Notes on Intuitionistic Fuzzy Sets
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Several significant equalities on intuitionistic fuzzy operators

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Abstract: There are various operators and operations in intuitionistic fuzzy set theory. The roles of these operators and operations are very significant as they show a deeper interconnection between the two ordinary modal logic operators. It can be well noted that modal operators can change intuitionistic fuzzy sets into fuzzy sets easily. Considering all of these, we establish several equalities on intuitionistic fuzzy sets.
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1 Introduction

The notion of fuzzy sets was introduced and developed by L. A. Zadeh [12] in 1965. Eighteen years later, in 1983, K. T. Atanassov [1] introduced the concept of intuitionistic fuzzy sets as an extension of fuzzy sets. The basic difference of these two concepts is that in fuzzy set theory only membership function has been taken into account while in intuitionistic fuzzy set theory membership function and non-membership function both are considered along with hesitation margin. Researchers [5, 7–11] are working hard to develop and improve this subject. There are some operators D_α, $F_{\alpha,\beta}$, $G_{\alpha,\beta}$, $H_{\alpha,\beta}$,$H^*_{\alpha,\beta}$, $J_{\alpha,\beta}$, and $J^*_{\alpha,\beta}$ as well as some operations like $*$, $\odot$, $\bowtie$, ∞, $\lhd$ and $\rhd$ in intuitionistic fuzzy set theory which are interconnected. In the main section of this paper we concentrate on the relations between these operators and operations with the help of modal operators and establish some significant equalities on intuitionistic fuzzy sets.

2 Preliminaries

Definition 2.1 [12] Let X be a nonempty set. A fuzzy set A drawn from X is defined as $A = \{\langle x, \mu_A(x)\rangle : x \in X\}$, where $\mu_A : x \to [0,1]$ is the membership function of the fuzzy set A. The fuzzy set is a collection of objects with graded membership, i.e., having degrees of membership.

Definition 2.2 [2] Let X be a nonempty set. An intuitionistic fuzzy set A in X is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where the functions $\mu_A, \nu_A : x \to [0,1]$ define, respectively, the degree of membership and degree of non-membership of the element $x \in X$ to the set A, which is a subset of X, and for every element $x \in X, 0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Furthermore, we have $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ called the intuitionistic fuzzy set index or hesitation margin of x in A. $\pi_A(x)$ is the degree of indeterminacy of $x \in X$ to the IFS A and $\pi_A(x) \in [0,1]$ that is $\pi_A : x \to [0,1]$ and $0 \leq \pi_A(x) \leq 1$ for every $x \in X$. $\pi_A(x)$ expresses the lack of knowledge of whether x belongs to IFS A or not.

Definition 2.3 [4] Let X be a nonempty set. If A is an IFS drawn from X, then,

(i) $\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$

(ii) $\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x)\rangle : x \in X\}$

For a proper IFS, $\Box A \subset A \subset \Diamond A$ and $\Box A \neq A \neq \Diamond A$.

Definition 2.4 [4] Let $\alpha, \beta \in [0,1]$ and $A \in$ IFS X. Then the following operators can be defined as:

(i) $D_\alpha(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + (1-\alpha)\pi_A(x)\rangle : x \in X\}$.

(ii) $F_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \nu_A(x) + \beta\pi_A(x)\rangle : x \in X\}$, where $\alpha + \beta \leq 1$.

(iii) $G_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \beta\nu_A(x)\rangle : x \in X\}$, where $\alpha + \beta \leq 1$.

(iv) $H_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \nu_A(x) + \beta\pi_A(x)\rangle : x \in X\}$, where $\alpha + \beta \leq 1$.

(v) $H^*_{\alpha,\beta}(A) = \{\langle x, \alpha\mu_A(x), \nu_A(x) + \beta(1 - \alpha\mu_A(x) - \nu_A(x))\rangle : x \in X\}$, where $\alpha + \beta \leq 1$.

(vi) $J_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha\pi_A(x), \beta\nu_A(x)\rangle : x \in X\}$, where $\alpha + \beta \leq 1$.

(vii) $J^*_{\alpha,\beta}(A) = \{\langle x, \mu_A(x) + \alpha(1 - \mu_A(x) - \beta\nu_A(x)), \beta\nu_A(x)\rangle : x \in X\}$, where $\alpha + \beta \leq 1$.

Definition 2.5 [3,6] Let X be a nonempty set. If A and B be two IFSs drawn from X, then,

(i) $A * B = \{\langle x, \frac{\mu_A(x)+\mu_B(x)}{2(\mu_A(x)+\mu_B(x)+1)}, \frac{\nu_A(x)+\nu_B(x)}{2(\nu_A(x)+\nu_B(x)+1)}\rangle : x \in X\}$,

(ii) $A \odot B = \{\langle x, \frac{\mu_A(x)\mu_B(x)}{2(\mu_A(x)\mu_B(x)+1)}, \frac{\nu_A(x)\nu_B(x)}{2(\nu_A(x)\nu_B(x)+1)}\rangle : x \in X\}$,

(iii) $A \bowtie B = \{\langle x, \frac{\mu_A(x)+\mu_B(x)}{2(\mu_A(x)+\mu_B(x))+1}, \frac{\nu_A(x)+\nu_B(x)}{2(\nu_A(x)+\nu_B(x))+1}\rangle : x \in X\}$,

(iv) $A \infty B = \{\langle x, \frac{\mu_A(x)\mu_B(x)}{2(\mu_A(x)\mu_B(x))+1}, \frac{\nu_A(x)\nu_B(x)}{2(\nu_A(x)\nu_B(x))+1}\rangle : x \in X\}$,

(v) $A \rhd B = \{\langle x, \frac{\mu_A(x)+\mu_B(x)}{\mu_A(x)+\mu_B(x)+1}, \frac{\nu_A(x)+\nu_B(x)}{\nu_A(x)+\nu_B(x)+1}\rangle : x \in X\}$,

(vi) $A \lhd B = \{\langle x, \frac{\mu_A(x)\mu_B(x)}{\mu_A(x)\mu_B(x)+1}, \frac{\nu_A(x)\nu_B(x)}{\nu_A(x)\nu_B(x)+1}\rangle : x \in X\}$.

3 Main results

Theorem 3.1. *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then*

(i) $[\Box(D_\alpha(A*B))]^C = \Diamond[D_\alpha(A*B)]^C$,

(ii) $[\Diamond(D_\alpha(A*B))]^C = \Box[D_\alpha(A*B)]^C$,

(iii) $[\Box(F_{\alpha,\beta}(A*B))]^C = \Diamond[F_{\alpha,\beta}(A*B)]^C$,

(iv) $[\Diamond(F_{\alpha,\beta}(A*B))]^C = \Box[F_{\alpha,\beta}(A*B)]^C$,

(v) $[\Box(G_{\alpha,\beta}(A*B))]^C = \Diamond[G_{\alpha,\beta}(A*B)]^C$,

(vi) $[\Diamond(G_{\alpha,\beta}(A*B))]^C = \Box[G_{\alpha,\beta}(A*B)]^C$,

(vii) $[\Box(H_{\alpha,\beta}(A*B))]^C = \Diamond[H_{\alpha,\beta}(A*B)]^C$,

(viii) $[\Diamond(H_{\alpha,\beta}(A*B))]^C = \Box[H_{\alpha,\beta}(A*B)]^C$,

(ix) $[\Box(H^*_{\alpha,\beta}(A*B))]^C = \Diamond[H^*_{\alpha,\beta}(A*B)]^C$,

(x) $[\Diamond(H^*_{\alpha,\beta}(A*B))]^C = \Box[H^*_{\alpha,\beta}(A*B)]^C$,

(xi) $[\Box(J_{\alpha,\beta}(A*B))]^C = \Diamond[J_{\alpha,\beta}(A*B)]^C$,

(xii) $[\Diamond(J_{\alpha,\beta}(A*B))]^C = \Box[J_{\alpha,\beta}(A*B)]^C$,

(xiii) $[\Box(J^*_{\alpha,\beta}(A*B))]^C = \Diamond[J^*_{\alpha,\beta}(A*B)]^C$,

(xiv) $[\Diamond(J^*_{\alpha,\beta}(A*B))]^C = \Box[J^*_{\alpha,\beta}(A*B)]^C$.

