# Counting the number of intuitionistic fuzzy subgroups of finite Abelian groups of different order 

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#### Abstract

In this paper, we have defined double keychain, double pinned flag and equivalence classes of intuitionistic fuzzy subgroups of a group by using an equivalence relation. We have also determined the formulae to count the number of distinct intuitionistic fuzzy subgroups of finite Abelian groups; in particular the intuitionistic fuzzy subgroups of $p$-groups and that of $Z_{p^{2}} \times Z_{q}$ where $p$ and $q$ are distinct primes.


Keywords: Double pins, Double keychain, Double pinned flag, Equivalence, Intuitionistic fuzzy subgroup.
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## 1 Introduction

Corresponding to the concept of fuzzy set given by L. Zadeh [12], A. Rosenfeld [7] gave the concept of fuzzy group. By using the concept of intuitionistic fuzzy sets given by K. Atanassov [1], K. Hur et al [3] introduced the concept of intuitionistic fuzzy subgroups. After that, much work has been done on fuzzy subgroups and intuitionistic fuzzy subgroups. V. Murali and B. B. Makamba [4, 5, 6] gave the concept of keychains, pins, pinned flags and used it in defining equivalence classes of fuzzy subgroups of finite Abelian groups of different order. After this, many researchers [2, 9, 10, 11] worked on counting the number of fuzzy subgroups of finite group of different order. On the basis of the concept used for fuzzy subgroups, in this paper we have defined equivalence relation on intuitionistic fuzzy subgroups to determine distinct equivalence classes. We have also counted the different equivalence classes of fuzzy subgroups of finite Abelian groups of different order.

## 2 Preliminaries

Definition 2.1 [4] Intuitionistic fuzzy set: Let $X$ be any set, then a set $A$ having the form $A=\left\{\left(x, \mu_{A}(x), v_{A}(x) \mid x \in X\right\}\right.$, where the function $\mu_{A}(x): X \rightarrow[0,1]$ and $v_{A}(x): X \rightarrow[0,1]$ denote the degree of membership and the degree of non-membership respectively of each element $x \in X$ to the set $A$ and $\mu_{A}(x), v_{A}(x) \leq 1$ for all $x \in X$.

Definition 2.2 [3] Intuitionistic fuzzy sub-group: Let $G$ be a group, then an intuitionistic fuzzy subset $A$ of $G$ is called intuitionistic fuzzy sub-group of $G$ if
(a) $\mu_{A}(x y) \geq \mu_{A}(x) \wedge \mu_{A}(y)$
(b) $\mu_{A}\left(x^{-1}\right) \geq \mu_{A}(x)$
(c) $v_{A}(x y) \leq v_{A}(x) \vee v_{A}(y)$
(d) $v_{A}\left(x^{-1}\right) \leq v_{A}(x), \forall x, y \in G$.

Definition 2.3 [8] ( $\alpha, \beta$ )-cut of Intuitionistic fuzzy sub-group: Let $A$ be an intuitionistic fuzzy sub-group of a group $G$, then $(\alpha, \beta)$-cut of $A$ denoted by $C_{\alpha, \beta}(A)$ defined by

$$
C_{\alpha, \beta}(A)=\left\{x \in G \mid \mu_{A}(x) \geq \alpha, v_{A}(x) \leq \beta .\right.
$$

where $\alpha, \beta \in[0,1]$ with $\alpha+\beta \leq 1$ is a crisp subgroup of $G$ provided $C_{\alpha, \beta}(A) \neq \varnothing$.
Definition 2.4. [10] A flag is a maximal chain of subgroups of the form

$$
G_{0} \subset G_{1} \subset G_{2} \subset \ldots \subset G_{n}=G,
$$

in which $G_{0}=\langle e\rangle$ and all $G_{i}$ 's are called the components of the flag.

## 3 Main results

Definition 3.1 Double keychain: Consider a paired set of real numbers ( $\alpha_{i}, \beta_{i}$ ), such that $\alpha_{i}, \beta_{i} \in$ [ 0,1 ] and $\alpha_{i}, \beta_{i} \leq 1$ for all $i=0,1,2, \ldots, n$, then we call a chain $\left(\alpha_{0}, \beta_{0}\right) \geq\left(\alpha_{1}, \beta_{1}\right) \geq \ldots \geq\left(\alpha_{n}, \beta_{n}\right)$ a double keychain iff $1=\alpha_{0} \geq \alpha_{1} \geq \ldots \geq \alpha_{n}$ and $0=\beta_{0} \leq \beta_{1} \leq \ldots \leq \beta_{n}$ and the pair ( $\alpha_{i}, \beta_{i}$ ) are called double pins.

