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Sugeno integral with respect to intuitionistic fuzzy-valued fuzzy measures

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Abstract We introduce the Sugeno integral of intuitionistic fuzzy-valued mappings with respect to intuitionistic fuzzy-valued fuzzy measures. A decomposition theorem to Sugeno integral of real-valued mappings with respect to fuzzy measures, some properties and examples of calculus are given.

1 Preliminaries

A fuzzy measure on a σ-algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ is a set function $m : \mathcal{A} \to [0,1]$ which satisfies $m(\emptyset) = 0, m(X) = 1$ and $A \subseteq B$ implies $m(A) \leq m(B)$. A fuzzy measure (or generally a set function) m defined on a σ-algebra $\mathcal{A}$ is called continuous from below if for every sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $A_n \subseteq A_{n+1}$, for every $n \in \mathbb{N}$, we have

$$m\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} m(A_n)$$

and continuous from above if for every sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $A_n \supseteq A_{n+1}$, for every $n \in \mathbb{N}$, we have

$$m\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} m(A_n).$$

The important set in intuitionistic fuzzy set theory (see [1], [2])

$$\mathcal{L} = \{(x_1, x_2); x_1, x_2 \in [0,1], x_1 + x_2 \leq 1\}$$

is a complete lattice if we consider

$$(x_1, x_2) \leq_{\mathcal{L}} (y_1, y_2) \text{ if and only if } x_1 \leq y_1 \text{ and } x_2 \geq y_2,$$

$$\begin{aligned}\sup\nolimits_{\mathcal{L}} A = (&\sup\{x \in [0,1] \mid \exists y \in [0,1] : (x,y) \in A\}, \\ &\inf\{y \in [0,1] \mid \exists x \in [0,1] : (x,y) \in A\}) \\ \inf\nolimits_{\mathcal{L}} A = (&\inf\{x \in [0,1] \mid \exists y \in [0,1] : (x,y) \in A\}, \\ &\sup\{y \in [0,1] \mid \exists x \in [0,1] : (x,y) \in A\}),\end{aligned}$$

for each $A \subseteq \mathcal{L}$. The standard negation on $\mathcal{L}$ is introduced by

$$(x_1, x_2)^* = (x_2, x_1).$$

It is obvious that $0_{\mathcal{L}} = (0,1)$ and $1_{\mathcal{L}} = (1,0)$.

The following concepts are introduced in the paper [3].

Definition 1.1 *An intuitionistic fuzzy-valued fuzzy measure over a measurable space* $(X, \mathcal{A})$ *is a map* $v : \mathcal{A} \longrightarrow \mathcal{L}$ *with the following properties:*

(i) $v(\emptyset) = (0,1)$;

(ii) $v(X) = (1,0)$;

(iii) $A \subseteq B$ *implies* $v(A) \leq_{\mathcal{L}} v(B)$.

Definition 1.2 *The intuitionistic fuzzy-valued fuzzy measure* $v : \mathcal{A} \longrightarrow \mathcal{L}, v(A) = (v_1(A), v_2(A))$ *is called:*

(i) continuous from below if v_1 *and* v_2 *are continuous from below;*

(ii) continuous from above if v_1 *and* v_2 *are continuous from above;*

(iii) continuous if it is continuous from below and continuous from above.

The Sugeno integral (in its original form) deals with functions having the range contained in $[0,1]$ and it is with respect to fuzzy measures on crisp sets. In several publications this integral is simply called a fuzzy integral.

Let $(X, \mathcal{A})$ be a measurable space and $m : \mathcal{A} \longrightarrow [0,1]$ be a fuzzy measure. For any function $f : X \longrightarrow [0,1]$ we write

$$F_\alpha = \{x \in X; f(x) \geq \alpha\},$$

where $\alpha \in [0,1]$.

Definition 1.3 *(see [5] or [6]) Let* $A \in \mathcal{A}$ *and* $f : X \longrightarrow [0,1]$ *be an* $\mathcal{A}$*-measurable function. The Sugeno integral of* f *on* A *with respect to* m*, which is denoted by* $(S)\int_A f dm$*, is defined by*

$$(S)\int_A f dm = \sup_{\alpha \in [0,1]} \min(\alpha, m(F_\alpha \cap A)).$$

The following theorem gives the most important properties of the Sugeno integral.

