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Common coupled fixed point theorems in generalized intuitionistic fuzzy metric spaces

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Abstract: In this paper the notion of generalized intuitionistic fuzzy metric space by using the idea of intuitionistic fuzzy set due to Atanassov. Some coupled coincidence point results for compatibility of two mappings. We prove two unique common coupled fixed point theorems for Junck type and for three mapping in symmetric generalized intuitionistic fuzzy metric spaces.
Keywords: t-norm, t-conorm, Coupled coincidence point, Compatible mappings, Generalized intuitionistic metric space.
AMS Classification: 47H10, 54H25.

1 Introduction

Park introduced and discussed in [6] a notion of intuitionistic fuzzy metric space which is based both on the idea of intuitionistic fuzzy set due to Atanassov [1], and the concept of a fuzzy

metric space given George & Veeramani [4]. Mustafa and Sims [7, 8] and Naidu et. al [11] demonstrated the most of the claims concerning the fundamental topological structure of D-metric introduced by Dhage [3] & hence all theorems are incorrect. Alternatively, Mustafa and Sims introduced a G-metric space and obtained some fixed point theorems in it. In this paper, we prove two unique common coupled fixed point theorems for Junck type for three mappings in symmetric generalized intuitionistic fuzzy metric spaces.

Definition 1.1.([15]) A 5-tuple $(X, G, H, *, \diamond)$ is said to be a generalized intuitionistic fuzzy metric space (Shortly GIFM) if X is an arbitrary non-empty set,$*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, G and H are fuzzy sets on $X^3 \times (0, \infty)$ satisfying the following conditions. For every $x, y, z, a \in X$ and $t, s > 0$

1. $G(x, y, z, t) + H(x, y, z, t) \leq 1$,
2. $G(x, x, y, t) > 0 for x \neq y$,
3. $G(x, x, y, t) \geq G(x, y, z, t) for y \neq z$,
4. $G(x, y, z, t) = 1$ if and only if $x = y = z$,
5. $G(x, y, z, t) = G(P(x, y, z), t)$, where p is a permutation function,
6. $G(x, a, a, t) * G(a, y, z, s) \leq G(x, y, z, t + s)$,
7. $G(x, y, z, .) : (0, \infty) \to [0, 1]$ is continuous,
8. G is a non-decreasing of $\Re^+$, $\lim_{t\to\infty} G(x, y, z, t) = 1$,
   $\lim_{t\to 0} G(x, y, z, t) = 0$ for all $x, y, z \in X, t > 0$,
9. $H(x, x, y.t) < 1$ for $x \neq y$,
10. $H(x, x, y, t) \leq H(x, y, z, t)$ for $y \neq z$,
11. $H(x, y, z, t) = 0$ if and only if $x = y = z$,
12. $H(x, y, z, t) = H(p(x, y, z), t)$, where p is a permutation function,
13. $H(x, a, a, t) \diamond H(a, y, z, s) \geq H(x, y, z, t + s)$,
14. $H(x, y, z, .) : (0, \infty) \to [0, 1]$ is continuous,
15. H is a non-increasing function on $\Re^+$, $\lim_{t\to\infty} H(x, y, z, t) = 0$,
    $\lim_{t\to 0} H(x, y, z, t) = 1$ for all $x, y, z \in X, t > 0$.
    In this case, the pair (G, H) is called an generalized intuitionistic fuzzy metric on X.

Definition 1.2. ([15]) Let X be a non-empty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 1.3. ([10]) Let X be a non-empty set. An element $(x, y) \in X \times X$ is called

1. a coupled coincidence point of $F : X \times X \to X$ and $g : X \to X$ if $gx = F(x, y)$ and $gy = F(y, x)$.

2. a common coupled fixed point of $F : X \times X \to X$ and $g : X \to X$ if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Lemma 1.4. ([10]) Let $(X, G, H, *, \diamond)$ be a generalized intuitionistic fuzzy metric space. Then, G and H are continuous function on $X^3 \times (0, \infty)$. Now onwards, we assume the following condition:

$$\lim_{t\to\infty} G(x, y, z, t) = 1 \text{ and } \lim_{t\to\infty} H(x, y, z, t) = 0 \text{ for all } x, y, z \in X. \tag{1}$$

Using (1), can prove the following lemma.

Lemma 1.5. ([10]) Let $(X, G, H, *, \diamond)$ be a generalized intuitionstic fuzzy metric space. If there exists $k \in (0, 1)$ such that

$$\begin{aligned} \min\{G(x, y, z, kt), G(u, v, w, kt)\} &\geq \min\{G(x, y, z, t), G(u, v, w, t)\}. \\ \max\{H(x, y, z, kt), H(u, v, w, kt)\} &\leq \max\{H(x, y, z, t), H(u, v, w, t)\}. \end{aligned}$$

for all $x, y, z, u, v, w \in X$ and $t > 0$, then $x = y = z$ and $u = v = w$.