Definition 3.2 Double pinned flag: With the combination of flag and double keychain, we denote the chain $\langle e\rangle^{(1,0)} \subset G_{1}^{\left(\alpha_{1}, \beta_{1}\right)} \subset G_{2}^{\left(\alpha_{2}, \beta_{2}\right)} \subset \ldots \subset G_{n}^{\left(\alpha_{n}, \beta_{n}\right)}$ as double pinned flag.

The purpose of defining the double pinned flag is to define Intuitionistic fuzzy sub-group in term of pinned flag.

It can be easily proved that corresponding to each pinned flag as defined above, we can find an intuitionistic fuzzy sub-group $A=\left\{\left(x, \mu_{A}(x), v_{A}(x) \mid x \in X\right\}\right.$ in short $\left(\mu_{A}(x), v_{A}(x)\right)$ as follows:

$$
\left(\mu_{A}(x), v_{A}(x)\right)= \begin{cases}(1,0) & : x \in\langle e\rangle \\ \left(\alpha_{1}, \beta_{1}\right) & : x \in G_{1} \mid\langle e\rangle \\ \left(\alpha_{2}, \beta_{2}\right) & : x \in G_{2} \mid G_{1} \\ \cdots & \\ \left(\alpha_{n}, \beta_{n}\right) & : x \in G_{n} \mid G_{n-1}\end{cases}
$$

The converse to the above result is also true. That is, given any intuitionistic fuzzy sub-group $A$ of $G$, then $A$ can be represented in the form of a double pinned flag.

Let $I^{G}$ be the collection of all Intuitionistic fuzzy sub-groups of a group $G$. We define a relation on $I^{G}$ as follows: Two intuitionistic fuzzy sub-groups $A$ and $B$ will be called related $(A \sim B)$ if and only if for all $x, y \in G$ :
(1) $\mu_{A}(x)>\mu_{A}(y)$ and $v_{A}(x)<v_{A}(y)$ if and only if $\mu_{B}(x)>\mu_{B}(y)$ and $v_{B}(x)<v_{B}(y)$
(2) $\mu_{A}(x)=0$ if and only if $\mu_{B}(x)=0$
(3) $\left(\mu_{A}(x), v_{A}(x)\right)=(0,0)$ if and only if $\left(\mu_{B}(x), v_{B}(x)\right)=(0,0)$

It can be easily checked that above relation is an equivalence relation.
Note: Condition (2) is introduced to preserve the support of membership value and Condition (3) is introduced because $(0,0)$ means we are completely uncertain about the support so we can say that the three cases $(0, \beta),(\alpha, \beta),(0,0)$ will be considered different.

Thus, we can say that if two intuitionistic fuzzy sub-groups $A$ and $B$ have double pinned flag representation as

$$
\langle e\rangle^{(1,0)} \subset G_{1}^{\left(\alpha_{1}, \beta_{1}\right)} \subset G_{2}^{\left(\alpha_{2}, \beta_{2}\right)} \subset \ldots \subset G_{n}^{\left(\alpha_{n}, \beta_{n}\right)}
$$

and

$$
\langle e\rangle^{(1,0)} \subset H_{1}^{\left(\gamma_{1}, \delta_{1}\right)} \subset H_{2}^{\left(\gamma, \delta_{2}\right)} \subset \ldots \subset H_{n}^{\left(\gamma_{n}, \delta_{n}\right)}
$$

respectively, then $A \sim B$ if and only if:

1. $n=m$;
2. $G_{i}=H_{i}$;
3. $\left(\alpha_{i}, \beta_{i}\right)>\left(\alpha_{j}, \beta_{j}\right)$ if and only if $\left(\gamma_{i}, \delta_{i}\right)>\left(\gamma_{j}, \delta_{j}\right), \alpha_{i}=0$ iff $\gamma_{i}=0,\left(\alpha_{i}, \beta_{i}\right)=(0,0)$ iff $\left(\gamma_{i}, \delta_{i}\right)=(0,0)$ for $1 \leq i, j \leq n$.

## 4 Counting the number of intuitionistic fuzzy subgroups

We will start with assigning a value $K_{H}$ to each subgroup $H$ of an Abelian group $G$ as follows:

$$
K_{H}=3\left(\sum_{i=1}^{n} K_{H_{i}}\right)+1
$$

where $\langle e\rangle=H_{1}, H_{2}, H_{3}, \ldots, H_{n}$ are all the possible subgroups of $H$, which are properly contained in $H$. Then, the number of distinct intuitionistic fuzzy subgroups of $H$ is given by

$$
K_{H}+\left(\sum_{i=1}^{n} K_{H_{i}}\right)=4\left(\sum_{i=1}^{n} K_{H_{i}}\right)+1
$$

