using the following formulas, we obtain the four new implications:

$$\langle a, b \rangle \to_{177} \langle c, d \rangle = \Box \langle a, b \rangle \to_{176} \Box \langle c, d \rangle,$$

$$\langle a, b \rangle \to_{178} \langle c, d \rangle = \Box \langle a, b \rangle \to_{176} \Diamond \langle c, d \rangle,$$

$$\langle a, b \rangle \to_{179} \langle c, d \rangle = \Diamond \langle a, b \rangle \to_{176} \Diamond \langle c, d \rangle,$$

$$\langle a, b \rangle \to_{180} \langle c, d \rangle = \Diamond \langle a, b \rangle \to_{176} \Box \langle c, d \rangle.$$

So, we obtain the explicit forms of the new four implications as follows:

$$\langle a, b \rangle \to_{177} \langle c, d \rangle = \langle \overline{\text{sg}}(a - c) + \text{sg}(a - c) \max(1 - a, c), \text{sg}(a - c) \min(a, 1 - c) \rangle,$$

$$\langle a, b \rangle \to_{178} \langle c, d \rangle = \langle \overline{\text{sg}}(a - 1 + d) + \text{sg}(a - 1 + d)(1 - \min(a, d)), \text{sg}(a - 1 + d) \min(a, d) \rangle,$$

$$\langle a, b \rangle \to_{179} \langle c, d \rangle = \langle \overline{\text{sg}}(1 - b - c) + \text{sg}(1 - b - c) \max(b, c), \text{sg}(1 - b - c)(1 - \max(b, c)) \rangle,$$

$$\langle a,b\rangle \rightarrow_{180} \langle c,d\rangle = \langle \overline{\mathrm{sg}}(d-b)+\mathrm{sg}(d-b)\max(b,1-d), \mathrm{sg}(d-b)\min(1-b,d)\rangle.$$

Let

$$X_{177} \equiv \overline{\mathrm{sg}}(a-c)+\mathrm{sg}(a-c)\max(1-a,c)+\mathrm{sg}(a-c)\min(a,1-c).$$

If $a \leq c$, then

$$X_{177} = 1+0.\max(1-a,c)+0.\min(a,1-c) = 1.$$

If $a > c$, then

$$X_{177} = 0+1.\max(1-a,c)+1.\min(a,1-c).$$

If $1-a \geq c$, then

$$X_{177} = 1-a+\min(a,1-c) \leq 1-a+a = 1.$$

If $1-a < c$, then

$$X_{177} = c+\min(a,1-c) leq c+1-c = 1.$$

Therefore, implication $\rightarrow_{177}$ is defined correctly. Analogously, we can prove also that the other implications are defined correctly.

Second, we check that for every $i = 177, 178, 179, 180$:

$$\langle 0,1\rangle \rightarrow_i \langle 0,1\rangle = \langle 1,0\rangle,$$

$$\langle 0,1\rangle \rightarrow_i \langle 1,0\rangle = \langle 1,0\rangle,$$

$$\langle 1,0\rangle \rightarrow_i \langle 0,1\rangle = \langle 0,1\rangle,$$

$$\langle 1,0\rangle \rightarrow_i \langle 1,0\rangle = \langle 1,0\rangle.$$

Using the definitions from [2, 4]

$$\langle a,b\rangle \geq \langle c,d\rangle \text{ if and only if } a \geq c \text{ and } b \leq d,$$

we can prove the validity of the following

Theorem 1. For every $a,b,c,d \in [0,1]$, so that $a+b \leq 1$ and $c+d \leq 1$ and for every $\lambda \geq 1$:

$$\langle a,b\rangle \rightarrow_{178} \langle c,d\rangle \geq \langle a,b\rangle \rightarrow_{177} \langle c,d\rangle, \tag{1}$$

$$\langle a,b\rangle \rightarrow_{178} \langle c,d\rangle \geq \langle a,b\rangle \rightarrow_{180} \langle c,d\rangle, \tag{2}$$

$$\langle a,b\rangle \rightarrow_{177} \langle c,d\rangle \geq \langle a,b\rangle \rightarrow_{179} \langle c,d\rangle, \tag{3}$$

$$\langle a,b\rangle \rightarrow_{180} \langle c,d\rangle \geq \langle a,b\rangle \rightarrow_{179} \langle c,d\rangle. \tag{4}$$

Proof: For example, let us check the validity of the fourth inequality.

First, we see, that

$$\max(b,1-d) \geq \max(b,c),$$

$$1-\max(b,c) \geq \min(1-b,d).$$

Therefore,

$$\langle \overline{\text{sg}}(d-b) + \text{sg}(d-b)\max(b, 1-d), \text{sg}(d-b)\min(1-b, d)\rangle$$

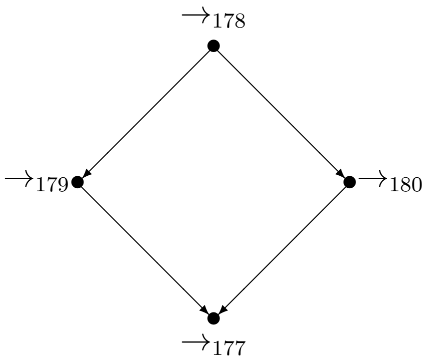
$$\geq \overline{\text{sg}}(1-b-c) + \text{sg}(1-b-c)\max(b,c), \text{sg}(1-b-c)(1-\max(b,c))\rangle,$$

i.e.,

$$\langle a, b\rangle \to_{180} \langle c, d\rangle \geq \langle a, b\rangle \to_{179} \langle c, d\rangle.$$

