An ergodic theorem on IF sets

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Outline

1. Basic notions
2. Ergodic theorem

Definition (IF-event)

Let (Ω, S) be a measurable space. By an IF-event we mean any pair $A = (\mu_A, \nu_A)$ of S-measurable functions, such that $\mu_A \geq 0, \nu_A \geq 0$ and $\mu_A + \nu_A \leq 1$.

The family $\mathcal{F}$ of all IF-events is ordered by the following way:

$$A \leq B \Leftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

Lukasiewicz connectives

We shall use the following Lukasiewicz connectives:

$$A \oplus B = (\mu_A \oplus \mu_B, \nu_A \otimes \nu_B)$$
$$A \otimes B = (\mu_A \otimes \mu_B, \nu_A \oplus \nu_B),$$

where the operations $\oplus, \otimes$ are defined by these equations:
$\mu_A \oplus \mu_B = (\mu_A + \mu_B) \wedge 1$, $\mu_A \otimes \mu_B = (\mu_A + \mu_B - 1) \vee 0$.

The smallest element in this structure is $(0, 1)$ and the largest is the element $(1, 0)$.

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Definition (The state on $\mathcal{F}$)

$m : \mathcal{F} \to [0,1]$:
(I) $m((1,0)) = 1$, $m((0,1)) = 0$
(II) if $A = B \oplus C$, then $m(A) = m(B) + m(C)$
(III) if $A_n \nearrow A$, then $m(A_n) \nearrow m(A)$

Definition (The observable on $\mathcal{F}$)

$x : \mathcal{B}(R) \to \mathcal{F}$:
(i) $x(R) = (1,0)$, $x(\emptyset) = (0,1)$;
(ii) if $A, B \in \mathcal{B}(R)$ and $A \cap B = \emptyset$, then
$x(A \cup B) = x(A) \oplus x(B)$;
(iii) if $A_n \in \mathcal{B}(R)$, $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$

The composite map $m \circ x = m_x : \mathcal{B}(R) \to [0,1]$ is a probability measure.

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Definition (The integrable observable)

$\mathcal{F}$ with state m

$x : \mathcal{B}(R) \to \mathcal{F}$ - observable

The mapping x is called integrable , if there exists the expected value of the observable defined by the equation:

$$E(x) = \int\limits_R t dm_x(t);$$

where $m_x : \mathcal{B}(R) \to [0, 1]$ is composite mapping $m_x = m \circ x$.

Definition (The m-almost everywhere convergence)

$\mathcal{F}$ with a state m

$(y_i)_{i=1}^{\infty}$ - sequence of an observables on the system $\mathcal{F}$

The sequence converges m-almost everywhere to 0, if it holds:

$$\lim_{l\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} m\left(\bigwedge_{n=1}^{k+i} y_n\left(-\frac{1}{l}, \frac{1}{l}\right)\right) = 1.$$

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Definition (The joint observable)

$x, y : \mathcal{B}(R) \to \mathcal{F}$-observables
joint observable $h : \mathcal{B}(R^2) \to \mathcal{F}$:
(i) $h(R^2) = (1,0), h(\emptyset) = (0,1)$;
(ii) $A \cap B = \emptyset \Rightarrow h(A \cup B) = h(A) \oplus (B)$;
(iii) $A_n \nearrow A \Rightarrow h(A_n) \nearrow (A)$;
(iv) $h(C \times D) = x(C) \otimes y(D),\ C, D \in \mathcal{B}(R)$.

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Kolmogorov construction

$(x_i)_1^\infty$ - a sequence of observables in the set $\mathcal{F}$ with a state m
h_n - the joint observable of the observables $x_1, x_2, .., x_n, \forall n \in N$
$C = \left\{\pi_n^{-1}(B), B \in \mathcal{B}(R^n), n \in N\right\}$ - set of all cylinders
$\pi_n\left((u_i)_1^\infty\right) = (u_1, ..., u_n)$ - n-th coordinate random vector
$\left(R^N, \sigma(C)\right)$

$$\mathbf{P}\left\{(u_i)_1^\infty \in R^N; u_1 \in A_1, ..., u_n \in A_n\right\} = m\left(h_n\left(A_1 \times A_2 \times ... \times A_n\right)\right)$$

for every $n \in N$ and every $A_1, ..., A_n \in \mathcal{B}(R)$.

$\forall n \in N$: $\xi_n : R^N \to R$ given by $\xi_n\left((u_i)_1^\infty\right) = u_n$ is called n-th coordinate random variable of $\left(R^N, \sigma(C), \mathbf{P}\right)$.

Outline

Definition (The m-state preserving map)

$\mathcal{F}$ with a state m

$\lambda : \mathcal{F} \to \mathcal{F}$

$$\forall A \in \mathcal{F} : m\left(\lambda\left(A\right)\right) = m\left(A\right).$$

Definition (The ergodic mapping according to observable x)

$\lambda : \mathcal{F} \to \mathcal{F}$:

(i) λ is m-state preserving map,

(ii) if for all $n \in N$ the mapping $h_n : \mathcal{B}\left(R^n\right) \to \mathcal{F}$ is the joint observable of $x, \lambda \circ x, ..., \lambda^{n-1} \circ x$ then the following equality holds:

$$m\left(h_n\left(A_1 \times A_2 ... \times A_n\right)\right) =$$
$$m\left(x_1\left(A_1\right) \otimes \left(\lambda \circ x\right)\left(A_2\right) \otimes ... \otimes \left(\lambda^{n-1} \circ x\right)\left(A_n\right)\right),$$

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Theorem

$(x_i)_{i=1}^{\infty}$ - *a sequence of observables on* $\mathcal{F}$ *with state* m

$h_n : \mathcal{B}(R^n) \to \mathcal{F}$ - *the joint observable of* $x_1, ..., x_n$

$P =$ *the probability measure generated by the Kolmogorov construction*

$\forall n \in N : g_n$ *is a Borel function from* R^n *to* R

The observable $y_n = g_n(x_1, x_2, ..., x_n) : \mathcal{B}(R) \to \mathcal{F}$*:*

$y_n = h_n \circ g_n^{-1}$

$\pi_n =$ *the projection of* R^N *to* R^n *:* $\pi_n\left((u_i)_{i=1}^{\infty}\right) = (u_1, u_2, ..., u_n)$

$\Longrightarrow$

$p \circ \pi_n^{-1} \circ g_n^{-1} = m \circ h_n \circ g_n^{-1} = m \circ y_n.$

In addition: If $(g_n \circ \pi_n)_{n=1}^{\infty}$ *converges* P*-almost everywhere to* 0

$\Longrightarrow$

the sequence $(y_i)_{i=1}^{\infty}$ *converges* m*-almost everywhere to* 0.

Theorem (The ergodic theorem)

x - integrable observable on $\mathcal{F}$ with a state m

$\lambda : \mathcal{F} \longrightarrow \mathcal{F}$ - ergodic mapping

$$y_n = \frac{1}{n} \sum_{i=0}^{n-1} \lambda^i \circ x - E(x)$$

$\Rightarrow$

converges m-almost everywhere to 0.