Proof. (i) Now

$$D_\alpha(A*B) = \langle \mu_{A*B}(x) + \alpha\pi_{A*B}(x), \nu_{A*B}(x) + (1-\alpha)\pi_{A*B}(x)\rangle$$

$$\Box(D_\alpha(A*B)) = \langle \mu_{A*B}(x) + \alpha\pi_{A*B}(x), 1 - (\mu_{A*B}(x) + \alpha\pi_{A*B}(x))\rangle$$

$$[\Box(D_\alpha(A*B))]^C = \langle 1 - (\mu_{A*B}(x) + \alpha\pi_{A*B}(x)), (\mu_{A*B}(x) + \alpha\pi_{A*B}(x))\rangle$$

Again,

$$[D_\alpha(A*B)]^C = \langle \nu_{A*B}(x) + (1-\alpha)\pi_{A*B}(x), \mu_{A*B}(x) + \alpha\pi_{A*B}(x)\rangle$$

$$\Diamond[D_\alpha(A*B)]^C = \langle 1 - (\mu_{A*B}(x) + \alpha\pi_{A*B}(x)), \mu_{A*B}(x) + \alpha\pi_{A*B}(x)\rangle.$$

Hence

$$[\Box(D_\alpha(A*B))]^C = \Diamond[D_\alpha(A*B)]^C.$$

Similarly (ii) to (xiv) can be proved. $\Box$

Theorem 3.2. *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then:*

(i) $[\Box(D_\alpha(A \odot B))]^C = \Diamond[D_\alpha(A \odot B)]^C$,

(ii) $[\Diamond(D_\alpha(A \odot B))]^C = \Box[D_\alpha(A \odot B)]^C$,

(iii) $[\Box(F_{\alpha,\beta}(A \odot B))]^C = \Diamond[F_{\alpha,\beta}(A \odot B)]^C$,

(iv) $[\Diamond(F_{\alpha,\beta}(A \odot B))]^C = \Box[F_{\alpha,\beta}(A \odot B)]^C$,

(v) $[\Box(G_{\alpha,\beta}(A \odot B))]^C = \Diamond[G_{\alpha,\beta}(A \odot B)]^C$,

(vi) $[\Diamond(G_{\alpha,\beta}(A \odot B))]^C = \Box[G_{\alpha,\beta}(A \odot B)]^C$,

(vii) $[\Box(H_{\alpha,\beta}(A \odot B))]^C = \Diamond[H_{\alpha,\beta}(A \odot B)]^C$,

(viii) $[\Diamond(H_{\alpha,\beta}(A \odot B))]^C = \Box[H_{\alpha,\beta}(A \odot B)]^C$,

(ix) $[\Box(H^*_{\alpha,\beta}(A \odot B))]^C = \Diamond[H^*_{\alpha,\beta}(A \odot B)]^C$,

(x) $[\Diamond(H^*_{\alpha,\beta}(A \odot B))]^C = \Box[H^*_{\alpha,\beta}(A \odot B)]^C$,

(xi) $[\Box(J_{\alpha,\beta}(A \odot B))]^C = \Diamond[J_{\alpha,\beta}(A \odot B)]^C$,

(xii) $[\Diamond(J_{\alpha,\beta}(A \odot B))]^C = \Box[J_{\alpha,\beta}(A \odot B)]^C$,

(xiii) $[\Box(J^*_{\alpha,\beta}(A \odot B))]^C = \Diamond[J^*_{\alpha,\beta}(A \odot B)]^C$,

(xiv) $[\Diamond(J^*_{\alpha,\beta}(A \odot B))]^C = \Box[J^*_{\alpha,\beta}(A \odot B)]^C$.

Proof. (ix) Let us have $\alpha + \beta \leq 1$. Now

$$\begin{aligned}
H^*_{\alpha,\beta}(A \odot B) &= \langle \alpha\mu_{A\odot B}(x), \nu_{A\odot B}(x) + \beta(1 - \alpha\mu_{A\odot B}(x) - \nu_{A\odot B}(x))\rangle \\
\Box(H^*_{\alpha,\beta}(A \odot B)) &= \langle \alpha\mu_{A\odot B}(x), 1 - \alpha\mu_{A\odot B}(x)\rangle \\
[\Box(H^*_{\alpha,\beta}(A \odot B))]^C &= \langle 1 - \alpha\mu_{A\odot B}(x), \alpha\mu_{A\odot B}(x)\rangle
\end{aligned}$$

Again,

$$\begin{aligned}
[H^*_{\alpha,\beta}(A \odot B)]^C &= \langle \nu_{A\odot B}(x) + \beta(1 - \alpha\mu_{A\odot B}(x) - \nu_{A\odot B}(x)), \alpha\mu_{A\odot B}(x)\rangle \\
\Diamond[H^*_{\alpha,\beta}(A \odot B)]^C &= \langle 1 - \alpha\mu_{A\odot B}(x), \alpha\mu_{A\odot B}(x)\rangle
\end{aligned}$$

Hence

$$[\Box(H^*_{\alpha,\beta}(A \odot B))]^C = \Diamond[H^*_{\alpha,\beta}(A \odot B)]^C.$$

Similarly (i) to (viii) and (x) to (xiv) can be proved. $\Box$

Theorem 3.3. *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0, 1]$, where $\alpha + \beta \leq 1$, then:*

(i) $[\Box(D_\alpha(A \bowtie B))]^C = \Diamond[D_\alpha(A \bowtie B)]^C$,

(ii) $[\Diamond(D_\alpha(A \bowtie B))]^C = \Box[D_\alpha(A \bowtie B)]^C$,

(iii) $[\Box(F_{\alpha,\beta}(A \bowtie B))]^C = \Diamond[F_{\alpha,\beta}(A \bowtie B)]^C$,

(iv) $[\Diamond(F_{\alpha,\beta}(A \bowtie B))]^C = \Box[F_{\alpha,\beta}(A \bowtie B)]^C$,

(v) $[\Box(G_{\alpha,\beta}(A \bowtie B))]^C = \Diamond[G_{\alpha,\beta}(A \bowtie B)]^C$,

(vi) $[\Diamond(G_{\alpha,\beta}(A \bowtie B))]^C = \Box[G_{\alpha,\beta}(A \bowtie B)]^C$,

(vii) $[\Box(H_{\alpha,\beta}(A \bowtie B))]^C = \Diamond[H_{\alpha,\beta}(A \bowtie B)]^C$,

(viii) $[\Diamond(H_{\alpha,\beta}(A \bowtie B))]^C = \Box[H_{\alpha,\beta}(A \bowtie B)]^C$,

(ix) $[\Box(H^*_{\alpha,\beta}(A \bowtie B))]^C = \Diamond[H^*_{\alpha,\beta}(A \bowtie B)]^C$,

(x) $[\Diamond(H^*_{\alpha,\beta}(A \bowtie B))]^C = \Box[H^*_{\alpha,\beta}(A \bowtie B)]^C$,

(xi) $[\Box(J_{\alpha,\beta}(A \bowtie B))]^C = \Diamond[J_{\alpha,\beta}(A \bowtie B)]^C$,

(xii) $[\Diamond(J_{\alpha,\beta}(A \bowtie B))]^C = \Box[J_{\alpha,\beta}(A \bowtie B)]^C$,

(xiii) $[\Box(J^*_{\alpha,\beta}(A \bowtie B))]^C = \Diamond[J^*_{\alpha,\beta}(A \bowtie B)]^C$,

(xiv) $[\Diamond(J^*_{\alpha,\beta}(A \bowtie B))]^C = \Box[J^*_{\alpha,\beta}(A \bowtie B)]^C$.

Proof. (iii) Let us have $\alpha + \beta \leq 1$. Now

$$
\begin{aligned}
F_{\alpha,\beta}(A \bowtie B) &= \langle \mu_{A\bowtie B}(x) + \alpha\pi_{A\bowtie B}(x), \nu_{A\bowtie B}(x) + \beta\pi_{A\bowtie B}(x)\rangle \\
\Box(F_{\alpha,\beta}(A \bowtie B)) &= \langle \mu_{A\bowtie B}(x) + \alpha\pi_{A\bowtie B}(x), 1 - (\mu_{A\bowtie B}(x) + \alpha\pi_{A\bowtie B}(x))\rangle \\
[\Box(F_{\alpha,\beta}(A \bowtie B))]^C &= \langle 1 - (\mu_{A\bowtie B}(x) + \alpha\pi_{A\bowtie B}(x)), (\mu_{A\bowtie B}(x) + \alpha\pi_{A\bowtie B}(x))\rangle
\end{aligned}
$$

Again,

$$
\begin{aligned}
[F_{\alpha,\beta}(A \bowtie B)]^C &= \langle \nu_{A\bowtie B}(x) + \beta\pi_{A\bowtie B}(x), \mu_{A\bowtie B}(x) + \alpha\pi_{A\bowtie B}(x)\rangle \\
\Diamond[F_{\alpha,\beta}(A \bowtie B)]^C &= \langle 1 - (\mu_{A\bowtie B}(x) + \alpha\pi_{A\bowtie B}(x)), \mu_{A\bowtie B}(x) + \alpha\pi_{A\bowtie B}(x)\rangle
\end{aligned}
$$

Hence

$$[\Box(F_{\alpha,\beta}(A \bowtie B))]^C = \Diamond[F_{\alpha,\beta}(A \bowtie B)]^C.$$

Similarly (i) to (ii) and (iv) to (xiv) can be proved. $\Box$

Theorem 3.4. *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0, 1]$, where $\alpha + \beta \leq 1$, then:*

(i) $[\Box(D_\alpha(A \infty B))]^C = \Diamond[D_\alpha(A \infty B)]^C$,

(ii) $[\Diamond(D_\alpha(A \infty B))]^C = \Box[D_\alpha(A \infty B)]^C$,

(iii) $[\Box(F_{\alpha,\beta}(A \infty B))]^C = \Diamond[F_{\alpha,\beta}(A \infty B)]^C$,

(iv) $[\Diamond(F_{\alpha,\beta}(A \infty B))]^C = \Box[F_{\alpha,\beta}(A \infty B)]^C$,