Theorem 1.4 *(see [6], pp. 135-136) Let* $m, m_1, m_2 : \mathcal{A} \longrightarrow [0,1]$ *be fuzzy measures,* $A, B \in \mathcal{A}$ *and* $f, f_1, f_2 : X \longrightarrow [0,1]$ *be* $\mathcal{A}$*-measurable functions.*

(i) If $m(A) = 0$ *then*

$$(S)\int_A f dm = 0;$$

(ii) If $f_1 \leq f_2$ *then*

$$(S)\int_A f_1 dm \leq (S)\int_A f_2 dm;$$

(iii)

$$(S)\int_A f dm = (S)\int_X f\chi_A dm,$$

where χ_A *is the characteristic function of* A;

(iv)

$$(S)\int_A a dm = \min(a, m(A)),$$

for any constant $a \in [0,1]$;

(v) If $A \subseteq B$ *then*

$$(S)\int_A f dm \leq (S)\int_B f dm;$$

(vi)

$$(S)\int_A \max(f_1, f_2)\, dm \geq \max\left((S)\int_A f_1 dm, (S)\int_A f_2 dm\right);$$

(vii)

$$(S)\int_A \min(f_1, f_2)\, dm \leq \min\left((S)\int_A f_1 dm, (S)\int_A f_2 dm\right);$$

(viii)

$$(S)\int_{A\cup B} f dm \geq \max\left((S)\int_A f dm, (S)\int_B f dm\right);$$

(ix)

$$(S)\int_{A\cap B} f dm \leq \min\left((S)\int_A f dm, (S)\int_B f dm\right);$$

(x) If $m_1 \leq m_2$ *then*

$$(S)\int_A f dm_1 \leq (S)\int_A f dm_2.$$

For any $\mathcal{A}$-measurable function $f : X \to [0,1]$ we define $f^c : X \to [0,1]$ by

$$f^c(x) = 1 - f(x),$$

for every $x \in X$. If m_* denotes the dual fuzzy measure of the fuzzy measure $m : \mathcal{A} \to [0,1]$, that is

$$m_*(A) = 1 - m(A^c),$$

for every $A \in \mathcal{A}$, then the following result is proved in [4].

Theorem 1.5 *If* m *is a continuous fuzzy measure then for every* $\mathcal{A}$*-measurable function* f *there holds*

$$(S)\int_X f dm = 1 - (S)\int_X f^c dm_*.$$

2 Measurability of intuitionistic fuzzy-valued functions

Taking into account the characterization of the measurability of real-valued functions, the following definition appears to be natural.

Definition 2.1 *Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a σ-algebra and $\widetilde{f} : X \to \mathcal{L}$. The function $\widetilde{f}$ is called $\mathcal{A}$-measurable if*

$$\left\{x \in X; \widetilde{f}(x) \geq_{\mathcal{L}} a\right\} \in \mathcal{A}$$

and

$$\left\{x \in X; a \geq_{\mathcal{L}} \widetilde{f}(x)\right\} \in \mathcal{A},$$

for every $a \in \mathcal{L}$.

The following result reduces the measurability of an intuitionistic fuzzy-valued function to the measurability of its components.

Theorem 2.2 *Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a σ-algebra and $\widetilde{f} : X \to \mathcal{L}$,*

$$\widetilde{f}(x) = (g(x), h(x)),$$

for every $x \in X$. The intuitionistic fuzzy-valued function $\widetilde{f}$ is $\mathcal{A}$-measurable if and only if $g, h : X \to [0,1]$ are $\mathcal{A}$-measurable.

Proof. If g is an $\mathcal{A}$-measurable function then

$$\{x \in X; g(x) \geq a_1\} \in \mathcal{A}$$

and

$$\{x \in X; g(x) \leq a_1\} \in \mathcal{A},$$

for every $a_1 \in [0,1]$. If h is an $\mathcal{A}$-measurable function then

$$\{x \in X; h(x) \geq a_2\} \in \mathcal{A}$$

and

$$\{x \in X; h(x) \leq a_2\} \in \mathcal{A},$$

for every $a_2 \in [0,1]$.

