**Notes on Intuitionistic Fuzzy Sets**
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# Four new intuitionistic fuzzy bimodal topological structures

**Krassimir T. Atanassov**

Dept. of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering,
Bulgarian Academy of Sciences
105 Acad. G. Bonchev Str., 1113 Sofia, Bulgaria,
and
Intelligent Systems Laboratory, "Prof. Asen Zlatarov" University
Burgas-8010, Bulgaria
e-mail: krat@bas.bg

*To Prof. Anthony Shannon for his 85th birthday!*

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**Abstract:** Two intuitionistic fuzzy modal operators from a second type are used for introducing of new intuitionistic fuzzy modal topological structures. After this, these structures, extended with the two standard intuitionistic fuzzy modal operators, generate four intuitionistic fuzzy bimodal topological structures. They are illustrated with examples.

**Keywords:** Intuitionistic fuzzy operation, Intuitionistic fuzzy operator, Intuitionistic fuzzy set, Intuitionistic fuzzy topological structure.
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# 1 Introduction

After introducing the concept of an Intuitionistic Fuzzy Modal Topological Structure (IFMTS) in [2] in series of papers, different forms of these structures were described in a series of author's papers. The basic definitions of these structures are extensions of Kazimierz Kuratowski's definitions in [4] of a topological structures with closure ($\mathcal{C}, cl$-) and interior ($\mathcal{I}, in$-) operators.

In the present paper, based on the definitions from [3], where for a first time the concept of an IFMTS was extended to Intuitionistic Fuzzy Bimodal Topological Structures (IF2MTSs), we construct examples of new four IFMTSs and four IF2MTSs.

All definitions, related to IFMTSs and IF2MTSs are bring from [3].

As was mentioned in some of the previous papers, the definitions of IFMTS are the first ones in which the idea for modal topological structures are discussed.

# 2 Preliminaries

Following [1], we define the third extension of the modal operators of second type:

$$\boxplus_{\alpha,\beta,\gamma}A = \{\langle x, \alpha\mu_A(x), \beta\nu_A(x) + \gamma\rangle | x \in E\},$$

$$\boxtimes_{\alpha,\beta,\gamma}A = \{\langle x, \alpha\mu_A(x) + \gamma, \beta\nu_A(x)\rangle | x \in E\},$$

where $\alpha, \beta, \gamma \in [0, 1]$ and $\max(\alpha, \beta) + \gamma \leq 1$.

For the needs of the next research, we see that if we like to be valid inclusions

$$\boxplus_{\alpha,\beta,\gamma}A \subseteq A \subseteq \boxtimes_{\alpha,\beta,\gamma}A,$$

then the inequalities

$$\mu_A(x) \leq \alpha\mu_A(x) + \gamma$$

and

$$\beta\nu_A(x) + \gamma \geq \nu_A(x)$$

must hold for each $x \in E$. Therefore

$$(1 - \alpha)\mu_A(x) \leq 1 - \alpha \leq \gamma \leq 1 - \max(\alpha, \beta) \leq 1 - \alpha,$$

i.e., $\gamma = 1 - \alpha$;

$$(1 - \beta)\nu_A(x) \leq 1 - \beta \leq \gamma \leq 1 - \max(\alpha, \beta) \leq 1 - \beta,$$

i.e., $\gamma = 1 - \beta$.

By this reason, below, we will use the two modal operators of second type in the forms:

$$\boxplus_{\alpha,\beta}A \equiv \boxplus_{\alpha,\beta,1-\beta}A = \{\langle x, \alpha\mu_A(x), \beta\nu_A(x) + 1 - \beta\rangle | x \in E\},$$

$$\boxtimes_{\alpha,\beta}A \equiv \boxtimes_{\alpha,\beta,1-\alpha}A = \{\langle x, \alpha\mu_A(x) + 1 - \alpha, \beta\nu_A(x)\rangle | x \in E\},$$

where $\alpha, \beta \in [0, 1]$. When we use the first operator, we must have in mind that the inequality

$$\max(\alpha, \beta) + 1 - \beta \leq 1,$$

must hold, i.e.,

$$\alpha \leq \beta,$$

while, when we use the second operator, the inequality

$$\max(\alpha, \beta) + 1 - \alpha \leq 1,$$

must hold, i.e.

$$\beta \leq \alpha.$$

We must mention that in these forms, both operators are more general than operators

$$\boxplus_{\alpha,\beta} A = \{\langle x, \alpha\mu_A(x), \alpha\nu_A(x) + \beta\rangle | x \in E\},$$

$$\boxtimes_{\alpha,\beta} A = \{\langle x, \alpha\mu_A(x) + \beta, \alpha\nu_A(x)\rangle | x \in E\},$$

where $\alpha, \beta, \alpha + \beta \in [0, 1]$ (see [1]). The latest operators have the original notation, but here we will use the same notation for the two previous operators.