(v) $[\Box(G_{\alpha,\beta}(A\infty B))]^C = \Diamond[G_{\alpha,\beta}(A\infty B)]^C$,

(vi) $[\Diamond(G_{\alpha,\beta}(A\infty B))]^C = \Box[G_{\alpha,\beta}(A\infty B)]^C$,

(vii) $[\Box(H_{\alpha,\beta}(A\infty B))]^C = \Diamond[H_{\alpha,\beta}(A\infty B)]^C$,

(viii) $[\Diamond(H_{\alpha,\beta}(A\infty B))]^C = \Box[H_{\alpha,\beta}(A\infty B)]^C$,

(ix) $[\Box(H^*_{\alpha,\beta}(A\infty B))]^C = \Diamond[H^*_{\alpha,\beta}(A\infty B)]^C$,

(x) $[\Diamond(H^*_{\alpha,\beta}(A\infty B))]^C = \Box[H^*_{\alpha,\beta}(A\infty B)]^C$,

(xi) $[\Box(J_{\alpha,\beta}(A\infty B))]^C = \Diamond[J_{\alpha,\beta}(A\infty B)]^C$,

(xii) $[\Diamond(J_{\alpha,\beta}(A\infty B))]^C = \Box[J_{\alpha,\beta}(A\infty B)]^C$,

(xiii) $[\Box(J^*_{\alpha,\beta}(A\infty B))]^C = \Diamond[J^*_{\alpha,\beta}(A\infty B)]^C$,

(xiv) $[\Diamond(J^*_{\alpha,\beta}(A\infty B))]^C = \Box[J^*_{\alpha,\beta}(A\infty B)]^C$.

Proof. (v) Let us have $\alpha + \beta \leq 1$. Now

$$\begin{aligned} G_{\alpha,\beta}(A\infty B) &= \langle \alpha\mu_{A\infty B}(x), \beta\nu_{A\infty B}(x)\rangle \\ \Box(G_{\alpha,\beta}(A\infty B)) &= \langle \alpha\mu_{A\infty B}(x), 1-(\alpha\mu_{A\infty B}(x))\rangle \\ [\Box(G_{\alpha,\beta}(A\infty B))]^C &= \langle 1-(\alpha\mu_{A\infty B}(x)), \alpha\mu_{A\infty B}(x)\rangle. \end{aligned}$$

Again,

$$\begin{aligned} [G_{\alpha,\beta}(A\infty B)]^C &= \langle \beta\nu_{A\infty B}(x), \alpha\mu_{A\infty B}(x)\rangle \\ \Diamond[G_{\alpha,\beta}(A\bowtie B)]^C &= \langle 1-(\alpha\mu_{A\infty B}(x)), \alpha\mu_{A\infty B}(x)\rangle. \end{aligned}$$

Hence

$$[\Box(G_{\alpha,\beta}(A\infty B))]^C = \Diamond[G_{\alpha,\beta}(A\infty B)]^C.$$

Similarly (i) to (iv) and (vi) to (xiv) can be proved. ∎

Theorem 3.5. *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then:*

(i) $[\Box(D_\alpha(A \rhd B))]^C = \Diamond[D_\alpha(A \rhd B)]^C$,

(ii) $[\Diamond(D_\alpha(A \rhd B))]^C = \Box[D_\alpha(A \rhd B)]^C$,

(iii) $[\Box(F_{\alpha,\beta}(A \rhd B))]^C = \Diamond[F_{\alpha,\beta}(A \rhd B)]^C$,

(iv) $[\Diamond(F_{\alpha,\beta}(A \rhd B))]^C = \Box[F_{\alpha,\beta}(A \rhd B)]^C$,

(v) $[\Box(G_{\alpha,\beta}(A \rhd B))]^C = \Diamond[G_{\alpha,\beta}(A \rhd B)]^C$,

(vi) $[\Diamond(G_{\alpha,\beta}(A \rhd B))]^C = \Box[G_{\alpha,\beta}(A \rhd B)]^C$,

(vii) $[\Box(H_{\alpha,\beta}(A \rhd B))]^C = \Diamond[H_{\alpha,\beta}(A \rhd B)]^C$,

(viii) $[\Diamond(H_{\alpha,\beta}(A \rhd B))]^C = \Box[H_{\alpha,\beta}(A \rhd B)]^C$,

(ix) $[\Box(H^*_{\alpha,\beta}(A \rhd B))]^C = \Diamond[H^*_{\alpha,\beta}(A \rhd B)]^C$,

(x) $[\Diamond(H^*_{\alpha,\beta}(A \rhd B))]^C = \Box[H^*_{\alpha,\beta}(A \rhd B)]^C$,

(xi) $[\Box(J_{\alpha,\beta}(A \rhd B))]^C = \Diamond[J_{\alpha,\beta}(A \rhd B)]^C$,

(xii) $[\Diamond(J_{\alpha,\beta}(A \rhd B))]^C = \Box[J_{\alpha,\beta}(A \rhd B)]^C$,

(xiii) $[\Box(J^*_{\alpha,\beta}(A \rhd B))]^C = \Diamond[J^*_{\alpha,\beta}(A \rhd B)]^C$,

(xiv) $[\Diamond(J^*_{\alpha,\beta}(A \rhd B))]^C = \Box[J^*_{\alpha,\beta}(A \rhd B)]^C$.

Proof. (vii) Let us have $\alpha + \beta \leq 1$. Now

$$\begin{aligned} H_{\alpha,\beta}(A \rhd B) &= \langle \alpha\mu_{A\rhd B}(x), \nu_{A\rhd B}(x) + \beta\pi_{A\rhd B}(x)\rangle \\ \Box(H_{\alpha,\beta}(A \rhd B)) &= \langle \alpha\mu_{A\rhd B}(x), 1 - (\alpha\mu_{A\rhd B}(x))\rangle \\ [\Box(H_{\alpha,\beta}(A \rhd B))]^C &= \langle 1 - (\alpha\mu_{A\rhd B}(x)), (\alpha\mu_{A\rhd B}(x))\rangle \end{aligned}$$

Again,

$$\begin{aligned} [H_{\alpha,\beta}(A \rhd B)]^C &= \langle \nu_{A\rhd B}(x) + \beta\pi_{A\rhd B}(x), \alpha\mu_{A\rhd B}(x)\rangle \\ \Diamond[H_{\alpha,\beta}(A \rhd B)]^C &= \langle 1 - (\alpha\mu_{A\rhd B}(x)), \alpha\mu_{A\rhd B}(x)\rangle \end{aligned}$$

Hence

$$[\Box(H_{\alpha,\beta}(A \rhd B))]^C = \Diamond[H_{\alpha,\beta}(A \rhd B)]^C.$$

Similarly (i) to (vi) and (viii) to (xiv) can be proved. $\Box$

Theorem 3.6. *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then:*

(i) $[\Box(D_\alpha(A \lhd B))]^C = \Diamond[D_\alpha(A \lhd B)]^C$,

(ii) $[\Diamond(D_\alpha(A \lhd B))]^C = \Box[D_\alpha(A \lhd B)]^C$,

(iii) $[\Box(F_{\alpha,\beta}(A \lhd B))]^C = \Diamond[F_{\alpha,\beta}(A \lhd B)]^C$,

(iv) $[\Diamond(F_{\alpha,\beta}(A \lhd B))]^C = \Box[F_{\alpha,\beta}(A \lhd B)]^C$,

(v) $[\Box(G_{\alpha,\beta}(A \lhd B))]^C = \Diamond[G_{\alpha,\beta}(A \lhd B)]^C$,

(vi) $[\Diamond(G_{\alpha,\beta}(A \lhd B))]^C = \Box[G_{\alpha,\beta}(A \lhd B)]^C$,

(vii) $[\Box(H_{\alpha,\beta}(A \lhd B))]^C = \Diamond[H_{\alpha,\beta}(A \lhd B)]^C$,

(viii) $[\Diamond(H_{\alpha,\beta}(A \lhd B))]^C = \Box[H_{\alpha,\beta}(A \lhd B)]^C$,

(ix) $[\Box(H^*_{\alpha,\beta}(A \lhd B))]^C = \Diamond[H^*_{\alpha,\beta}(A \lhd B)]^C$,

(x) $[\Diamond(H^*_{\alpha,\beta}(A \lhd B))]^C = \Box[H^*_{\alpha,\beta}(A \lhd B)]^C$,

(xi) $[\Box(J_{\alpha,\beta}(A \lhd B))]^C = \Diamond[J_{\alpha,\beta}(A \lhd B)]^C$,

(xii) $[\Diamond(J_{\alpha,\beta}(A \lhd B))]^C = \Box[J_{\alpha,\beta}(A \lhd B)]^C$,

(xiii) $[\Box(J^*_{\alpha,\beta}(A \lhd B))]^C = \Diamond[J^*_{\alpha,\beta}(A \lhd B)]^C$,

(xiv) $[\Diamond(J^*_{\alpha,\beta}(A \lhd B))]^C = \Box[J^*_{\alpha,\beta}(A \lhd B)]^C$.