Then obviously $K_{\langle e\rangle}=1$.
And if we talk about intuitionistic fuzzy subgroups of $Z_{p}$, then as $Z_{p}$ contains only one subgroup $\langle e\rangle, K_{Z p}=3\left(K_{\langle e\rangle}\right)+1=4=2^{2}$ and number of distinct intuitionistic fuzzy subgroups is given by $K_{Z p}+K_{\langle e\rangle}=5$. As the double pinned flag representation of any intuitionistic fuzzy subgroup of $Z_{p}$ is $\langle e\rangle^{(1,0)} \subset\langle p\rangle^{(\alpha, \beta)}$, there can be 5 representations of double pins as follows:

1. $\{(1,0)>(1,0)\}$
2. $\{(1, \alpha)>(0,0)\}$
3. $\{(1, \alpha)>(0, \beta)\}$
4. $\{(1,0)>(0,0)\}$
5. $\{(1,0)>(0, \beta)\}$

Thus there can be 5 distinct equivalence classes of intuitionistic fuzzy subgroups of a cyclic group of order $p$. Now, if $G$ is a cyclic group of order $p^{2}$ say $Z_{p^{2}}$, then by the given formula $K_{z_{p^{2}}}=3\left(K_{Z_{p}}+K_{\langle e\rangle}\right)+1=16=2^{4}$ and the number of distinct intuitionistic fuzzy subgroups of $Z_{p^{2}}=K_{Z_{p^{2}}}+\left(K_{Z_{p}}+K_{\langle e\rangle}\right)=21$. As the double pinned flag representation of any intuitionistic fuzzy subgroup of $Z_{p^{2}}$ is $\langle e\rangle^{(1,0)} \subset\langle p\rangle^{\left(\alpha_{1}, \beta_{1}\right)} \subset\langle p\rangle^{\left(\alpha_{2}, \beta_{2}\right)}$ there can be 21 representations of double pins, as follows:

1. $\{(1,0)>(1,0)>(1,0)\}$
2. $\left\{(1,0)>(1,0)>\left(\alpha_{1}, 0\right)\right\}$
3. $\left\{(1,0)>(1,0)>\left(\alpha_{1}, \beta_{1}\right)\right\}$
4. $\{(1,0)>(1,0)>(0,0)\}$
5. $\left\{(1,0)>(1,0)>\left(0, \beta_{1}\right)\right\}$
6. $\left\{(1,0)>\left(\alpha_{1}, 0\right)>\left(\alpha_{1}, 0\right)\right\}$
7. $\left\{(1,0)>\left(\alpha_{1}, 0\right)>\left(\alpha_{1}, \beta_{1}\right)\right\}$
8. $\left\{(1,0)>\left(\alpha_{1}, \beta_{1}\right)>\left(\alpha_{1}, \beta_{1}\right)\right\}$
9. $\left\{(1,0)>\left(\alpha_{1}, \beta_{1}\right)>\left(\alpha_{1}, \beta_{2}\right)\right\}$
10. $\left\{(1,0)>\left(\alpha_{1}, 0\right)>\left(\alpha_{2}, 0\right)\right\}$
11. $\left\{(1,0)>\left(\alpha_{1}, 0\right)>\left(\alpha_{2}, \beta_{1}\right)\right\}$
12. $\left\{(1,0)>\left(\alpha_{1}, \beta_{1}\right)>\left(\alpha_{2}, \beta_{1}\right)\right\}$
13. $\left\{(1,0)>\left(\alpha_{1}, \beta_{1}\right)>\left(\alpha_{2}, \beta_{2}\right)\right\}$
14. $\left\{(1,0)>\left(\alpha_{1}, 0\right)>(0,0)\right\}$
15. $\left\{(1,0)>\left(\alpha_{1}, 0\right)>\left(0, \beta_{1}\right)\right\}$
16. $\left\{(1,0)>\left(\alpha_{1}, \beta_{1}\right)>\left(0, \beta_{1}\right)\right\}$
17. $\left\{(1,0)>\left(\alpha_{1}, \beta_{1}\right)>\left(0, \beta_{2}\right)\right\}$
18. $\{(1,0)>(0,0)>(0,0)\}$
19. $\left\{(1,0)>(0,0)>\left(0, \beta_{1}\right)\right\}$
20. $\left\{(1,0)>\left(0, \beta_{1}\right)>\left(0, \beta_{1}\right)\right\}$
21. $\left\{(1,0)>\left(0, \beta_{1}\right)>\left(0, \beta_{2}\right)\right\}$

Thus, there can be 21 distinct equivalence classes of intuitionistic fuzzy subgroups of a cyclic group of order $p^{2}$. One more important thing to note is that $K_{z_{p^{2}}}=2^{2 n}$.