Let $a \in \mathcal{L}, a = (a_1, a_2)$. Because

$$\left\{x \in X; \widetilde{f}(x) \geq_{\mathcal{L}} a\right\}$$
$$= \{x \in X; g(x) \geq a_1\} \cap \{x \in X; h(x) \leq a_2\} \in \mathcal{A}$$

and

$$\left\{x \in X; a \geq_{\mathcal{L}} \widetilde{f}(x)\right\}$$
$$= \{x \in X; g(x) \leq a_1\} \cap \{x \in X; h(x) \geq a_2\} \in \mathcal{A}$$

we obtain $\widetilde{f}$ is $\mathcal{A}$-measurable.

Conversely, if $\widetilde{f}$ is $\mathcal{A}$-measurable then

$$\{x \in X; g(x) \geq a_1\} \cap \{x \in X; h(x) \leq a_2\} \in \mathcal{A},$$

for every $a_1, a_2 \in [0,1], a_1 + a_2 \leq 1$. If $a_1 = 0$ then $\alpha := a_2 \in [0,1]$ and

$$\{x \in X; g(x) \geq a_1\} = X.$$

We have

$$\{x \in X; h(x) \leq \alpha\} \in \mathcal{A},$$

for every $\alpha \in [0,1]$, therefore h is $\mathcal{A}$-measurable. If $\widetilde{f}$ is $\mathcal{A}$-measurable then

$$\{x \in X; g(x) \leq a_1\} \cap \{x \in X; h(x) \geq a_2\} \in \mathcal{A},$$

for every $a_1, a_2 \in [0,1], a_1 + a_2 \leq 1$. If $a_2 = 0$ then $\alpha := a_1 \in [0,1]$ and

$$\{x \in X; h(x) \geq a_2\} = X.$$

We have

$$\{x \in X; g(x) \leq \alpha\} \in \mathcal{A},$$

for every $\alpha \in [0,1]$, therefore g is $\mathcal{A}$-measurable. ∎

A consequence of the previous theorem is the easy transfer of the concepts and results from real-valued measurable functions to intuitionistic fuzzy-valued measurable functions.

3 Sugeno integral with respect to intuitionistic fuzzy-valued fuzzy measures

We introduce the Sugeno integral of an intuitionistic fuzzy-valued mapping, on a crisp set, with respect to an intuitionistic fuzzy-valued fuzzy measure as follows.

Definition 3.1 *Let $(X, \mathcal{A})$ be a measurable space, $v : \mathcal{A} \to \mathcal{L}$ be an intuitionistic fuzzy-valued fuzzy measure and $\widetilde{f} : X \to \mathcal{L}$ be an $\mathcal{A}$-measurable intuitionistic fuzzy-valued mapping. The Sugeno integral type of $\widetilde{f}$ on $A \in \mathcal{A}$ with respect to v, denoted by*

$$(S_I) \int_A \widetilde{f} dv,$$

is defined by

$$(S_I)\int_A \widetilde{f}dv = \underset{a\in\mathcal{L}}{sup_{\mathcal{L}}} inf_{\mathcal{L}}\left(a, v\left(\widetilde{F}_a \cap A\right)\right),$$

where

$$\widetilde{F}_a = \left\{x \in X; \widetilde{f}(x) \geq_{\mathcal{L}} a\right\},$$

for every $a \in \mathcal{L}$.

The idea in Definition 3.1 was used in [7] to introduce a lattice-valued fuzzy integral of Sugeno type for lattice-valued functions.

To simplify the calculus of the integral we prove the following theorem.