# 3 Four new intuitionistic fuzzy modal topological structures

## 3.1 *cl*-*cl*-IFMTS

**Theorem 1.** *For each universe $E$ and for every two $\alpha, \beta \in [0, 1], \alpha \geq \beta$: $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \boxtimes_{\alpha,\beta}\rangle$ is a cl-cl-IFMTS.*

*Proof.* Let $\alpha, \beta \in [0, 1]$ and the IFSs $A, B \in \mathcal{P}(E^*)$. The check of conditions CC1 - CC4 coincide with those in [2] and we omit them.

CC5.

$$\begin{aligned}
\boxtimes_{\alpha,\beta}(A \cap B) &= \boxtimes_{\alpha,\beta}(\{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \cap \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in E\}) \\
&= \boxtimes_{\alpha,\beta}\{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\} \\
&= \{\langle x, \alpha \min(\mu_A(x), \mu_B(x)) + 1 - \alpha, \beta \max(\nu_A(x), \nu_B(x))\rangle | x \in E\} \\
&= \{\langle x, \min(\alpha\mu_A(x) + 1 - \alpha, \alpha\mu_B(x) + 1 - \alpha), \\
&\qquad \max(\beta\nu_A(x), \beta\nu_B(x))\rangle | x \in E\} \\
&= \boxtimes_{\alpha,\beta} A \cap \boxtimes_{\alpha,\beta} B;
\end{aligned}$$

CC6.

$$\begin{aligned}
A &= \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\} \\
&\subseteq \{\langle x, \alpha\mu_A(x) + 1 - \alpha, \beta\nu_A(x)\rangle | x \in E\} \\
&= \boxtimes_{\alpha,\beta} A;
\end{aligned}$$

CC7.

$$\begin{aligned}\boxtimes_{\alpha,\beta}E^* &= \boxtimes_{\alpha,\beta}\{\langle x,1,0\rangle|x\in E\}\\ &= \{\langle x,\alpha+1-\alpha,0\rangle|x\in E\}\\ &= E^*;\end{aligned}$$

CC8.

$$\begin{aligned}\boxtimes_{\alpha,\beta}\boxtimes_{\alpha,\beta}A &= \boxtimes_{\alpha,\beta}\{\langle x,\alpha\mu_A(x)+1-\alpha,\beta\nu_A(x)\rangle|x\in E\}\\ &= \{\langle x,\alpha^2\mu_A(x)+\alpha-\alpha^2+1-\alpha,\beta^2\nu_A(x)\rangle|x\in E\}\\ &= \{\langle x,\alpha^2\mu_A(x)+1-\alpha^2,\beta^2\nu_A(x)\rangle|x\in E\}\\ &= \boxtimes_{\alpha^2,\beta^2}A;\end{aligned}$$

Really, here the result is not exactly $\boxtimes_{\alpha,\beta}$, but the analogy with condition CC8 from [2, 3] is that the composition of the two modal operators is changed with only one operator from the same type.

CC9.

$$\begin{aligned}\boxtimes_{\alpha,\beta}\mathcal{C}(A) &= \boxtimes_{\alpha,\beta}\{\langle x,\sup_{y\in E}\mu_A(y),\inf_{y\in E}\nu_A(y)\rangle|x\in E\})\\ &= \{\langle x,\alpha\sup_{y\in E}\mu_A(y)+1-\alpha,\beta\inf_{y\in E}\nu_A(y)\rangle|x\in E\})\\ &= \{\langle x,\sup_{y\in E}(\alpha\mu_A(y)+1-\alpha),\inf_{y\in E}\beta\nu_A(y)\rangle|x\in E\})\\ &= \mathcal{C}(\{\langle x,\alpha\mu_A(y)+1-\alpha,\beta\nu_A(y)\rangle|x\in E\})\\ &= \mathcal{C}(\boxtimes_{\alpha,\beta}A).\end{aligned}$$

This completes the proof. □

## 3.2 $in$-$in$-IFMTS

**Theorem 2.** *For each universe $E$ and for every two $\alpha,\beta\in[0,1],\alpha\le\beta$: $\langle\mathcal{P}(E^*),\mathcal{I},\cap,\boxplus_{\alpha,\beta}\rangle$ is an $in$-$in$-IFMTS.*

*Proof.* Let the IFSs $A,B\in\mathcal{P}(E^*)$ and $\alpha,\beta\in[0,1]$. The check of conditions II1 - II4 coincide with those in [2] and we omit them.