Definition 1.6. ([10]) Let X be a non-empty set. The mapping $F : X \times X \to X$ and $g : X \to X$ are called w-compatible if $g(F(x, y)) = F(gx, gy)$ and $g(F(y, x)) = F(gy, gx)$, whenever $gx = F(x, y)$ and $gy = F(y, x)$ for some $(x, y) \in X \times X$.

Now, we give our main results.

2 Main results

Theorem 2.1. Let $(X, G, H, *, \diamond)$ be a generalized intuitionistic fuzzy metric space with $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$, for all $a, b \in [0, 1]$ and $S : X \times X \to X$ and $f : X \to X$ be mappings satisfying

$$\begin{aligned} G(S(x, y), S(u, v), S(u, v), kt) &\geq \min\{G(fx, fu, fu, t), G(fy, fv, fv, t)\} \\ H(S(x, y), S(u, v).S(u, v), kt) &\leq \max\{H(fx, fu, fu, t), H(fy, fv, fv, t)\}. \end{aligned} \tag{2}$$

for all $x, y, u, v, \in X$ where $0 \leq k < 1$. $S(X \times X) \subseteq f(X)$ and $f(X)$ is a complete subspace of X, the pair (f, S) is w-compatible. Then S and f have a unique common coupled fixed point of the form (α, α) in $X \times X$.

Proof. Let $x_0, y_0 \in X$ and denote $z_n = S(x_n, y_n) = fx_{n+1}, p_n = S(y_n, x_n) = fy_{n+1}, n = 0, 1, 2, ...$

Let $d_n(t) = G(z_n, z_{n+1}, z_{n+1}, t)\rho_n(t) = H(z_n, z_{n+1}, z_{n+1}, t)e_n(t) = G(p_n, p_{n+1}, p_{n+1}, t)$ and $r_n(t) = H(p_n, p_{n+1}, p_{n+1}, t)$ From (2) we have

$$\begin{aligned} d_{n+1}(kt) &= G\left(z_{n+1}, z_{n+2}, z_{n+2}, kt\right) \\ &= G\left(S(x_{n+1}, y_{n+1}), S(x_{n+2}, y_{n+2}), S(x_{n+2}, y_{n+2}), kt\right)) \\ &\geq \min\left\{G(z_n, z_{n+1}, z_{n+1}, t), G(p_n, p_{n+1}, p_{n+1}, t)\right\} \\ &\geq \min\{d_n(t), e_n(t)\}. \end{aligned} \tag{3}$$

$$\begin{aligned} \rho_{n+1}(kt) &= H\left(z_{n+1}, z_{n+2}, z_{n+2}, kt\right) \\ &= H\left(S(x_{n+1}, y_{n+1}), S(x_{n+2}, y_{n+2}), S(x_{n+2}, y_{n+2}), kt\right)) \\ &\leq \max\left\{H(z_n, z_{n+1}, z_{n+1}, t), H(p_n, p_{n+1}, p_{n+1}, t)\right\} \\ &\leq \max\{\rho_n(t), r_n(t)\}. \end{aligned} \tag{4}$$

$$\begin{aligned} e_{n+1}(kt) &= G\left(p_{n+1}, p_{n+2}, p_{n+2}, kt\right) \\ &= G\left(S(y_{n+1}, x_{n+1}), S(y_{n+2}, x_{n+2}), S(y_{n+2}, x_{n+2}), kt\right)) \\ &\geq \min\left\{G(p_n, p_{n+1}, p_{n+1}, t), G(z_n, z_{n+1}, z_{n+1}, t)\right. \\ &\geq \min\{e_n(t), d_n(t)\}. \end{aligned} \tag{5}$$

$$\begin{aligned} r_{n+1}(kt) &= H\left(p_{n+1}, p_{n+2}, p_{n+2}, kt\right) \\ &= H\left(S(y_{n+1}, x_{n+1}), S(y_{n+2}, x_{n+2}), S(y_{n+2}, x_{n+2}), kt\right)) \\ &\leq \max\left\{H(p_n, p_{n+1}, p_{n+1}, t), H(z_n, z_{n+1}, z_{n+1}, t)\right\} \\ &\leq \max\{r_n(t), \rho_n(t)\}. \end{aligned} \tag{6}$$