Proof. (xi) Let us have $\alpha + \beta \leq 1$. Now

$$\begin{aligned} J_{\alpha,\beta}(A \lhd B) &= \langle \mu_{A \lhd B}(x) + \alpha\pi_{A \lhd B}(x), \beta\nu_{A \lhd B}(x)\rangle \\ \Box(J_{\alpha,\beta}(A \lhd B)) &= \langle \mu_{A \lhd B}(x) + \alpha\pi_{A \lhd B}(x), 1 - (\mu_{A \lhd B}(x) + \alpha\pi_{A \lhd B}(x))\rangle \\ [\Box(J_{\alpha,\beta}(A \lhd B))]^C &= \langle 1 - (\mu_{A \lhd B}(x) + \alpha\pi_{A \lhd B}(x)), (\mu_{A \lhd B}(x) + \alpha\pi_{A \lhd B}(x))\rangle. \end{aligned}$$

Again,

$$\begin{aligned} [J_{\alpha,\beta}(A \lhd B)]^C &= \langle \beta\nu_{A \lhd B}(x), \mu_{A \lhd B}(x) + \alpha\pi_{A \lhd B}(x)\rangle \\ \Diamond[J_{\alpha,\beta}(A \lhd B)]^C &= \langle 1 - (\mu_{A \lhd B}(x) + \alpha\pi_{A \lhd B}(x)), \mu_{A \lhd B}(x) + \alpha\pi_{A \lhd B}(x)\rangle. \end{aligned}$$

Hence

$$[\Box(J_{\alpha,\beta}(A \lhd B))]^C = \Diamond[J_{\alpha,\beta}(A \lhd B)]^C.$$

Similarly (i) to (x) and (xii) to (xiv) can be proved. □

Theorem 3.7. *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0, 1]$, where $\alpha + \beta \leq 1$, then:*

(i) $[(\Box D_\alpha(A)) * (\Diamond D_\alpha(B))]^C = \Diamond[D_\alpha(A)]^C * \Box[D_\alpha(B)]^C$,

(ii) $[(\Diamond D_\alpha(A)) * (\Box D_\alpha(B))]^C = \Box[D_\alpha(A)]^C * \Diamond[D_\alpha(B)]^C$,

(iii) $[(\Box F_{\alpha,\beta}(A)) * (\Diamond F_{\alpha,\beta}(B))]^C = \Diamond[F_{\alpha,\beta}(A)]^C * \Box[F_{\alpha,\beta}(B)]^C$,

(iv) $[(\Diamond F_{\alpha,\beta}(A)) * (\Box F_{\alpha,\beta}(B))]^C = \Box[F_{\alpha,\beta}(A)]^C * \Diamond[F_{\alpha,\beta}(B)]^C$,

(v) $[(\Box G_{\alpha,\beta}(A)) * (\Diamond G_{\alpha,\beta}(B))]^C = \Diamond[G_{\alpha,\beta}(A)]^C * \Box[G_{\alpha,\beta}(B)]^C$,

(vi) $[(\Diamond G_{\alpha,\beta}(A)) * (\Box G_{\alpha,\beta}(B))]^C = \Box[G_{\alpha,\beta}(A)]^C * \Diamond[G_{\alpha,\beta}(B)]^C$,

(vii) $[(\Box H_{\alpha,\beta}(A)) * (\Diamond H_{\alpha,\beta}(B))]^C = \Diamond[H_{\alpha,\beta}(A)]^C * \Box[H_{\alpha,\beta}(B)]^C$,

(viii) $[(\Diamond H_{\alpha,\beta}(A)) * (\Box H_{\alpha,\beta}(B))]^C = \Box[H_{\alpha,\beta}(A)]^C * \Diamond[H_{\alpha,\beta}(B)]^C$,

(ix) $[(\Box H^*_{\alpha,\beta}(A)) * (\Diamond H^*_{\alpha,\beta}(B))]^C = \Diamond[H^*_{\alpha,\beta}(A)]^C * \Box[H^*_{\alpha,\beta}(B)]^C$,

(x) $[(\Diamond H^*_{\alpha,\beta}(A)) * (\Box H^*_{\alpha,\beta}(B))]^C = \Box[H^*_{\alpha,\beta}(A)]^C * \Diamond[H^*_{\alpha,\beta}(B)]^C$,

(xi) $[(\Box J_{\alpha,\beta}(A)) * (\Diamond J_{\alpha,\beta}(B))]^C = \Diamond[J_{\alpha,\beta}(A)]^C * \Box[J_{\alpha,\beta}(B)]^C$,

(xii) $[(\Diamond J_{\alpha,\beta}(A)) * (\Box J_{\alpha,\beta}(B))]^C = \Box[J_{\alpha,\beta}(A)]^C * \Diamond[J_{\alpha,\beta}(B)]^C$,

(xiii) $[(\Box J^*_{\alpha,\beta}(A)) * (\Diamond J^*_{\alpha,\beta}(B))]^C = \Diamond[J^*_{\alpha,\beta}(A)]^C * \Box[J^*_{\alpha,\beta}(B)]^C$,

(xiv) $[(\Diamond J^*_{\alpha,\beta}(A)) * (\Box J^*_{\alpha,\beta}(B))]^C = \Box[J^*_{\alpha,\beta}(A)]^C * \Diamond[J^*_{\alpha,\beta}(B)]^C$.

Proof. (i) Now

$$\Box(D_\alpha(A)) = \langle \mu_A(x) + \alpha\pi_A(x), 1\text{–}(\mu_A(x) + \alpha\pi_A(x))\rangle$$
$$\Diamond(D_\alpha(B)) = \langle 1\text{–}(\nu_B(x) + (1-\alpha)\pi_B(x)), \nu_B(x) + (1-\alpha)\pi_B(x)\rangle$$

So

$$[(\Box D_\alpha(A)) * (\Diamond D_\alpha(B))] = \langle \frac{((\mu_A(x) + \alpha\pi_A(x)) + 1\text{–}(\nu_B(x) + (1-\alpha)\pi_B(x)))}{(2((\mu_A(x) + \alpha\pi_A(x)) + 1\text{–}(\nu_B(x) + (1-\alpha)\pi_B(x)) + 1))}, \frac{(1\text{–}(\mu_A(x) + \alpha\pi_A(x)) + \nu_B(x) + (1-\alpha)\pi_B(x))}{(2(1-(\mu_A(x) + \alpha\pi_A(x)) + \nu_B(x) + (1-\alpha)\pi_B(x) + 1))}\rangle$$

Therefore,

$$[(\Box D_\alpha(A)) * (\Diamond D_\alpha(B))]^C = \langle \frac{(1-(\mu_A(x) + \alpha\pi_A(x)) + \nu_B(x) + (1-\alpha)\pi_B(x))}{(2(1-(\mu_A(x) + \alpha\pi_A(x)) + \nu_B(x) + (1-\alpha)\pi_B(x) + 1))}, \frac{((\mu_A(x) + \alpha\pi_A(x)) + 1 - (\nu_B(x) + (1-\alpha)\pi_B(x)))}{(2((\mu_A(x) + \alpha\pi_A(x)) + 1 - (\nu_B(x) + (1-\alpha)\pi_B(x)) + 1))}\rangle$$

Again,

$$[D_\alpha(A)]^C = \langle \nu_A(x) + (1-\alpha)\pi_A(x), \mu_A(x) + \alpha\pi_A(x)\rangle$$
$$\Diamond[D_\alpha(A)]^C = \langle 1\text{–}(\mu_A(x) + \alpha\pi_A(x)), \mu_A(x) + \alpha\pi_A(x)\rangle$$

And

$$[D_\alpha(B)]^C = \langle \nu_B(x) + (1-\alpha)\pi_B(x), \mu_B(x) + \alpha\pi_B(x)\rangle$$
$$\Box[D_\alpha(B)]^C = \langle \nu_B(x) + (1-\alpha)\pi_B(x), 1 - (\nu_B(x) + (1-\alpha)\pi_B(x))\rangle$$

So

$$\Diamond[D_\alpha(A)]^C * \Box[D_\alpha(B)]^C = \langle \frac{(1\text{–}(\mu_A(x) + \alpha\pi_A(x)) + \nu_B(x) + (1-\alpha)\pi_B(x))}{(2(1\text{–}(\mu_A(x) + \alpha\pi_A(x)) + \nu_B(x) + (1-\alpha)\pi_B(x) + 1))}, \frac{((\mu_A(x) + \alpha\pi_A(x)) + 1\text{–}(\nu_B(x) + (1-\alpha)\pi_B(x)))}{(2((\mu_A(x) + \alpha\pi_A(x)) + 1\text{–}(\nu_B(x) + (1-\alpha)\pi_B(x)) + 1))}\rangle$$

Hence

$$[(\Box D_\alpha(A)) * (\Diamond D_\alpha(B))]^C = \Diamond[D_\alpha(A)]^C * \Box[D_\alpha(B)]^C.$$

Similarly (ii) to (xiv) can be proved. $\Box$

Theorem 3.8. *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then:*

(i) $[(\Box D_\alpha(A)) \odot (\Diamond D_\alpha(B))]^C = \Diamond[D_\alpha(A)]^C \odot \Box[D_\alpha(B)]^C$,