II5.

$$\begin{aligned}\boxplus_{\alpha,\beta}(A\cup B) &= \boxplus_{\alpha,\beta}(\{\langle x,\mu_A(x),\nu_A(x)\rangle|x\in E\}\cup\{\langle x,\mu_B(x),\nu_B(x)\rangle|x\in E\})\\ &= \boxplus_{\alpha,\beta}\{\langle x,\max(\mu_A(x),\mu_B(x)),\min(\nu_A(x),\nu_B(x))\rangle|x\in E\}\\ &= \{\langle x,\alpha\max(\mu_A(x),\mu_B(x)),\beta\min(\nu_A(x),\nu_B(x))+1-\beta\rangle|x\in E\}\\ &= \{\langle x,\max(\alpha\mu_A(x),\alpha\mu_B(x)),\\ &\qquad \min(\beta\nu_A(x)+1-\beta,\beta\nu_B(x)+1-\beta)\rangle|x\in E\}\\ &= \{\langle x,\alpha\mu_A(x),\beta\nu_A(x)+1-\beta\rangle|x\in E\}\\ &\quad \cup\ \{\langle x,\alpha\mu_B(x),\beta\nu_B(x)+1-\beta\rangle|x\in E\}\\ &= \boxplus_{\alpha,\beta}A\cup\boxplus_{\alpha,\beta}B;\end{aligned}$$

II6.
$$\begin{aligned}\boxplus_{\alpha,\beta}A &= \{\langle x, \alpha\mu_A(x), \beta\nu_A(x)+1-\beta\rangle | x \in E\}\\ &\subseteq \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}\\ &= A;\end{aligned}$$

II7.
$$\begin{aligned}\boxplus_{\alpha,\beta}O^* &= \boxplus_{\alpha,\beta}\{\langle x, 0, 1\rangle | x \in E\}\\ &= \{\langle x, 0, \beta+1-\beta\rangle | x \in E\}\\ &= O^*;\end{aligned}$$

II8.
$$\begin{aligned}\boxplus_{\alpha,\beta}\boxplus_{\alpha,\beta}A &= \boxplus_{\alpha,\beta}\{\langle x, \alpha\mu_A(x), \beta\nu_A(x)+1-\beta\rangle | x \in E\}\\ &= \{\langle x, \alpha^2\mu_A(x), \beta^2\nu_A(x)+1-\beta^2\rangle | x \in E\}\\ &= \boxplus_{\alpha^2,\beta^2}A;\end{aligned}$$

II9.
$$\begin{aligned}\boxplus_{\alpha,\beta}\mathcal{I}(A) &= \boxplus_{\alpha,\beta}\{\langle x, \inf_{y\in E}\mu_A(y), \sup_{y\in E}\nu_A(y)\rangle | x \in E\})\\ &= \{\langle x, \alpha\inf_{y\in E}\mu_A(y), \beta\sup_{y\in E}\nu_A(y)+1-\beta\rangle | x \in E\})\\ &= \{\langle x, \inf_{y\in E}\alpha\mu_A(y), \sup_{y\in E}(\beta\nu_A(y)+1-\beta)\rangle | x \in E\})\\ &= \mathcal{I}(\{\langle x, \alpha\mu_A(x), \beta\nu_A(x)+1-\beta\rangle | x \in E\})\\ &= \mathcal{I}(\boxplus_{\alpha,\beta}A).\end{aligned}$$

This completes the proof. □

## 3.3 $cl$-$in$-IFMTS

**Theorem 3.** *For each universe $E$ and for every two $\alpha, \beta \in [0,1], \alpha \le \beta$: $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \boxplus_{\alpha,\beta}\rangle$ is a $cl$-$in$-IFMTS.*

*Proof.* Let the IFSs $A, B \in \mathcal{P}(E^*)$ and $\alpha, \beta \in [0,1]$. The check of conditions CC1 - CC4 coincide with those in [2] and we omit them. The checks of the conditions CI6– CI8 coincide with the proofs of conditions II6–II8 in Theorem 2. Hence, it is enough only to show the validity of the conditions CI5 and CI9.