Hence, (4) is valid. (1)–(3) are proved by analogical manner. □

Now, we can construct the following diagram



The new intuitionistic fuzzy implications generate the following intuitionistic fuzzy negations

$$\neg_{177}\langle a, b\rangle = \langle a, b\rangle \to_{177} \langle 0, 1\rangle = \langle \overline{\text{sg}}(a) + \text{sg}(a)(1-a), a\rangle,$$

$$\neg_{178}\langle a, b\rangle = \langle a, b\rangle \to_{178} \langle 0, 1\rangle = \langle \overline{\text{sg}}(a) + \text{sg}(a)(1-a), a\rangle,$$

$$\neg_{179}\langle a, b\rangle = \langle a, b\rangle \to_{179} \langle 0, 1\rangle = \langle \overline{\text{sg}}(1-b) + \text{sg}(1-b)b, 1-b\rangle,$$

$$\neg_{180}\langle a, b\rangle = \langle a, b\rangle \to_{180} \langle 0, 1\rangle = \langle \overline{\text{sg}}(1-b) + \text{sg}(1-b)b, 1-b\rangle.$$

Third, we give the 17 axioms of the intuitionistic logic (see, e.g. [7]). If A, B and C are arbitrary propositional forms, then:

(IL1) $A \to A$,

(IL2) $A \to (B \to A)$,

(IL3) $A \to (B \to (A \& B))$,

(IL4) $(A \to (B \to C)) \to (B \to (A \to C))$,

(IL5) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$,

(IL6) $A \to \neg\neg A$,

(IL7) $\neg(A \& \neg A)$,

(IL8) $(\neg A \vee B) \to (A \to B)$,

(IL9) $\neg(A \vee B) \to (\neg A \& \neg B)$, (IL10) $(\neg A \& \neg B) \to \neg(A \vee B)$,

(IL11) $(\neg A \vee \neg B) \to \neg(A \& B)$,

(IL12) $(A \to B) \to (\neg B \to \neg A)$,

(IL13) $(A \to \neg B) \to (B \to \neg A)$,

(IL14) $\neg\neg\neg A \to \neg A$,

(IL15) $\neg A \to \neg\neg\neg A$,

(IL16) $\neg\neg(A \to B) \to (A \to \neg\neg B)$,

(IL17) $(C \to A) \to ((C \to (A \to B)) \to (C \to B))$.

Theorem 2. Intuitionistic fuzzy implications $\to_{177}$ and $\to_{180}$ satisfy all axioms, implication $\to_{178}$ satisfies axioms (IL1), ..., (IL4), (IL6), ..., (IL11), (IL13), ..., (IL15) and implication $\to_{179}$ satisfies axioms (IL4), ..., (IL7), (IL9), ..., (IL17) as IFSs.

Theorem 3. Intuitionistic fuzzy implications $\to_{177}$ and $\to_{180}$ satisfy axioms (IL1), ..., (IL4), (IL6), (IL8), ..., (IL16), implication $\to_{178}$ satisfies axioms (IL1), ..., (IL4), (IL6), (IL8), ..., (IL11), (IL13), ..., (IL15), and implication $\to_{179}$ satisfies axioms (IL4), (IL6), (IL9), ..., (IL16) as tautologies.

Fourth, we check the validity of Kolmogorov's axioms of logic (see, e.g., [8]). They are

(K1) $A \to (B \to A)$,

(K2) $(A \to (A \to B)) \to (A \to B))$,

(K3) $(A \to (B \to C)) \to (B \to (A \to C))$,

(K4) $(B \to C) \to ((A \to B) \to (A \to C))$,

(K5) $(A \to B) \to ((A \to \neg B) \to \neg A)$.

Theorem 4. Intuitionistic fuzzy implications $\to_{177}$ and $\to_{180}$ satisfy all axioms, implication $\to_{178}$ satisfies axioms (K1), ..., (K3), and implication $\to_{179}$ satisfies axioms (K2), ..., (K4) as IFSs.

Theorem 5. Intuitionistic fuzzy implications $\to_{177}$ and $\to_{180}$ satisfy axioms (K1), (K3) and (K4), implication $\to_{178}$ satisfies axioms (K1), (K3) and implication $\to_{179}$ satisfies axioms (K3), (K4) as tautologies.

Fifth, we check the validity of Łukasiewisz–Tarski's axioms of logic (see, e.g., [8]). They are

(LT1) $A \to (B \to A)$,

(LT2) $(A \to B) \to ((B \to C) \to (A \to C))$, (LT3) $\neg A \to (\neg B \to (B \to A))$,

(LT4) $((A \to \neg A) \to A) \to A$.

Theorem 6. Intuitionistic fuzzy implications $\to_{177}$ and $\to_{180}$ satisfy all axioms, implication $\to_{178}$ satisfies axioms (LT1), (LT3), (LT4), and implication $\to_{179}$ satisfies axioms (LT2), (LT3) as IFSs.

Theorem 7. Intuitionistic fuzzy implications $\to_{177}$ and $\to_{180}$ satisfy axioms (LT1), ..., (LT3) and (K4), implication $\to_{178}$ satisfies axioms (LT1), (LT3) and implication $\to_{179}$ satisfies axioms (LT2), (LT3) as tautologies.

3 Conclusion

In a next research other properties of the new implications will be introduced and studied. Some possible applications of them with be discussed.

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