Theorem 3.2 *Let* $(X, \mathcal{A})$ *be a measurable space,* $v : \mathcal{A} \to \mathcal{L}$ *be an intuitionistic fuzzy-valued fuzzy measure,* $v = (v_1, v_2)$ *and* $\widetilde{f} : X \to \mathcal{L}$,

$$\widetilde{f}(x) = (g(x), h(x))$$

be an $\mathcal{A}$*-measurable intuitionistic fuzzy-valued mapping. Then*

$$(S_I)\int_X \widetilde{f}dv = \left(\sup_{a\in[0,1]} \min(a, v_1(G_a)), \inf_{a\in[0,1]} \max(a, v_2(H^a))\right),$$

where

$$G_a = \{x \in X; g(x) \geq a\}$$

and

$$H^a = \{x \in X; h(x) \leq a\}.$$

Proof. The definitions of the infimum and supremum in the lattice $\mathcal{L}$ imply

$$\begin{aligned}(S_I)\int_X \widetilde{f}dv &= \underset{a\in\mathcal{L}}{\sup_{\mathcal{L}}} \inf_{\mathcal{L}}\left(a, v\left(\widetilde{F}_a\right)\right)\\
&= \sup_{\mathcal{L}\,(a_1,a_2)\in\mathcal{L}} \inf_{\mathcal{L}}\left((a_1, a_2), \left(v_1\left(\widetilde{F}_{(a_1,a_2)}\right), v_2\left(\widetilde{F}_{(a_1,a_2)}\right)\right)\right)\\
&= \left(\sup_{\substack{a_1,a_2\in[0,1]\\ a_1+a_2\leq 1}} \min\left(a_1, v_1\left(\left\{x \in X; \widetilde{f}(x) \geq_{\mathcal{L}} (a_1, a_2)\right\}\right)\right),\right.\\
&\quad \left.\inf_{\substack{a_1,a_2\in[0,1]\\ a_1+a_2\leq 1}} \max\left(a_2, v_2\left(\left\{x \in X; \widetilde{f}(x) \geq_{\mathcal{L}} (a_1, a_2)\right\}\right)\right)\right).\end{aligned}$$

Because

$$\left\{x \in X; \widetilde{f}(x) \geq_{\mathcal{L}} a\right\} = \{x \in X; (g(x), h(x)) \geq_{\mathcal{L}} (a_1, a_2)\}$$
$$= \{x \in X; g(x) \geq a_1\} \cap \{x \in X; h(x) \leq a_2\},$$

for every $a = (a_1, a_2) \in \mathcal{L}$ and v_1 is non-decreasing we obtain

$$\begin{aligned}
&\sup_{\substack{a_1,a_2\in[0,1]\\a_1+a_2\leq 1}} \min\left(a_1, v_1\left(\left\{x \in X; \widetilde{f}(x) \geq_{\mathcal{L}} (a_1, a_2)\right\}\right)\right) \\
&= \sup_{\substack{a_1,a_2\in[0,1]\\a_1+a_2\leq 1}} \min\left(a_1, v_1\left(\{x \in X; g(x) \geq a_1\} \cap \{x \in X; h(x) \leq a_2\}\right)\right) \\
&\leq \sup_{\substack{a_1,a_2\in[0,1]\\a_1+a_2\leq 1}} \min\left(a_1, v_1\left(\{x \in X; g(x) \geq a_1\}\right)\right) \\
&= \sup_{a\in[0,1]} \min\left(a, v_1\left(\{x \in X; g(x) \geq a\}\right)\right).
\end{aligned}$$

Analogously, we get

$$\begin{aligned}
&\inf_{\substack{a_1,a_2\in[0,1]\\a_1+a_2\leq 1}} \max\left(a_2, v_2\left(\left\{x \in X; \widetilde{f}(x) \geq_{\mathcal{L}} (a_1, a_2)\right\}\right)\right) \\
&= \inf_{\substack{a_1,a_2\in[0,1]\\a_1+a_2\leq 1}} \max\left(a_2, v_2\left(\{x \in X; g(x) \geq a_1\} \cap \{x \in X; h(x) \leq a_2\}\right)\right) \\
&\geq \inf_{\substack{a_1,a_2\in[0,1]\\a_1+a_2\leq 1}} \max\left(a_2, v_2\left(\{x \in X; h(x) \leq a_2\}\right)\right) \\
&= \inf_{a\in[0,1]} \max\left(a, v_2\left(\{x \in X; h(x) \leq a\}\right)\right),
\end{aligned}$$

because v_2 is non-increasing. As a first conclusion,

$$(S_I)\int_X \widetilde{f}dv \leq_{\mathcal{L}} \left(\sup_{a\in[0,1]} \min\left(a, v_1\left(G_a\right)\right), \inf_{a\in[0,1]} \max\left(a, v_2\left(H^a\right)\right)\right).$$