For the validity of condition (CI5) we obtain:

$$\begin{aligned}\boxplus_{\alpha,\beta}(A \cap B) &= \boxplus_{\alpha,\beta}\{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\}\\ &= \{\langle x, \alpha\min(\mu_A(x), \mu_B(x)), \beta\max(\nu_A(x), \nu_B(x))+1-\beta\rangle | x \in E\}\\ &= \{\langle x, \min(\alpha\mu_A(x), \alpha\mu_B(x)), \max(\beta\nu_A(x)+1-\beta, \beta\nu_B(x)+1-\beta)\rangle | x \in E\}\\ &= \{\langle x, \alpha\mu_A(x), \beta\nu_A(x)+1-\beta\rangle | x \in E\}\\ &\quad \cap\ \{\langle x, \alpha\mu_B(x), \beta\nu_B(x)+1-\beta\rangle | x \in E\}\\ &= \boxplus_{\alpha,\beta}A \cap \boxplus_{\alpha,\beta}B.\end{aligned}$$

For the validity of condition CI9, we obtain

$$\begin{aligned}
\boxplus_{\alpha,\beta}\mathcal{C}(A) &= \boxplus_{\alpha,\beta}\{\langle x, \sup_{y\in E}\mu_A(y), \inf_{y\in E}\nu_A(y)\rangle | x \in E\}) \\
&= \{\langle x, \alpha\sup_{y\in E}\mu_A(y), \beta\inf_{y\in E}\nu_A(y) + 1 - \beta\rangle | x \in E\}) \\
&= \{\langle x, \sup_{y\in E}\alpha\mu_A(y), \inf_{y\in E}(\beta\nu_A(y) + 1 - \beta)\rangle | x \in E\}) \\
&= \mathcal{C}(\{\langle x, \alpha\mu_A(x), \beta\nu_A(x) + 1 - \beta\rangle | x \in E\}) \\
&= \mathcal{C}(\boxplus_{\alpha,\beta}A).
\end{aligned}$$

This completes the proof. □

## 3.4 *in*-*cl*-IFMTS

**Theorem 4.** *For each universe $E$ and for every two $\alpha, \beta \in [0,1], \alpha \geq \beta$: $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \boxtimes_{\alpha,\beta}\rangle$ is an $in$-$cl$-IFMTS.*

*Proof.* Let the IFS $A \in \mathcal{P}(E^*)$ be given and $\alpha, \beta \in [0,1]$.

The checks of the conditions IC1–IC4 coincide with the proofs of conditions II1–II4 in Theorem 2. The checks of the conditions IC6–IC8 coincide with the proofs of conditions CC6–CC8 from Theorem 1.

Hence, it remains that we check only the validity of the conditions IC5 and IC9.

For the validity of condition (IC5) we obtain:

$$\begin{aligned}
\boxtimes_{\alpha,\beta}(A \cup B) &= \boxtimes_{\alpha,\beta}\{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\} \\
&= \{\langle x, \alpha\max(\mu_A(x), \mu_B(x)) + 1 - \alpha, \beta\min(\nu_A(x), \nu_B(x))\rangle | x \in E\} \\
&= \{\langle x, \max(\alpha\mu_A(x) + 1 - \alpha, \alpha\mu_B(x) + 1 - \alpha), \\
&\qquad \min(\beta\nu_A(x), \beta\nu_B(x))\rangle | x \in E\} \\
&= \{\langle x, \alpha\mu_A(x) + 1 - \alpha, \beta\mu_B(x))\rangle | x \in E\} \\
&\quad \cup\ \{\langle x, \alpha\mu_B(x) + 1 - \alpha, \beta\nu_B(x))\rangle | x \in E\} \\
&= \boxtimes_{\alpha,\beta}A \cup \boxtimes_{\alpha,\beta}B;
\end{aligned}$$

For the validity of condition IC9, we obtain

$$\begin{aligned}
\boxtimes_{\alpha,\beta}\mathcal{I}(A) &= \boxtimes_{\alpha,\beta}\{\langle x, \inf_{y\in E}\mu_A(y), \sup_{y\in E}\nu_A(y)\rangle | x \in E\}) \\
&= \{\langle x, \alpha\inf_{y\in E}\mu_A(y) + 1 - \alpha, \beta\sup_{y\in E}\nu_A(y)\rangle | x \in E\}) \\
&= \{\langle x, \inf_{y\in E}`(\alpha\mu_A(y) + 1 - \alpha), \sup_{y\in E}\beta\nu_A(y)\rangle | x \in E\}) \\
&= \mathcal{I}(\{\langle x, \alpha\mu_A(x) + 1 - \alpha, \beta\nu_A(x)\rangle | x \in E\}) \\
&= \mathcal{I}(\boxtimes_{\alpha,\beta}A).
\end{aligned}$$