Thus

$$\min\{d_{n+1}(kt), e_{n+1}(kt)\} \geq \min\{d_n(t), e_n(t)\}$$

Hence,

$$\begin{aligned} \min\{d_n(t), e_n(t)\} &\geq \min\left\{d_{n-1}\left(\frac{t}{k}\right), e_{n-1}\left(\frac{t}{k}\right)\right\} \\ &\geq \min\left\{d_{n-2}\left(\frac{t}{k^2}\right), e_{n-2}\left(\frac{t}{k^2}\right)\right\} \\ &\vdots \\ &\geq \left\{d_0\left(\frac{t}{k^n}\right), e_0\left(\frac{t}{k^n}\right)\right\} \\ &= \min\left\{G(z_0, z_1, z_1, \frac{t}{k^n}), G(p_0, p_1, p_1, \frac{t}{k^n})\right\}. \end{aligned} \tag{7}$$

$$
\begin{aligned}
\max\{\rho_{n+1}(kt), r_{n+1}(kt)\} &\leq \max\{\rho_n(t), r_n(t)\} \\
\max\{\rho_n(t), r_n(t)\} &\leq \max\left\{\rho_{n-1}\left(\frac{t}{k}\right), r_{n-1}\left(\frac{t}{k}\right)\right\} \\
&\leq \max\left\{\rho_{n-2}\left(\frac{t}{k^2}\right), r_{n-2}\left(\frac{t}{k^2}\right)\right\} \\
&\vdots \\
&\leq \left\{\rho_0\left(\frac{t}{k^n}\right), r_0\left(\frac{t}{k^n}\right)\right\} \\
&= \max\left\{H(z_0, z_1, z_1, \frac{t}{k^n}), H(p_0, p_1, p_1, \frac{t}{k^n})\right\}.
\end{aligned} \tag{8}
$$

For any positive integer n and fixed positive integer p, we have

$$
\begin{aligned}
G(z_n, z_{n+p}, z_{n+p}, t) \geq\ & G\left(z_{n+p-1}, z_{n+p}, z_{n+p}, \frac{t}{p}\right) * G\left(z_{n+p-2}, z_{n+p-1}, z_{n+p-1}, \frac{t}{p}\right) \\
& * \ldots * G\left(z_n, z_{n+1}, z_{n+1}, \frac{t}{p}\right)
\end{aligned}
$$

$$
\begin{aligned}
G(z_n, z_{n+p}, z_{n+p}, t) \geq\ & min\left\{G(z_0, z_1, z_1, \frac{t}{pk^{n+p-1}}), G(p_0, p_1, p_1, \frac{t}{pk^{n+p-1}})\right\} \\
& * \min\left\{G(z_0, z_1, z_1, \frac{t}{pk^{n+p-2}}), G(p_0, p_1, p_1, \frac{t}{pk^{n+p-2}}\right\} \\
& * \ldots * \min\left\{G(z_0, z_1, z_1, \frac{t}{pk^n}), G(p_0, p_1, p_1, \frac{t}{pk^n})\right\}
\end{aligned}
$$

$$
\begin{aligned}
H(z_n, z_{n+p}, z_{n+p}, t) \leq\ & H\left(z_{n+p-1}, z_{n+p}, z_{n+p}, \frac{t}{p}\right) \diamond H\left(z_{n+p-2}, z_{n+p-1}, z_{n+p-1}, \frac{t}{p}\right) \\
& \diamond \ldots \diamond H\left(z_n, z_{n+1}, z_{n+1}, \frac{t}{p}\right)
\end{aligned}
$$

$$
\begin{aligned}
H(z_n, z_{n+p}, z_{n+p}, t) \leq\ & \max\left\{H(z_0, z_1, z_1, \frac{t}{pk^{n+p-1}}), H(p_0, p_1, p_1, \frac{t}{pk^{n+p-1}})\right\} \\
& \diamond \max\left\{H(z_0, z_1, z_1, \frac{t}{pk^{n+p-2}}), H(p_0, p_1, p_1, \frac{t}{pk^{n+p-2}})\right\} \\
& \diamond \ldots \diamond \max\left\{H(z_0, z_1, z_1, \frac{t}{pk^n}), G(p_0, p_1, p_1, \frac{t}{pk^n})\right\}
\end{aligned}
$$

Letting $n \to \infty$ and using (1) we get

$$
\lim_{n\to\infty} G(z_n, z_{n+p}, z_{n+p}, t) \geq 1 * 1 * \ldots * 1 = 1
$$

$$
\lim_{n\to\infty} H(z_n, z_{n+p}, z_{n+p}, t) \leq 0 \diamond 0 \diamond \ldots \diamond 0 = 0.
$$

Hence, $\lim_{n\to\infty} G(z_n, z_{n+p}, z_{n+p}, t) = 1$, and $\lim_{n\to\infty} H(z_n, z_{n+p}, z_{n+p}, t) = 0$. Thus, $\{z_n\}$ is G and H Cauchy in X. Similarly, we can show that $\{p_n\}$ is G and H Cauchy in X.

Since $f(X)$ is G and H complete $\{z_n\}$ and $\{p_n\}$ converges to some α and β in $f(X)$, respectively.