(ii) $[(\Diamond D_\alpha(A)) \odot (\Box D_\alpha(B))]^C = \Box[D_\alpha(A)]^C \odot \Diamond[D_\alpha(B)]^C$,

(iii) $[(\Box F_{\alpha,\beta}(A)) \odot (\Diamond F_{\alpha,\beta}(B))]^C = \Diamond[F_{\alpha,\beta}(A)]^C \odot \Box[F_{\alpha,\beta}(B)]^C$,

(iv) $[(\Diamond F_{\alpha,\beta}(A)) \odot (\Box F_{\alpha,\beta}(B))]^C = \Box[F_{\alpha,\beta}(A)]^C \odot \Diamond[F_{\alpha,\beta}(B)]^C$,

(v) $[(\Box G_{\alpha,\beta}(A)) \odot (\Diamond G_{\alpha,\beta}(B))]^C = \Diamond[G_{\alpha,\beta}(A)]^C \odot \Box[G_{\alpha,\beta}(B)]^C$,

(vi) $[(\Diamond G_{\alpha,\beta}(A)) \odot (\Box G_{\alpha,\beta}(B))]^C = \Box[G_{\alpha,\beta}(A)]^C \odot \Diamond[G_{\alpha,\beta}(B)]^C$,

(vii) $[(\Box H_{\alpha,\beta}(A)) \odot (\Diamond H_{\alpha,\beta}(B))]^C = \Diamond[H_{\alpha,\beta}(A)]^C \odot \Box[H_{\alpha,\beta}(B)]^C$,

(viii) $[(\Diamond H_{\alpha,\beta}(A)) \odot (\Box H_{\alpha,\beta}(B))]^C = \Box[H_{\alpha,\beta}(A)]^C \odot \Diamond[H_{\alpha,\beta}(B)]^C$,

(ix) $[(\Box H^*_{\alpha,\beta}(A)) \odot (\Diamond H^*_{\alpha,\beta}(B))]^C = \Diamond[H^*_{\alpha,\beta}(A)]^C \odot \Box[H^*_{\alpha,\beta}(B)]^C$,

(x) $[(\Diamond H^*_{\alpha,\beta}(A)) \odot (\Box H^*_{\alpha,\beta}(B))]^C = \Box[H^*_{\alpha,\beta}(A)]^C \odot \Diamond[H^*_{\alpha,\beta}(B)]^C$,

(xi) $[(\Box J_{\alpha,\beta}(A)) \odot (\Diamond J_{\alpha,\beta}(B))]^C = \Diamond[J_{\alpha,\beta}(A)]^C \odot \Box[J_{\alpha,\beta}(B)]^C$,

(xii) $[(\Diamond J_{\alpha,\beta}(A)) \odot (\Box J_{\alpha,\beta}(B))]^C = \Box[J_{\alpha,\beta}(A)]^C \odot \Diamond[J_{\alpha,\beta}(B)]^C$,

(xiii) $[(\Box J^*_{\alpha,\beta}(A)) \odot (\Diamond J^*_{\alpha,\beta}(B))]^C = \Diamond[J^*_{\alpha,\beta}(A)]^C \odot \Box[J^*_{\alpha,\beta}(B)]^C$,

(xiv) $[(\Diamond J^*_{\alpha,\beta}(A)) \odot (\Box J^*_{\alpha,\beta}(B))]^C = \Box[J^*_{\alpha,\beta}(A)]^C \odot \Diamond[J^*_{\alpha,\beta}(B)]^C$.

Proof. (ix) Let us have $\alpha + \beta \leq 1$. Now

$$H^*_{\alpha,\beta}(A \odot B) = \langle \alpha\mu_{A\odot B}(x), \nu_{A\odot B}(x) + \beta(1 - \alpha\mu_{A\odot B}(x) - \nu_{A\odot B}(x))\rangle$$
$$\Box(H^*_{\alpha,\beta}(A)) = \langle \alpha\mu_A(x), 1{-}\alpha\mu_A(x)\rangle$$
$$\Diamond(H^*_{\alpha,\beta}(B)) = \langle 1 - (\nu_B(x) + \beta(1{-}\alpha\mu_B(x){-}\nu_B(x))), \nu_B(x){+}\beta(1{-}\alpha\mu_B(x){-}\nu_B(x))\rangle$$
$$[(\Box H^*_{\alpha,\beta}(A)) \odot (\Diamond H^*_{\alpha,\beta}(B))] = \langle \frac{([\alpha\mu_A(x)][1 - (\nu_B(x) + \beta(1 - \alpha\mu_B(x) - \nu_B(x)))])}{(2[(\alpha\mu_A(x))][1 - (\nu_B(x) + \beta(1 - \alpha\mu_B(x) - \nu_B(x)))] + 1])}, \frac{((1{-}\alpha\mu_A(x))(\nu_B(x) + \beta(1 - \alpha\mu_B(x) - \nu_B(x)))}{(2[(1{-}\alpha\mu_A(x))(\nu_B(x) + \beta(1 - \alpha\mu_B(x) - \nu_B(x)) + 1])}\rangle$$

So

$$[(\Box H^*_{\alpha,\beta}(A)) \odot (\Diamond H^*_{\alpha,\beta}(B))]^C = \langle \frac{((1{-}\alpha\mu_A(x))(\nu_B(x) + \beta(1 - \alpha\mu_B(x) - \nu_B(x)))}{(2[(1{-}\alpha\mu_A(x))(\nu_B(x) + \beta(1 - \alpha\mu_B(x) - \nu_B(x)) + 1])}, \frac{([\alpha\mu_A(x)][1 - (\nu_B(x) + \beta(1 - \alpha\mu_B(x) - \nu_B(x)))])}{(2[(\alpha\mu_A(x))][1 - (\nu_B(x) + \beta(1 - \alpha\mu_B(x) - \nu_B(x)))] + 1])}\rangle$$

Again,

$$\begin{aligned}
[H^*_{\alpha,\beta}(A)]^C &= \langle \nu_A(x) + \beta(1 - \alpha\mu_A(x) - \nu_A(x)), \alpha\mu_A(x)\rangle \\
\Diamond[H^*_{\alpha,\beta}(A)]^C &= \langle 1\text{–}\alpha\mu_A(x), \alpha\mu_A(x)\rangle \\
[H^*_{\alpha,\beta}(B)]^C &= \langle \nu_B(x) + \beta(1 - \alpha\mu_B(x)\text{–}\nu_B(x)), \alpha\mu_B(x)\rangle \\
\Box[H^*_{\alpha,\beta}(B)]^C &= \langle \nu_B(x) + \beta(1 - \alpha\mu_B(x)\text{–}\nu_B(x)), 1 - (\nu_B(x) + \beta(1 - \alpha\mu_B(x)\text{–}\nu_B(x)))\rangle
\end{aligned}$$

Then,

$$\Diamond[H^*_{\alpha,\beta}(A)]^C \odot \Box[H^*_{\alpha,\beta}(B)]^C = \langle \frac{((1\text{–}\alpha\mu_A(x))(\nu_B(x) + \beta(1 - \alpha\mu_B(x) - \nu_B(x)))}{(2[(1\text{–}\alpha\mu_A(x))(\nu_B(x) + \beta(1 - \alpha\mu_B(x) - \nu_B(x)) + 1])}, \frac{([\alpha\mu_A(x)][1 - (\nu_B(x) + \beta(1 - \alpha\mu_B(x) - \nu_B(x)))])}{(2[(\alpha\mu_A(x))][1 - (\nu_B(x) + \beta(1 - \alpha\mu_B(x) - \nu_B(x)))] + 1])}\rangle.$$

Hence

$$[(\Box H^*_{\alpha,\beta}(A)) \odot (\Diamond H^*_{\alpha,\beta}(B))]^C = \Diamond[H^*_{\alpha,\beta}(A)]^C \odot \Box[H^*_{\alpha,\beta}(B)]^C.$$

Similarly (i) to (viii) and (x) to (xiv) can be proved. $\Box$

Theorem 3.9. *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then:*

(i) $[(\Box D_\alpha(A)) \bowtie (\Diamond D_\alpha(B))]^C = \Diamond[D_\alpha(A)]^C \bowtie \Box[D_\alpha(B)]^C$,

(ii) $[(\Diamond D_\alpha(A)) \bowtie (\Box D_\alpha(B))]^C = \Box[D_\alpha(A)]^C \bowtie \Diamond[D_\alpha(B)]^C$,

(iii) $[(\Box F_{\alpha,\beta}(A)) \bowtie (\Diamond F_{\alpha,\beta}(B))]^C = \Diamond[F_{\alpha,\beta}(A)]^C \bowtie \Box[F_{\alpha,\beta}(B)]^C$,

(iv) $[(\Diamond F_{\alpha,\beta}(A)) \bowtie (\Box F_{\alpha,\beta}(B))]^C = \Box[F_{\alpha,\beta}(A)]^C \bowtie \Diamond[F_{\alpha,\beta}(B)]^C$,

(v) $[(\Box G_{\alpha,\beta}(A)) \bowtie (\Diamond G_{\alpha,\beta}(B))]^C = \Diamond[G_{\alpha,\beta}(A)]^C \bowtie \Box[G_{\alpha,\beta}(B)]^C$,