To prove the converse inequality we need the following inclusion

$$\{x \in X; g(x) \geq a\} \subseteq \{x \in X; h(x) \leq 1 - a\},$$

for every $a \in [0, 1]$. Indeed, if the point $x_0 \in X$ verifies

$$g(x_0) \geq a$$

for a fixed $a \in [0, 1]$ then

$$h(x_0) \leq 1 - g(x_0) \leq 1 - a.$$

Then

$$\sup_{\substack{a_1,a_2\in[0,1]\\ a_1+a_2\leq 1}} \min\left(a_1, v_1\left(\left\{x\in X; \widetilde{f}(x) \geq_{\mathcal{L}} (a_1,a_2)\right\}\right)\right)$$

$$= \sup_{\substack{a_1,a_2\in[0,1]\\ a_1+a_2\leq 1}} \min\left(a_1, v_1\left(\{x\in X; g(x)\geq a_1\} \cap \{x\in X; h(x)\leq a_2\}\right)\right)$$

$$\geq \text{(taking } a_1 = a \text{ and } a_2 = 1-a\text{)}$$

$$\geq \sup_{a\in[0,1]} \min\left(a, v_1\left(\{x\in X; g(x)\geq a\} \cap \{x\in X; h(x)\leq 1-a\}\right)\right)$$

$$= \sup_{a\in[0,1]} \min\left(a, v_1\left(\{x\in X; g(x)\geq a\}\right)\right).$$

Finally,

$$\inf_{\substack{a_1,a_2\in[0,1]\\ a_1+a_2\leq 1}} \max\left(a_2, v_2\left(\left\{x\in X; \widetilde{f}(x) \geq_{\mathcal{L}} (a_1,a_2)\right\}\right)\right)$$

$$= \inf_{\substack{a_1,a_2\in[0,1]\\ a_1+a_2\leq 1}} \max\left(a_2, v_2\left(\{x\in X; g(x)\geq a_1\} \cap \{x\in X; h(x)\leq a_2\}\right)\right)$$

$$\leq \text{(taking } a_1 = 0 \text{ and } a_2 = a\text{)}$$

$$\leq \inf_{a\in[0,1]} \max\left(a, v_2\left(\{x\in X; g(x)\geq 0\} \cap \{x\in X; h(x)\leq a\}\right)\right)$$

$$= \inf_{a\in[0,1]} \max\left(a, v_2\left(\{x\in X; h(x)\leq a\}\right)\right),$$

therefore

$$(S_I)\int_X \widetilde{f}dv \geq_{\mathcal{L}} \left(\sup_{a\in[0,1]} \min\left(a, v_1(G_a)\right), \inf_{a\in[0,1]} \max\left(a, v_2(H^a)\right)\right)$$

and the proof is complete. ∎

The following corollary is an obvious consequence of Theorem 3.2.

Corollary 3.3 *In the hypothesis of Theorem 3.2,*

$$(S_I)\int_A \widetilde{f}dv = \left(\sup_{a\in[0,1]} \min\left(a, v_1(G_a\cap A)\right), \inf_{a\in[0,1]} \max\left(a, v_2(H^a\cap A)\right)\right),$$

for every $A \in \mathcal{A}$.

4 Properties and calculus of Sugeno integral with respect to intuitionistic fuzzy-valued fuzzy measures

The Sugeno integral of the intuitionistic fuzzy-valued function $\widetilde{f} = (g,h)$ can be expressed with the Sugeno integral of the components g and h as follows.

Theorem 4.1 *Let $(X, \mathcal{A})$ be a measurable space, $v : \mathcal{A} \longrightarrow \mathcal{L}, v = (v_1, v_2)$ be an intuitionistic fuzzy-valued fuzzy measure and $\widetilde{f} : X \longrightarrow \mathcal{L}, \widetilde{f} = (g, h)$ be an $\mathcal{A}$-measurable intuitionistic fuzzy-valued mapping. Then*

$$(S_I) \int_A \widetilde{f} dv = \left((S) \int_A g dv_1, \left((S) \int_A h^c dv_2^c \right)^c \right),$$

where $h^c : X \longrightarrow [0, 1]$ is defined by

$$h^c(x) = 1 - h(x),$$

$v_2^c : \mathcal{A} \longrightarrow [0, 1]$ is defined by

$$v_2^c(A) = 1 - v_2(A)$$

and, for symmetry, we denote $\alpha^c = 1 - \alpha$, for every $\alpha \in \mathbb{R}$.