Hence, there exists x and y in X such that $\alpha = fx$, $\beta = fy$.

$$\begin{aligned} G\left(z_n, S(x,y), S(x,y), kt\right) &= G(S(x_n,y_n), S(x,y), S(x,y), kt) \\ &\geq \min\left\{G(z_{n-1}, fx, fx, t), G(p_{n-1}, fy, fy, t)\right\} \end{aligned}$$

Letting $n \to \infty$ we get, $G(fx, S(x,y), Sx, y), kt) \geq \min\{1,1\} = 1$,

$$\begin{aligned} H\left(z_n, S(x,y), S(x,y), kt\right) &= H(S(x_n,y_n), S(x,y), S(x,y), kt) \\ &\leq \max\left\{H(z_{n-1}, fx, fx, t), H(p_{n-1}, fy, fy, t)\right\} \end{aligned}$$

Letting $n \to \infty$ we get, $H(fx, S(x,y), Sx, y), kt) \leq \max\{0,0\} = 0$,

Hence, $S(x,y) = fx$. Similarly, it can be shown that $S(y,x) = fy$. Since (f,S) is w-compatible, we have

$$f\alpha = ffx = f(S(x,y)) = S(fx, fy) = S(\alpha, \beta).$$
$$f\beta = ffy = f(S(y,x)) = S(fy, fx) = S(\beta, \alpha).$$

$$\begin{aligned} G(z_n, f\alpha, f\alpha, kt) &= G(S(x_n,y_n), S(\alpha,\beta), S(\alpha,\beta), kt) \\ &\geq \min\left\{G(z_{n-1}, f\alpha, f\alpha, t), G(p_{n-1}, f\beta, f\beta, t)\right\}. \end{aligned}$$

Letting $n \to \infty$ we get,

$$G(\alpha, f\alpha, f\alpha, kt) \geq \min\left\{G(\alpha, f\alpha, f\alpha, t), G(\beta, f\beta, f\beta, t)\right\}. \tag{9}$$

Similarly, we can show that,

$$G(\beta, f\beta, f\beta, kt) \geq \min\left\{G(\alpha, f\alpha, f\alpha, t), G(\beta, f\beta, f\beta, t)\right\}. \tag{10}$$

Thus,

$$\min\left\{G(\alpha, f\alpha, f\alpha, kt), G(\beta, f\beta, f\beta, kt)\right\} \geq \min\left\{G(\alpha, f\alpha, f\alpha, t), G(\beta, f\beta, f\beta, t)\right\}. \tag{11}$$

and

$$\begin{aligned} H(z_n, f\alpha, f\alpha, kt) &= H(S(x_n,y_n), S(\alpha,\beta), S(\alpha,\beta), kt) \\ &\leq \max\left\{H(z_{n-1}, f\alpha, f\alpha, t), H(p_{n-1}, f\beta, f\beta, t)\right\}. \end{aligned}$$

Letting $n \to \infty$ we get,

$$H(\alpha, f\alpha, f\alpha, kt) \leq \max\left\{H(\alpha, f\alpha, f\alpha, t), G(\beta, f\beta, f\beta, t)\right\}. \tag{12}$$

Similarly, we can show that,

$$H(\beta, f\beta, f\beta, kt) \leq \max\left\{H(\alpha, f\alpha, f\alpha, t), H(\beta, f\beta, f\beta, t)\right\}. \tag{13}$$

Thus,

$$\max\{H(\alpha, f\alpha, f\alpha, kt), H(\beta, f\beta, f\beta, kt)\} \leq \max\{H(\alpha, f\alpha, f\alpha, t), H(\beta, f\beta, f\beta, t)\}. \quad (14)$$

From Lemma 1.5, we have $f\alpha = \alpha$ and $f\beta = \beta$. Thus, $\alpha = f\alpha = S(\alpha, \beta)$ and $\beta = f\beta = S(\beta, \alpha)$, Hence (α, β) is a common coupled fixed point of S and f.

Suppose (α', β') is another common coupled fixed point of S and f.

$$G(\alpha, \alpha', \alpha', kt) = G(S(\alpha, \beta), S(\alpha', \beta'), S(\alpha', \beta'), kt) \geq \min\left\{G(\alpha, \alpha', \alpha', t), G(\beta, \beta', \beta', t)\right\}.$$

[Similarly,]

$$G(\beta, \beta', \beta', kt) = G(S(\beta, \alpha), S(\beta', \alpha'), S(\beta', \alpha'), kt) \geq \min\left\{G(\alpha, \alpha', \alpha', t), G(\beta, \beta', \beta', t)\right\}.$$

[Thus,] $\min\left\{G(\alpha, \alpha', \alpha', kt), G(\beta, \beta', \beta', kt)\right\} \geq \min\left\{G(\alpha, \alpha', \alpha', t), G(\beta, \beta', \beta', t)\right\}.$ (15)

and

$$H(\alpha, \alpha', \alpha', kt) = H(S(\alpha, \beta), S(\alpha', \beta'), S(\alpha', \beta'), kt) \leq \max\left\{G(\alpha, \alpha', \alpha', t), H(\beta, \beta', \beta', t)\right\}.$$

[Similarly,]

$$H(\beta, \beta', \beta', kt) = H(S(\beta, \alpha), S(\beta', \alpha'), S(\beta', \alpha'), kt) \leq \max\left\{H(\alpha, \alpha', \alpha', t), H(\beta, \beta', \beta', t)\right\}.$$

[Thus,]

$$\max\left\{H(\alpha, \alpha', \alpha', kt), H(\beta, \beta', \beta', kt)\right\} \leq \max\left\{H(\alpha, \alpha', \alpha', t), H(\beta, \beta', \beta', t)\right\}. \quad (16)$$

From Lemma 1.5, $\alpha' = \alpha$ and $\beta' = \beta$. Thus, (α, β) is the unique common coupled fixed point of S and f.