(vi) $[(\Diamond G_{\alpha,\beta}(A)) \bowtie (\Box G_{\alpha,\beta}(B))]^C = \Box[G_{\alpha,\beta}(A)]^C \bowtie \Diamond[G_{\alpha,\beta}(B)]^C$,

(vii) $[(\Box H_{\alpha,\beta}(A)) \bowtie (\Diamond H_{\alpha,\beta}(B))]^C = \Diamond[H_{\alpha,\beta}(A)]^C \bowtie \Box[H_{\alpha,\beta}(B)]^C$,

(viii) $[(\Diamond H_{\alpha,\beta}(A)) \bowtie (\Box H_{\alpha,\beta}(B))]^C = \Box[H_{\alpha,\beta}(A)]^C \bowtie \Diamond[H_{\alpha,\beta}(B)]^C$,

(ix) $[(\Box H^*_{\alpha,\beta}(A)) \bowtie (\Diamond H^*_{\alpha,\beta}(B))]^C = \Diamond[H^*_{\alpha,\beta}(A)]^C \bowtie \Box[H^*_{\alpha,\beta}(B)]^C$,

(x) $[(\Diamond H^*_{\alpha,\beta}(A)) \bowtie (\Box H^*_{\alpha,\beta}(B))]^C = \Box[H^*_{\alpha,\beta}(A)]^C \bowtie \Diamond[H^*_{\alpha,\beta}(B)]^C$,

(xi) $[(\Box J_{\alpha,\beta}(A)) \bowtie (\Diamond J_{\alpha,\beta}(B))]^C = \Diamond[J_{\alpha,\beta}(A)]^C \bowtie \Box[J_{\alpha,\beta}(B)]^C$,

(xii) $[(\Diamond J_{\alpha,\beta}(A)) \bowtie (\Box J_{\alpha,\beta}(B))]^C = \Box[J_{\alpha,\beta}(A)]^C \bowtie \Diamond[J_{\alpha,\beta}(B)]^C$,

(xiii) $[(\Box J^*_{\alpha,\beta}(A)) \bowtie (\Diamond J^*_{\alpha,\beta}(B))]^C = \Diamond[J^*_{\alpha,\beta}(A)]^C \bowtie \Box[J^*_{\alpha,\beta}(B)]^C$,

(xiv) $[(\Diamond J^*_{\alpha,\beta}(A)) \bowtie (\Box J^*_{\alpha,\beta}(B))]^C = \Box[J^*_{\alpha,\beta}(A)]^C \bowtie \Diamond[J^*_{\alpha,\beta}(B)]^C$.

Proof. (ii) Now

$$\Diamond(D_\alpha(A)) = \langle 1–(\nu_A(x) + (1-\alpha)\pi_A(x)), \nu_A(x) + (1-\alpha)\pi_A(x)\rangle$$
$$\Box(D_\alpha(B)) = \langle \mu_B(x) + \alpha\pi_B(x), 1–(\mu_B(x) + \alpha\pi_B(x))\rangle.$$

So

$$(\Diamond D_\alpha(A)) \bowtie (\Box D_\alpha(B)) = \langle \frac{((1–(\nu_A(x) + (1-\alpha)\pi_A(x)) + (\mu_B(x) + \alpha\pi_B(x))}{(2((1–(\nu_A(x) + (1-\alpha)\pi_A(x)) + (\mu_B(x) + \alpha\pi_B(x)) + 1}, \frac{(\nu_A(x) + (1-\alpha)\pi_A(x) + (1–(\mu_B(x) + \alpha\pi_B(x)))}{(2(\nu_A(x) + (1-\alpha)\pi_A(x)) + (1–(\mu_B(x) + \alpha\pi_B(x)) + 1}\rangle.$$

Therefore,

$$[(\Diamond D_\alpha(A)) \bowtie (\Box D_\alpha(B))]^C = \langle \frac{(\nu_A(x) + (1-\alpha)\pi_A(x) + (1–(\mu_B(x) + \alpha\pi_B(x)))}{(2(\nu_A(x) + (1-\alpha)\pi_A(x)) + (1–(\mu_B(x) + \alpha\pi_B(x)) + 1}, \frac{((1–(\nu_A(x) + (1-\alpha)\pi_A(x)) + (\mu_B(x) + \alpha\pi_B(x))}{(2((1–(\nu_A(x) + (1-\alpha)\pi_A(x)) + (\mu_B(x) + \alpha\pi_B(x)) + 1}\rangle.$$

Again,

$$[D_\alpha(A)]^C = \langle \nu_A(x) + (1-\alpha)\pi_A(x), \mu_A(x) + \alpha\pi_A(x)\rangle$$
$$\Box[D_\alpha(A)]^C = \langle \nu_A(x) + (1-\alpha)\pi_A(x), 1 - (\nu_A(x) + (1-\alpha)\pi_A(x))\rangle.$$
$$[D_\alpha(B)]^C = \langle \nu_B(x) + (1-\alpha)\pi_B(x), \mu_B(x) + \alpha\pi_B(x)\rangle$$
$$\Diamond[D_\alpha(B)]^C = \langle 1 - (\mu_B(x) + \alpha\pi_B(x)), \mu_B(x) + \alpha\pi_B(x)\rangle.$$

So,

$$\Box[D_\alpha(A)]^C \bowtie \Diamond[D_\alpha(B)]^C = \langle \frac{(\nu_A(x) + (1-\alpha)\pi_A(x)) + 1 - (\mu_B(x) + \alpha\pi_B(x))}{(2(\nu_A(x) + (1-\alpha)\pi_A(x)) + 1 - (\mu_B(x) + \alpha\pi_B(x)) + 1}, \frac{((1 - (\nu_A(x) + (1-\alpha)\pi_A(x))) + (\mu_B(x) + \alpha\pi_B(x))}{(2((1 - (\nu_A(x) + (1-\alpha)\pi_A(x)) + (\mu_B(x) + \alpha\pi_B(x)) + 1}\rangle.$$

Hence

$$[(\Diamond D_\alpha(A)) \bowtie (\Box D_\alpha(B))]^C = \Box[D_\alpha(A)]^C \bowtie \Diamond[D_\alpha(B)]^C.$$

Similarly the other parts of the theorem can be proved. $\Box$

Theorem 3.10. *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0, 1]$, where $\alpha + \beta \leq 1$, then:*

(i) $[(\Box D_\alpha(A))\infty(\Diamond D_\alpha(B))]^C = \Diamond[D_\alpha(A)]^C\infty\Box[D_\alpha(B)]^C$,

(ii) $[(\Diamond D_\alpha(A))\infty(\Box D_\alpha(B))]^C = \Box[D_\alpha(A)]^C\infty\Diamond[D_\alpha(B)]^C$,

(iii) $[(\Box F_{\alpha,\beta}(A))\infty(\Diamond F_{\alpha,\beta}(B))]^C = \Diamond[F_{\alpha,\beta}(A)]^C\infty\Box[F_{\alpha,\beta}(B)]^C$,

(iv) $[(\Diamond F_{\alpha,\beta}(A))\infty(\Box F_{\alpha,\beta}(B))]^C = \Box[F_{\alpha,\beta}(A)]^C\infty\Diamond[F_{\alpha,\beta}(B)]^C$,

(v) $[(\Box G_{\alpha,\beta}(A))\infty(\Diamond G_{\alpha,\beta}(B))]^C = \Diamond[G_{\alpha,\beta}(A)]^C\infty\Box[G_{\alpha,\beta}(B)]^C$,

(vi) $[(\Diamond G_{\alpha,\beta}(A))\infty(\Box G_{\alpha,\beta}(B))]^C = \Box[G_{\alpha,\beta}(A)]^C\infty\Diamond[G_{\alpha,\beta}(B)]^C$,

(vii) $[(\Box H_{\alpha,\beta}(A))\infty(\Diamond H_{\alpha,\beta}(B))]^C = \Diamond[H_{\alpha,\beta}(A)]^C\infty\Box[H_{\alpha,\beta}(B)]^C$,

(viii) $[(\Diamond H_{\alpha,\beta}(A))\infty(\Box H_{\alpha,\beta}(B))]^C = \Box[H_{\alpha,\beta}(A)]^C\infty\Diamond[H_{\alpha,\beta}(B)]^C$,

(ix) $[(\Box H^*_{\alpha,\beta}(A))\infty(\Diamond H^*_{\alpha,\beta}(B))]^C = \Diamond[H^*_{\alpha,\beta}(A)]^C\infty\Box[H^*_{\alpha,\beta}(B)]^C$,

(x) $[(\Diamond H^*_{\alpha,\beta}(A))\infty(\Box H^*_{\alpha,\beta}(B))]^C = \Box[H^*_{\alpha,\beta}(A)]^C\infty\Diamond[H^*_{\alpha,\beta}(B)]^C$,

(xi) $[(\Box J_{\alpha,\beta}(A))\infty(\Diamond J_{\alpha,\beta}(B))]^C = \Diamond[J_{\alpha,\beta}(A)]^C\infty\Box[J_{\alpha,\beta}(B)]^C$,

(xii) $[(\Diamond J_{\alpha,\beta}(A))\infty(\Box J_{\alpha,\beta}(B))]^C = \Box[J_{\alpha,\beta}(A)]^C\infty\Diamond[J_{\alpha,\beta}(B)]^C$,

(xiii) $[(\Box J^*_{\alpha,\beta}(A))\infty(\Diamond J^*_{\alpha,\beta}(B))]^C = \Diamond[J^*_{\alpha,\beta}(A)]^C\infty\Box[J^*_{\alpha,\beta}(B)]^C$,

(xiv) $[(\Diamond J^*_{\alpha,\beta}(A))\infty(\Box J^*_{\alpha,\beta}(B))]^C = \Box[J^*_{\alpha,\beta}(A)]^C\infty\Diamond[J^*_{\alpha,\beta}(B)]^C$.