Theorem 4.1 The number of distinct equivalence classes of intuitionistic fuzzy subgroups of a cyclic group of order $p^{n}$ is $\sum_{i=0}^{n} 2^{2 i}$.
Proof: Let $S^{1}, S^{2}, S^{3}, \ldots, S^{n}$ denote the number of distinct equivalence classes of intuitionistic fuzzy subgroups of cyclic groups of order $p, p^{2}, p^{3}, \ldots, p^{n}$, respectively.

We will prove the result by induction.
The number of distinct equivalence classes of intuitionistic fuzzy subgroups of cyclic group of order p is given by:
$S^{1}=5=2^{0}+2^{2}$. Thus, the result is true for $n=1$. Suppose the result is true for $n=t$, that is:

$$
S^{t}=\sum_{i=0}^{t} 2^{2 i} .
$$

Now, to prove the result for $n=t+1$.
Hence the result is proved.

Now consider the group $Z_{p} \times Z_{q}$ where $p$ and $q$ are distinct primes. $Z_{p} \times Z_{q}$ has two maximal chains which can be identified with the maximal chain of $Z_{p^{2}}$. Each of these chains will yield 21 intuitionistic fuzzy subgroups but out of these 21,5 will yield identical intuitionistic fuzzy subgroups viz:

1. $\{(1,0)>(1,0)>(1,0)\}$,
2. $\left\{(1,0)>\left(\alpha_{1}, 0\right)>\left(\alpha_{1}, 0\right)\right\}$
3. $\left\{(1,0)>\left(\alpha_{1}, \beta_{1}\right)>\left(\alpha_{1}, \beta_{1}\right)\right\}$
4. $\{(1,0)>(0,0)>(0,0)\}$
5. $\left\{(1,0)>\left(0, \beta_{1}\right)>\left(0, \beta_{1}\right)\right\}$

The other 16 will yield distinct intuitionistic fuzzy subgroups. So the number of distinct intuitionistic fuzzy subgroups of $Z_{p} \times Z_{q}$ will be $16+16+5=37$. Another way to count distinct intuitionistic fuzzy subgroups of $Z_{p} \times Z_{q}$ is given by $4\left(K_{\langle e\rangle}+K_{Z p}+K_{Z q}\right)+1=4(1+4+4)+1=$ 37. As $Z_{p} \times Z_{q}$ contains $\langle e\rangle, Z_{p}, Z_{q}$ properly.

Theorem 4.2 The number of distinct equivalence classes of intuitionistic fuzzy subgroups of $Z_{p^{n}} \times Z_{q}$ is equal to:

$$
(n+1) 2^{2(n+1)}+\sum_{i=0}^{n} 2^{2 i}
$$

Proof. By induction on $n$. For $n=1$, the number of distinct intuitionistic fuzzy subgroups of

$$
Z_{p} \times Z_{q} \text { is }(1+1) 2^{2(1+1)}+\sum_{i=0}^{1} 2^{2 i} .=37
$$

So the result is true for $n=1$.
Let the result be true for $n=k$, that is, the number of distinct intuitionistic fuzzy subgroups of $Z_{p^{k}} \times Z_{q}$ be

$$
\left.(k+1) 2^{2(k+1)}+\sum_{i=0}^{k} 2^{2 i}=M \quad \text { (say }\right) .
$$

Now to prove the result for $n=k+1$.
$Z_{p^{k}} \times Z_{q}$ contains all the subgroups of $Z_{p^{k+1}} \times Z_{q}$ except $Z_{p^{k+1}}$. So the number of distinct intuitionistic fuzzy subgroups of $Z_{p^{k+1}} \times Z_{q}$ is equal to:

$$
\begin{aligned}
4\left[M+K_{Z_{p^{k+1}}}\right]+1 & =4\left[(k+1) 2^{2(k+1)}+\sum_{i=0}^{k} 2^{2 i}+2^{2(k+1)}\right]+1 \\
& =2^{2}\left[(k+2) 2^{2(k+1)}+\sum_{i=0}^{k} 2^{2 i}\right]+1=\left[(k+2) 2^{2(k+2)}+\sum_{i=0}^{k} 2^{2(i+1)}\right]+1 \\
& =\left[(k+2) 2^{2(k+2)}+\sum_{i=1}^{k+1} 2^{2 i}\right]+2^{0}=\left[(k+2) 2^{2(k+2)}+\sum_{i=0}^{k+1} 2^{2 i}\right] .
\end{aligned}
$$

Hence the result is true for $n=k+1$.

## Conclusion

Thus in this paper we have generalized the concept of fuzzy group theory to intuitionistic fuzzy group theory. It is not the end but only a start to find distinct equivalence classes of intuitionistic fuzzy subgroups of a finite group of different order.

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