Now, we will show that $\alpha = \beta$.

$$\begin{aligned}
G(\alpha, \alpha, \beta, kt) &= G(S(\alpha, \beta), S(\alpha, \beta), S(\beta, \alpha), kt). \\
&\geq \min\{G(\alpha, \alpha, \beta, t), G(\beta, \beta, \alpha, t)\}. \\
G(\alpha, \beta, \beta, kt) &= G(S(\alpha, \beta), S(\beta, \alpha), S(\beta, \alpha), kt). \\
&\geq \min\{G(\alpha, \beta, \beta, t), G(\beta, \alpha, \alpha, t)\}. \text{[Thus,]} \\
\min\{G(\alpha, \alpha, \beta, kt), G(\alpha, \beta, \beta, kt)\} &\geq \min\{G(\alpha, \alpha, \beta, t), G(\alpha, \beta, \beta, t)\}. \text{[and]} \\
H(\alpha, \alpha, \beta, kt) &= H(S(\alpha, \beta), S(\alpha, \beta), S(\beta, \alpha), kt) \\
&\leq \max\{H(\alpha, \alpha, \beta, t), H(\beta, \beta, \alpha, t)\} \\
H(\alpha, \beta, \beta, kt) &= H(S(\alpha, \beta), S(\beta, \alpha), S(\beta, \alpha), kt) \\
&\leq \max\{H(\alpha, \beta, \beta, t), H(\beta, \alpha, \alpha, t)\} \\
&\text{[Thus,]} \\
\max\{H(\alpha, \alpha, \beta, kt), H(\alpha, \beta, \beta, kt)\} &\leq \max\{H(\alpha, \alpha, \beta, t), H(\alpha, \beta, \beta, t)\}.
\end{aligned}$$

Thus, we have $\alpha = \beta$. Thus, α is a common fixed point of S and f. That is, $\alpha = f\alpha =$

$S(\alpha,\alpha)$. Suppose $\alpha^{'}$ is another common fixed point of S and f.

$$\begin{aligned}
G(\alpha^{'},\alpha,\alpha,t) &= G\left(S(\alpha^{'},\alpha^{'},S(\alpha,\alpha),S(\alpha,\alpha),t\right)\\
&\geq \min\left\{G\left(\alpha^{'},\alpha,\alpha,\frac{t}{k}\right),G\left(\alpha^{'},\alpha,\alpha,\frac{t}{k}\right)\right\}.\\
&\geq G\left(\alpha^{'},\alpha,\alpha,\frac{t}{k^2}\right)\\
&\vdots\\
&\geq G\left(\alpha^{'},\alpha,\alpha,\frac{t}{k^n}\right)\to 1.
\end{aligned}$$

$$\begin{aligned}
H(\alpha^{'},\alpha,\alpha,t) &= H\left(S(\alpha^{'},\alpha^{'},S(\alpha,\alpha),S(\alpha,\alpha),t\right)\\
&\leq \min\left\{H\left(\alpha^{'},\alpha,\alpha,\frac{t}{k}\right),H\left(\alpha^{'},\alpha,\alpha,\frac{t}{k}\right)\right\}.\\
&\leq H\left(\alpha^{'},\alpha,\alpha,\frac{t}{k^2}\right)\\
&\vdots\\
&\leq H\left(\alpha^{'},\alpha,\alpha,\frac{t}{k^n}\right)\to 0.
\end{aligned}$$

Hence $\alpha^{'}=\alpha$. Thus S and f have a unique common coupled fixed point of the form (α,α). Finally, we prove a common coupled fixed point theorem for three mappings in symmetric generalized intuitionstic fuzzy metric spaces. □

Theorem 2.2. Let $(X,G,H,*,\diamond)$ be a symmetric generalized intuitionistic complete fuzzy metric space with $a*b=\min\{a,b\}$ and $a\diamond b=\max\{a,b\}$ for all $a,b\in[0,1]$ and let $S,T,R:X\times X\to X$ be mappings satisfying

$$\begin{aligned}
G\left(S(x,y),T(u,v),R(p,q),kt\right)\geq \min\{&G(x,u,p,t),G(y,v,q,t),G(x,x,S(x,y),t),\\
&G(u,u,T(u,v),t),G(p,p,R(p,q),t)\}
\end{aligned}$$

$$\begin{aligned}
H\left(S(x,y),T(u,v),R(p,q),kt\right)\leq \max\{&H(x,u,p,t),H(y,v,q,t),H(x,x,S(x,y),t),\\
&H(u,u,T(u,v),t),H(p,p,R(p,q),t)\}.
\end{aligned}$$

for all $x,y,u,v,p,q\in X$ where $0\leq k<1$. Then there exists $(x,y)\in X\times X$ such that

$$x=S(x,y)=T(x,y)=R(x,y)$$

and

$$y=S(y,x)=T(y,x)=R(y,x)$$

S,T and R have a unique common coupled fixed point of the form $(x,x)\in X\times X$.