This completes the proof. □

## 4 Four new intuitionistic fuzzy bimodal topological structures

First we will mention that the following equalities are valid for each IFS $A$ and for every two $\alpha, \beta \in [0,1], \alpha \geq \beta$:

$$\begin{aligned}
\square \boxtimes_{\alpha,\beta} A &= \square \{\langle x, \alpha\mu_A(x) + 1 - \alpha, \beta\nu_A(x)\rangle | x \in E\} \\
&= \{\langle x, \alpha\mu_A(x) + 1 - \alpha, 1 - (\alpha\mu_A(x) + 1 - \alpha)\rangle | x \in E\} \\
&= \{\langle x, \alpha\mu_A(x) + 1 - \alpha, \alpha(1 - \mu_A(x))\rangle | x \in E\} \\
&\subseteq \{\langle x, \alpha\mu_A(x) + 1 - \alpha, \beta(1 - \mu_A(x))\rangle | x \in E\} \\
&= \boxtimes_{\alpha,\beta}\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\} \\
&= \boxtimes_{\alpha,\beta} \square A;
\end{aligned}$$

$$\begin{aligned}
\boxtimes_{\alpha,\beta}\diamond A &= \boxtimes_{\alpha,\beta}\{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle | x \in E\} \\
&= \{\langle x, \alpha - \alpha\nu_A(x) + 1 - \alpha, \beta\nu_A(x)\rangle | x \in E\} \\
&= \{\langle x, 1 - \alpha\nu_A(x), \beta\nu_A(x)\rangle | x \in E\} \\
&\subseteq \{\langle x, 1 - \beta\nu_A(x), \beta\nu_A(x)\rangle | x \in E\} \\
&= \diamond\{\langle x, \alpha\mu_A(x) + 1 - \alpha, \beta\nu_A(x)\rangle | x \in E\} \\
&= \diamond\boxtimes_{\alpha,\beta} A.
\end{aligned}$$

$$\begin{aligned}
\square A &= \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\} \\
&\subseteq \{\langle x, \alpha\mu_A(x) + 1 - \alpha, \beta\nu_A(x)\rangle | x \in E\} \\
&= \boxtimes_{\alpha,\beta} A.
\end{aligned}$$

When $\alpha \leq \beta$, by analogy we can check that

$$\square \boxplus_{\alpha,\beta} A \subseteq \boxtimes_{\alpha,\beta} \square A,$$

$$\boxplus_{\alpha,\beta}\diamond A \subseteq \diamond\boxtimes_{\alpha,\beta} A,$$

$$\boxplus_{\alpha,\beta} A \subseteq \diamond A.$$

Therefore, the four operators satisfy conditions CC15, II15, CI15 and IC15 from [3].

Now, by analogy with [3], we can formulate the following four assertions and their proofs are based on the proofs of Theorems 1–4.

**Theorem 5.** For each universe $E$, $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \diamond, \boxplus_{\alpha,\beta}\rangle$ is an $cl - (cl, in)$-IFMTS.

**Theorem 6.** For each universe $E$, $\langle \mathcal{P}(E^*), \mathcal{C}, \cup, \square, \boxtimes_{\alpha,\beta}\rangle$ is an $cl - (in, cl)$-IFMTS.

**Theorem 7.** For each universe $E$, $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \diamond, \boxplus_{\alpha,\beta}\rangle$ is an $in - (cl, in)$-IFMTS.

**Theorem 8.** For each universe $E$, $\langle \mathcal{P}(E^*), \mathcal{I}, \cap, \square, \boxtimes_{\alpha,\beta}\rangle$ is an $in - (in, cl)$-IFMTS.

# 5 Conclusion

As it was mentioned in [3], there, for a first time we introduced idea not only for IF2MTS, but in general, for bimodal topological structures. They have the axioms (conditons) CC1–CC15, II1–II15, CI1–CI15, IC1–IC15. In the present paper, new examples of IF2MTSs were given.

Finally, in future we will search for topological structures having more than two modal operators, i.e., for multimodal topological structures.


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