Proof. (iv) Let us have $\alpha + \beta \leq 1$. Now

$$\Diamond(F_{\alpha,\beta}(A)) = \langle 1-(\nu_A(x) + \beta\pi_A(x)), \nu_A(x) + \beta\pi_A(x)\rangle$$
$$\Box(F_{\alpha,\beta}(B)) = \langle \mu_B(x) + \alpha\pi_B(x), 1-(\mu_B(x) + \alpha\pi_B(x))\rangle.$$

So

$$[(\Diamond F_{\alpha,\beta}(A))\infty(\Box F_{\alpha,\beta}(B))] = \langle \frac{(1-(\nu_A(x) + \beta\pi_A(x)))(\mu_B(x) + \alpha\pi_B(x))}{(2(1-(\nu_A(x) + \beta\pi_A(x))(\mu_B(x) + \alpha\pi_B(x)) + 1}, \frac{(\nu_A(x) + \beta\pi_A(x))(1-(\mu_B(x) + \alpha\pi_B(x)))}{(2(\nu_A(x) + \beta\pi_A(x))(1-(\mu_B(x) + \alpha\pi_B(x))) + 1}\rangle$$

Therefore,

$$[(\Diamond F_{\alpha,\beta}(A))\infty(\Box F_{\alpha,\beta}(B))]^C = \langle \frac{(\nu_A(x) + \beta\pi_A(x))(1-(\mu_B(x) + \alpha\pi_B(x)))}{(2(\nu_A(x) + \beta\pi_A(x))(1-(\mu_B(x) + \alpha\pi_B(x))) + 1}, \frac{(1-(\nu_A(x) + \beta\pi_A(x)))(\mu_B(x) + \alpha\pi_B(x))}{(2(1-(\nu_A(x) + \beta\pi_A(x)))(\mu_B(x) + \alpha\pi_B(x)) + 1}\rangle$$

Again,

$$[(F_{\alpha,\beta}(A))]^C = \langle \nu_A(x) + \beta\pi_A(x), \mu_A(x) + \alpha\pi_A(x)\rangle$$
$$[(F_{\alpha,\beta}(B))]^C = \langle \nu_B(x) + \beta\pi_B(x), \mu_B(x) + \alpha\pi_B(x)\rangle$$
$$\Box[(F_{\alpha,\beta}(A))]^C = \langle \nu_A(x) + \beta\pi_A(x), 1-(\nu_A(x) + \beta\pi_A(x))\rangle,$$
$$\Diamond[(F_{\alpha,\beta}(B))]^C = \langle 1-(\mu_B(x) + \alpha\pi_B(x)), \mu_B(x) + \alpha\pi_B(x)\rangle$$

So

$$\Box[(F_{\alpha,\beta}(A))]^C\infty\Diamond[(F_{\alpha,\beta}(B))]^C = \langle \frac{(\nu_A(x) + \beta\pi_A(x))(1-(\mu_B(x) + \alpha\pi_B(x)))}{(2(\nu_A(x) + \beta\pi_A(x))(1-(\mu_B(x) + \alpha\pi_B(x))) + 1}, \frac{(1-(\nu_A(x) + \beta\pi_A(x)))(\mu_B(x) + \alpha\pi_B(x))}{(2(1-(\nu_A(x) + \beta\pi_A(x)))(\mu_B(x) + \alpha\pi_B(x)) + 1}\rangle.$$

Hence

$$[(\Diamond F_{\alpha,\beta}(A))\infty(\Box F_{\alpha,\beta}(B))]^C = \Box[F_{\alpha,\beta}(A)]^C\infty\Diamond[F_{\alpha,\beta}(B)]^C.$$

Similarly the other parts of the theorem can be proved. □

Theorem 3.11. *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0,1]$, where $\alpha + \beta \leq 1$, then:*

(i) $[(\Box D_\alpha(A)) \rhd (\Diamond D_\alpha(B))]^C = \Diamond[D_\alpha(A)]^C \rhd \Box[D_\alpha(B)]^C$,

(ii) $[(\Diamond D_\alpha(A)) \rhd (\Box D_\alpha(B))]^C = \Box[D_\alpha(A)]^C \rhd \Diamond[D_\alpha(B)]^C$,

(iii) $[(\Box F_{\alpha,\beta}(A)) \rhd (\Diamond F_{\alpha,\beta}(B))]^C = \Diamond[F_{\alpha,\beta}(A)]^C \rhd \Box[F_{\alpha,\beta}(B)]^C$,

(iv) $[(\Diamond F_{\alpha,\beta}(A)) \rhd (\Box F_{\alpha,\beta}(B))]^C = \Box[F_{\alpha,\beta}(A)]^C \rhd \Diamond[F_{\alpha,\beta}(B)]^C$,

(v) $[(\Box G_{\alpha,\beta}(A)) \rhd (\Diamond G_{\alpha,\beta}(B))]^C = \Diamond[G_{\alpha,\beta}(A)]^C \rhd \Box[G_{\alpha,\beta}(B)]^C$,

(vi) $[(\Diamond G_{\alpha,\beta}(A)) \rhd (\Box G_{\alpha,\beta}(B))]^C = \Box[G_{\alpha,\beta}(A)]^C \rhd \Diamond[G_{\alpha,\beta}(B)]^C$,

(vii) $[(\Box H_{\alpha,\beta}(A)) \rhd (\Diamond H_{\alpha,\beta}(B))]^C = \Diamond[H_{\alpha,\beta}(A)]^C \rhd \Box[H_{\alpha,\beta}(B)]^C$,

(viii) $[(\Diamond H_{\alpha,\beta}(A)) \rhd (\Box H_{\alpha,\beta}(B))]^C = \Box[H_{\alpha,\beta}(A)]^C \rhd \Diamond[H_{\alpha,\beta}(B)]^C$,

(ix) $[(\Box H^*_{\alpha,\beta}(A)) \rhd (\Diamond H^*_{\alpha,\beta}(B))]^C = \Diamond[H^*_{\alpha,\beta}(A)]^C \rhd \Box[H^*_{\alpha,\beta}(B)]^C$,

(x) $[(\Diamond H^*_{\alpha,\beta}(A)) \rhd (\Box H^*_{\alpha,\beta}(B))]^C = \Box[H^*_{\alpha,\beta}(A)]^C \rhd \Diamond[H^*_{\alpha,\beta}(B)]^C$,

(xi) $[(\Box J_{\alpha,\beta}(A)) \rhd (\Diamond J_{\alpha,\beta}(B))]^C = \Diamond[J_{\alpha,\beta}(A)]^C \rhd \Box[J_{\alpha,\beta}(B)]^C$,

(xii) $[(\Diamond J_{\alpha,\beta}(A)) \rhd (\Box J_{\alpha,\beta}(B))]^C = \Box[J_{\alpha,\beta}(A)]^C \rhd \Diamond[J_{\alpha,\beta}(B)]^C$,

(xiii) $[(\Box J^*_{\alpha,\beta}(A)) \rhd (\Diamond J^*_{\alpha,\beta}(B))]^C = \Diamond[J^*_{\alpha,\beta}(A)]^C \rhd \Box[J^*_{\alpha,\beta}(B)]^C$,

(xiv) $[(\Diamond J^*_{\alpha,\beta}(A)) \rhd (\Box J^*_{\alpha,\beta}(B))]^C = \Box[J^*_{\alpha,\beta}(A)]^C \rhd \Diamond[J^*_{\alpha,\beta}(B)]^C$.