Proof. Let $x_0, y_0 \in X$. Define the sequence $\{x_n\}$ and $\{y_n\}$ in X as follows:

$$x_{3n+1} = S(x_{3n}, y_{3n}), y_{3n+1} = S(y_{3n}, x_{3n}); x_{3n+2} = T(x_{3n+1}, y_{3n+1}), y_{3n+2} = T(y_{3n+1}, x_{3n+1});$$
$$x_{3n+3} = R(x_{3n+2}, y_{3n+2}), y_{3n+3} = R(y_{3n+2}, x_{3n+2}), n = 0, 1, 2, ...$$

Suppose $x_{3n+1} = x_{3n}$ for some n. Then $S(x, y) = x$, where $x = x_{3n}, y = y_{3n}$. Suppose $T(x, y) \neq R(x, y)$. Then

$$\begin{aligned} G(x, T(x, y), R(x, y), kt) &= G(S(x, y), T(x, y), R(x, y), kt) \\ &\geq \min\{1, 1, 1, G(x, x, T(x, y), t), G(x, x, R(x, y), t)\}. \\ &\geq G(x, T(x, y), R(x, y), t) \end{aligned} \tag{17}$$

and

$$\begin{aligned} H(x, T(x, y), R(x, y), kt) &= H(S(x, y), T(x, y), R(x, y), kt) \\ &\leq \max\{0, 0, 0, H(x, x, T(x, y), t), H(x, x, R(x, y), t)\}. \\ &\leq H(x, T(x, y), R(x, y), t) \end{aligned} \tag{18}$$

It is contradiction. Hence $T(x, y) = R(x, y)$. X is symmetric,

$$\begin{aligned} G(x, T(x, y), T(x, y), kt) &\geq G(x, x, T(x, y), t) \\ &= G(x, T(x, y), T(x, y), t). \end{aligned} \tag{19}$$

$$\begin{aligned} H(x, T(x, y), T(x, y), kt) &\leq H(x, x, T(x, y), t) \\ &= H(x, T(x, y), T(x, y), t). \end{aligned} \tag{20}$$

We have $T(x, y) = x$. Thus $S(x, y) = T(x, y) = R(x, y) = x$.

Similarly, if $x_{3n+1} = x_{3n+2}$ or $x_{3n+2} = x_{3n+3}$, then also we can show that $S(x, y) = T(x, y) = R(x, y) = x$ for some x, y in X. Similarly it can be shown that if $y_{3n} = y_{3n+1}$ or $y_{3n+1} = y_{3n+2}$ or $y_{3n+2} = y_{3n+3}$ then there exists $(x, y) \in X \times X$ such that $S(y, x) = T(y, x) = R(y, x) = y$.

Now, assume that $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1}$ for all n, write $d_n(t) = G(x_n, x_{n+1}, x_{n+2}, t)\rho_n(t) = H(x_n, x_{n+1}, x_{n+2}, t), e_n(t) = G(y_n, y_{n+1}, y_{n+2}, t), r_n(t) = H(y_n, y_{n+1}, y_{n+2}, t)$.

$$\begin{aligned} d_{3n}(kt) &= G(x_{3n}, x_{3n+1}, x_{3n+2}, kt) \\ &= G(S(x_{3n}, y_{3n}), T(x_{3n+1}), y_{3n+1}, R(x_{3n-1}, y_{3n-1}), kt) \\ &\geq \min\{d_{3n-1}(t), e_{3n-1}(t), G(x_{3n}, x_{3n}, x_{3n+1}, t), \\ &\quad G(x_{3n+1}, x_{3n+1}, x_{3n+2}, t), G(x_{3n-1}, x_{3n-1}, x_{3n}, t)\} \\ &\geq \min\{d_{3n-1}(t), e_{3n-1}(t), d_{3n}(t), d_{3n}(t), d_{3n-1}(t)\}. \end{aligned} \tag{21}$$

and

$$\begin{aligned} \rho_{3n}(kt) &= H(x_{3n}, x_{3n+1}, x_{3n+2}, kt) \\ &= H(S(x_{3n}, y_{3n}), T(x_{3n+1}, y_{3n+1}), R(x_{3n-1}, y_{3n-1}), kt) \\ &\leq \max\{\rho_{3n-1}(t), r_{3n-1}(t), H(x_{3n}, x_{3n}, x_{3n+1}, t), \\ &\quad H(x_{3n+1}, x_{3n+1}, x_{3n+2}, t), H(x_{3n-1}, x_{3n-1}, x_{3n}, t)\} \\ &\leq \max\{\rho_{3n-1}(t), r_{3n-1}(t), \rho_{3n}(t), \rho_{3n}(t), \rho_{3n-1}(t)\}. \end{aligned} \tag{22}$$

Thus, $d_{3n}(kt) \geq \min\{d_{3n-1}(t), e_{3n-1}(t)\}$ and $\rho_{3n}(kt) \leq max\{\rho_{3n-1}(t), r_{3n-1}(t)\}$. Similarly, we have $e_{3n}(kt) \geq \min\{d_{3n-1}(kt), e_{3n-1}(kt)\}$ and $r_{3n}(kt) \leq \max\{\rho_{3n-1}(t), r_{3n-1}(t)\}$.

Thus, $\min\{d_{3n}(kt), e_{3n}(kt)\} \geq \min\{d_{3n-1}(kt), e_{3n-1}(kt)\}$ and $\max\{\rho_{3n}(kt), r_{3n}(kt)\} \leq \max\{\rho_{3n-1}(kt), r_{3n-1}(kt)\}$. Similarly, we can show that
$\min\{d_{3n+1}(kt), e_{3n+1}(kt)\} \geq \min\{d_{3n}(t), e_{3n}(t)\}$
$\min\{d_{3n+2}(kt), e_{3n+2}(kt)\} \geq \min\{d_{3n+1}(t), e_{3n+1}(t)\}$ and
$\max\{\rho_{3n+1}(kt), r_{3n+1}(kt)\} \leq \max\{\rho_{3n}(t), r_{3n}(t)\}$
$\max\{\rho_{3n+2}(kt), r_{3n+2}(kt)\} \leq \max\{\rho_{3n+1}(t), r_{3n+1}(t)\}$.
Thus, $\min\{d_{n+1}(kt), e_{n+1}(kt)\} \geq \min\{d_n(t), e_n(t)\}$ and
$\max\{\rho_{n+1}(kt), r_{n+1}(kt)\} \leq \max\{\rho_n(t), r_n(t)\}$

$$\begin{aligned}
\min\{d_n(t), e_n(t)\} &\geq \min\{d_n\left(\frac{t}{k}\right), e_n\left(\frac{t}{k}\right)\} \\
&\geq \min\{d_{n-1}\left(\frac{t}{k^2}\right), e_{n-1}\left(\frac{t}{k^2}\right)\} \\
&\vdots \\
&\geq \min\{d_n\left(\frac{t}{k^n}\right), e_n\left(\frac{t}{k^n}\right)\} \\
&= \min\{G(x_0, x_1, x_2, \left(\frac{t}{k^n}\right)), G(y_0, y_1, y_2, \left(\frac{t}{k^n}\right))\}.
\end{aligned}$$

$$\begin{aligned}
\max\{\rho_n(t), r_n(t)\} &\leq \max\{\rho_n\left(\frac{t}{k}\right), r_n\left(\frac{t}{k}\right)\} \\
&\leq \max\{\rho_{n-1}\left(\frac{t}{k^2}\right), r_{n-1}\left(\frac{t}{k^2}\right)\} \\
&\vdots \\
&\leq \max\{\rho_0\left(\frac{t}{k^n}\right), r_0\left(\frac{t}{k^n}\right)\} \\
&= \{H(x_0, x_1, x_2, \left(\frac{t}{k^n}\right)), H(y_0, y_1, y_2, \left(\frac{t}{k^n}\right))\}.
\end{aligned}$$

$$\begin{aligned}
\text{Thus,} G(x_n, x_{n+1}, x_{n+2}, t) &\geq \min\{G(x_0, x_1, x_2, \frac{t}{k^n}), G(y_0, y_1, y_2, \frac{t}{k^n})\} \\
H(x_n, x_{n+1}, x_{n+2}, t) &\leq \max\{H(x_0, x_1, x_2, \frac{t}{k^n}), H(y_0, y_1, y_2, \frac{t}{k^n})\}.
\end{aligned}$$

$$\begin{aligned}
\text{We have,} G(x_n, x_n, x_{n+1}, t) &\geq G(x_n, x_{n+!}, x_{n+2}, t) \\
&\geq \min\{G(x_0, x_1, x_2, \frac{t}{k^n}), G(y_0, y_1, y_2, \frac{t}{k^n})\}. \\
\text{and} \qquad H(x_n, x_n, x_{n+1}, t) &\leq H(x_n, x_{n+!}, x_{n+2}, t) \\
&\leq \max\{H(x_0, x_1, x_2, \frac{t}{k^n}), H(y_0, y_1, y_2, \frac{t}{k^n})\}
\end{aligned}$$

As in previous Theorem 2.1, we can show that $\{x_n\}$ and $\{y_n\}$ are G and H Cauchy sequence in X. Since X is G and H complete, there exists $x, y \in X$ such that $x_n \to x$ and $y_n \to y$.