Proof. (vi) Let us have $\alpha + \beta \leq 1$. Now

$$\Diamond(G_{\alpha,\beta}(A)) = \langle 1\text{–}(\beta\nu_A(x)), \beta\nu_A(x)\rangle$$
$$\Box(F_{\alpha,\beta}(B)) = \langle \alpha\mu_B(x), 1\text{–}(\alpha\mu_B(x))\rangle$$

So

$$[(\Diamond G_{\alpha,\beta}(A)) \rhd (\Box G_{\alpha,\beta}(B))] = \langle \frac{(1-(\beta\nu_A(x))) + \alpha\mu_B(x)}{(1-(\beta\nu_A(x))) + (\alpha\mu_B(x)) + 1}, \frac{(\beta\nu_A(x)) + 1 - (\alpha\mu_B(x))}{(\beta\nu_A(x) + (1-(\alpha\mu_B(x))) + 1}\rangle.$$

Therefore,

$$[(\Diamond G_{\alpha,\beta}(A)) \rhd (\Box G_{\alpha,\beta}(B))]^C = \langle \frac{(\beta\nu_A(x)) + 1\text{–}(\alpha\mu_B(x))}{(\beta\nu_A(x) + (1\text{–}(\alpha\mu_B(x))) + 1}, \frac{(1\text{–}(\beta\nu_A(x))) + \alpha\mu_B(x)}{(1\text{–}(\beta\nu_A(x))) + (\alpha\mu_B(x)) + 1}\rangle.$$

Again,

$$[(G_{\alpha,\beta}(A))]^C = \langle \beta\nu_A(x), \alpha\mu_A(x)\rangle$$
$$[(G_{\alpha,\beta}(B))]^C = \langle \beta\nu_B(x), \alpha\mu_B(x)\rangle$$

$$\Box[(G_{\alpha,\beta}(A))]^C = \langle \beta\nu_A(x), 1–(\beta\nu_A(x))\rangle$$
$$\Diamond[(G_{\alpha,\beta}(B))]^C = \langle 1–(\alpha\mu_B(x)), \alpha\mu_B(x)\rangle.$$

So

$$\Box[(G_{\alpha,\beta}(A))]^C \rhd \Diamond[(G_{\alpha,\beta}(B))]^C = \langle \frac{(\beta\nu_A(x)) + 1–(\alpha\mu_B(x))}{(\beta\nu_A(x)+(1–(\alpha\mu_B(x)))+1}, \frac{(1–(\beta\nu_A(x))) + \alpha\mu_B(x)}{(1–(\beta\nu_A(x)))+(\alpha\mu_B(x))+1}\rangle.$$

Hence

$$[(\Diamond G_{\alpha,\beta}(A)) \rhd (\Box G_{\alpha,\beta}(B))]^C = \Box[G_{\alpha,\beta}(A)]^C \rhd \Diamond[G_{\alpha,\beta}(B)]^C.$$

Similarly the other parts of the theorem can be proved. $\Box$

Theorem 3.12. *Let X be a nonempty set. If A and B be any two IFSs drawn from X and $\alpha, \beta \in [0, 1]$, where $\alpha + \beta \leq 1$, then:*

(i) $[(\Box D_\alpha(A)) \lhd (\Diamond D_\alpha(B))]^C = \Diamond[D_\alpha(A)]^C \lhd \Box[D_\alpha(B)]^C$,

(ii) $[(\Diamond D_\alpha(A)) \lhd (\Box D_\alpha(B))]^C = \Box[D_\alpha(A)]^C \lhd \Diamond[D_\alpha(B)]^C$,

(iii) $[(\Box F_{\alpha,\beta}(A)) \lhd (\Diamond F_{\alpha,\beta}(B))]^C = \Diamond[F_{\alpha,\beta}(A)]^C \lhd \Box[F_{\alpha,\beta}(B)]^C$,

(iv) $[(\Diamond F_{\alpha,\beta}(A)) \lhd (\Box F_{\alpha,\beta}(B))]^C = \Box[F_{\alpha,\beta}(A)]^C \lhd \Diamond[F_{\alpha,\beta}(B)]^C$,

(v) $[(\Box G_{\alpha,\beta}(A)) \lhd (\Diamond G_{\alpha,\beta}(B))]^C = \Diamond[G_{\alpha,\beta}(A)]^C \lhd \Box[G_{\alpha,\beta}(B)]^C$,

(vi) $[(\Diamond G_{\alpha,\beta}(A)) \lhd (\Box G_{\alpha,\beta}(B))]^C = \Box[G_{\alpha,\beta}(A)]^C \lhd \Diamond[G_{\alpha,\beta}(B)]^C$,

(vii) $[(\Box H_{\alpha,\beta}(A)) \lhd (\Diamond H_{\alpha,\beta}(B))]^C = \Diamond[H_{\alpha,\beta}(A)]^C \lhd \Box[H_{\alpha,\beta}(B)]^C$,

(viii) $[(\Diamond H_{\alpha,\beta}(A)) \lhd (\Box H_{\alpha,\beta}(B))]^C = \Box[H_{\alpha,\beta}(A)]^C \lhd \Diamond[H_{\alpha,\beta}(B)]^C$,

(ix) $[(\Box H^*_{\alpha,\beta}(A)) \lhd (\Diamond H^*_{\alpha,\beta}(B))]^C = \Diamond[H^*_{\alpha,\beta}(A)]^C \lhd \Box[H^*_{\alpha,\beta}(B)]^C$,

(x) $[(\Diamond H^*_{\alpha,\beta}(A)) \lhd (\Box H^*_{\alpha,\beta}(B))]^C = \Box[H^*_{\alpha,\beta}(A)]^C \lhd \Diamond[H^*_{\alpha,\beta}(B)]^C$,

(xi) $[(\Box J_{\alpha,\beta}(A)) \lhd (\Diamond J_{\alpha,\beta}(B))]^C = \Diamond[J_{\alpha,\beta}(A)]^C \lhd \Box[J_{\alpha,\beta}(B)]^C$,

(xii) $[(\Diamond J_{\alpha,\beta}(A)) \lhd (\Box J_{\alpha,\beta}(B))]^C = \Box[J_{\alpha,\beta}(A)]^C \lhd \Diamond[J_{\alpha,\beta}(B)]^C$,

(xiii) $[(\Box J^*_{\alpha,\beta}(A)) \lhd (\Diamond J^*_{\alpha,\beta}(B))]^C = \Diamond[J^*_{\alpha,\beta}(A)]^C \lhd \Box[J^*_{\alpha,\beta}(B)]^C$,

(xiv) $[(\Diamond J^*_{\alpha,\beta}(A)) \lhd (\Box J^*_{\alpha,\beta}(B))]^C = \Box[J^*_{\alpha,\beta}(A)]^C \lhd \Diamond[J^*_{\alpha,\beta}(B)]^C$.

Proof. (viii) Let us have $\alpha + \beta \leq 1$. Now

$$\Diamond(H_{\alpha,\beta}(A)) = \langle 1 - (\nu_A(x) + \beta\pi_A(x)), \nu_A(x) + \beta\pi_A(x)\rangle$$
$$\Box(H_{\alpha,\beta}(B)) = \langle \alpha\mu_B(x), 1–(\alpha\mu_B(x))\rangle.$$

So

$$[(\Diamond H_{\alpha,\beta}(A)) \lhd (\Box H_{\alpha,\beta}(B))] = \langle \frac{(1–(\nu_A(x) + \beta\pi_A(x))(\alpha\mu_B(x))}{(1–(\nu_A(x) + \beta\pi_A(x))(\alpha\mu_B(x)) + 1}, \frac{(\nu_A(x) + \beta\pi_A(x))(1–(\alpha\mu_B(x)))}{(\nu_A(x) + \beta\pi_A(x)(1–(\alpha\mu_B(x))) + 1}\rangle.$$

Therefore,

$$[(\Diamond H_{\alpha,\beta}(A)) \lhd (\Box H_{\alpha,\beta}(B))]^C = \langle \frac{(\nu_A(x) + \beta\pi_A(x))(1–(\alpha\mu_B(x)))}{(\nu_A(x) + \beta\pi_A(x)(1–(\alpha\mu_B(x))) + 1}, \frac{(1–(\nu_A(x) + \beta\pi_A(x))(\alpha\mu_B(x))}{(1–(\nu_A(x) + \beta\pi_A(x))(\alpha\mu_B(x)) + 1}\rangle.$$

Again,

$$[(H_{\alpha,\beta}(A))]^C = \langle \nu_A(x) + \beta\pi_A(x), \alpha\mu_A(x)\rangle$$
$$[(H_{\alpha,\beta}(B))]^C = \langle \nu_B(x) + \beta\pi_B(x), \alpha\mu_B(x)\rangle$$
$$\Box[(H_{\alpha,\beta}(A))]^C = \langle \nu_A(x) + \beta\pi_A(x), 1–(\nu_A(x) + \beta\pi_A(x))\rangle$$
$$\Diamond[(H_{\alpha,\beta}(B))]^C = \langle 1–(\alpha\mu_B(x)), \alpha\mu_B(x)\rangle.$$

So,

$$\Box[(H_{\alpha,\beta}(A))]^C \lhd \Diamond[(H_{\alpha,\beta}(B))]^C = \langle \frac{(\nu_A(x) + \beta\pi_A(x))(1–(\alpha\mu_B(x)))}{(\nu_A(x) + \beta\pi_A(x)(1–(\alpha\mu_B(x))) + 1}, \frac{(1–(\nu_A(x) + \beta\pi_A(x))(\alpha\mu_B(x))}{(1–(\nu_A(x) + \beta\pi_A(x))(\alpha\mu_B(x)) + 1}\rangle.$$

Hence

$$[(\Diamond H_{\alpha,\beta}(A)) \lhd (\Box H_{\alpha,\beta}(B))]^C = \Box[H_{\alpha,\beta}(A)]^C \lhd \Diamond[H_{\alpha,\beta}(B)]^C.$$

Similarly the other parts of the theorem can be proved. $\Box$

4 Conclusion

Some new equalities are established in intuitionistic fuzzy sets with the help of certain operations together with the modal operators. These will certainly help us to investigate many other properties in connection to intuitionistic fuzzy operators in near future.

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