$$\begin{aligned} G(S(x,y), x_{3n+2}, x_{3n+3}, kt) &= G(S(x,y), T(x_{3n+1}, y_{3n+1}), R(x_{3n+2}, y_{3n+2}), kt) \\ &\geq \min\{G(x, x_{3n+1}, x_{3n+2}, t), G(y, y_{3n+1}, y_{3n+2}, t), G(x, x, S(x,y), t), \\ &\qquad G(x_{3n+1}, x_{3n+1}, x_{3n+2}, t), G(x_{3n+2}, x_{3n+2}, x_{3n+3}, t)\}. \end{aligned}$$

Letting $n \to \infty$

$$\begin{aligned} G(S(x,y), x, x, kt) &\geq \min\{1, 1, G(x, x, S(x,y), t), 1, 1\} \\ &= G(x, x, S(x,y), t). \end{aligned}$$

and
$$\begin{aligned} H(S(x,y), x_{3n+2}, x_{3n+3}, kt) &= H(S(x,y), T(x_{3n+1}, y_{3n+1}), R(x_{3n+2}, y_{3n+2}), kt) \\ &\leq \max\{H(x, x_{3n+1}, x_{3n+2}, t), H(y, y_{3n+1}, y_{3n+2}, t), H(x, x, S(x,y), t), \\ &\qquad H(x_{3n+1}, x_{3n+1}, x_{3n+2}, t), H(x_{3n+2}, x_{3n+2}, x_{3n+3}, t)\} \end{aligned}$$

Letting $n \to \infty$

$$\begin{aligned} H(S(x,y), x, x, kt) &\leq \max\{0, 0, H(x, x, S(x,y), t), 0, 0\} \\ &= H(x, x, S(x,y), t) \end{aligned}$$

From this, we have $S(x,y) = x$. As in the first part of the proof, we can show that $S(x,y) = T(x,y) = R(x,y) = x$.
Similarly, it can be shown that $S(y,x) = T(y,x) = R(y,x) = y$.

Thus, (x,y) is a common coupled fixed point of S, T and R. Suppose $(x^{'}, y^{'})$ is another common coupled fixed point of S, T and R. Consider,

$$\begin{aligned} G(x, x, x^{'}, kt) &= G(S(x,y), T(x,y), R(x^{'}, y^{'}), kt) \\ &\geq \min\{G(x, x, x^{'}t), G(y, y, y^{'}, t), 1, 1.1\} \\ &= \min\{G(x, x, x^{'}t), G(y, y, y^{'}t)\} \end{aligned}$$

$$\begin{aligned} H(x, x, x^{'}, kt) &= H(S(x,y), T(x,y), R(x^{'}, y^{'}), kt) \\ &\leq \max\{H(x, x, x^{'}t), H(y, y, y^{'}, t), 0, 0, 0\} \\ &= \mathit{max}\{H(x, x, x^{'}t), H(y, y, y^{'}t)\}. \end{aligned}$$

Also,

$$\begin{aligned} G(y, y, y^{'}, kt) &= G(S(y,x), T(y,x), R(y^{'}, x^{'}), kt) \\ &\geq \min\{G(x, x, x^{'}t), G(y, y, y^{'}, t), 1, 1, 1, \} \\ &= \min\{G(x, x, x^{'}t), G(y, y, y^{'}t)\}. \end{aligned}$$

$$\begin{aligned} H(y, y, y^{'}, kt) &= H(S(y,x), T(y,x), R(y^{'}, x^{'}), kt) \\ &\leq \max\{H(x, x, x^{'}t), H(y, y, y^{'}, t), 0, 0, 0\} \\ &= \max\{H(x, x, x^{'}t), H(y, y, y^{'}t)\}. \end{aligned}$$

$$\begin{aligned} \text{Thus, } \min\{G(x,x,x^{'},kt),G(y,y,y^{'},kt)\} &\geq \min\{G(x,x,x^{'},t),G(y,y,y^{'},t)\} \\ \max\{H(x,x,x^{'},kt),H(y,y,y^{'},kt)\} &\leq \max\{H(x,x,x^{'},t),H(y,y,y^{'},t)\}. \end{aligned}$$

From Lemma 1.5 we have $x^{'} = x$ and $y^{'} = y$. Thus (x, y) is the unique common coupled fixed point of S, T and R. Now, we will show that $x = y$. Consider

$$\begin{aligned} G(x,x,y,kt) &= G(S(x,y),T(x,y),R(y,x),kt) \\ &\geq \min\{G(x,x,y,t),G(y,y,x,t),1,1,1\} \\ &= G(x,x,y,t) \\ \text{[and]}H(x,x,y,kt) &= H(S(x,y),T(x,y),R(y,x),kt) \\ &\leq \max\{H(x,x,y,t),H(y,y,x,t),0,0,0\} \\ &= H(x,x,y,t). \end{aligned}$$

Hence $x = y$.

Thus S, T and R have a unique common coupled fixed point of the form (x, x). □

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