Proof. Firstly, let us remark that v_2^c is a fuzzy measure on $\mathcal{A}$ and h^c is $\mathcal{A}$-measurable if h is $\mathcal{A}$-measurable. Below we use the notations in Theorem 3.2. The equality

$$\sup_{a \in [0,1]} \min(a, v_1(G_a \cap A)) = (S) \int_A g dv_1$$

is the definition of Sugeno integral (Definition 1.3). The same definition and the properties of inf and sup imply

$$\begin{aligned} \left((S) \int_A h^c dv_2^c \right)^c &= \left(\sup_{a \in [0,1]} \min(a, v_2^c(H_a^* \cap A)) \right)^c \\ &= \inf_{a \in [0,1]} \max(1 - a, v_2(H_a^* \cap A)) \\ &= \inf_{a \in [0,1]} \max\left(1 - a, v_2\left(H^{1-a} \cap A\right)\right) \\ &= \inf_{a \in [0,1]} \max(a, v_2(H^a \cap A)), \end{aligned}$$

where $H_a^* = \{x \in X; h^c(x) \geq a\}$, and the proof is complete. ∎

It is obvious that in the fuzzy case, that is

$$v_1(A) + v_2(A) = 1,$$

for every $A \in \mathcal{A}$ and

$$g(x) + h(x) = 1,$$

for every $x \in X$, the Sugeno integral of intuitionistic fuzzy-valued functions reduces to Sugeno integral.

The following theorem gives the most important properties of the introduced integral.

Theorem 4.2 *Let $v : \mathcal{A} \to \mathcal{L}$ be an intuitionistic fuzzy-valued fuzzy measure, $\widetilde{f}, \widetilde{f}_1, \widetilde{f}_2 : X \to \mathcal{L}$ be $\mathcal{A}$-measurable intuitionistic fuzzy-valued mappings and $A, B \in \mathcal{A}$, where $\mathcal{A}$ is a σ-algebra on X. Then*

(i) If $v(A) = (0,1)$ then

$$(S_I)\int_A \widetilde{f}dv = (0,1);$$

(ii) If $\widetilde{f}_1 \leq_{\mathcal{L}} \widetilde{f}_2$, that is $\widetilde{f}_1(x) \leq_{\mathcal{L}} \widetilde{f}_2(x)$, for all $x \in X$, then

$$(S_I)\int_A \widetilde{f}_1 dv \leq_{\mathcal{L}} (S_I)\int_A \widetilde{f}_2 dv;$$

(iii)

$$(S_I)\int_A \widetilde{f}dv = (S_I)\int_X \widetilde{f}\mathcal{T}_P\widetilde{\chi}_A dv,$$

where, if $\widetilde{f}_1 = (g_1, h_1), \widetilde{f}_2 = (g_2, h_2)$ then $\widetilde{f}_1\mathcal{T}_P\widetilde{f}_2 : X \to \mathcal{L}$ is defined by

$$\left(\widetilde{f}_1\mathcal{T}_P\widetilde{f}_2\right)(x) = \left(g_1(x)\,g_2(x), h_1(x) + h_2(x) - h_1(x)\,h_2(x)\right),$$

for every $x \in X$, and

$$\widetilde{\chi}_A(x) = \left(\chi_A(x), 1 - \chi_A(x)\right)$$

for every $x \in X$;

(iv)

$$(S_I)\int_A a dv = inf_{\mathcal{L}}(a, v(A)),$$

for any $a \in \mathcal{L}$;

(v) If $A \subseteq B$ then

$$(S_I)\int_A \widetilde{f}dv \leq_{\mathcal{L}} (S_I)\int_B \widetilde{f}dv;$$

(vi)

$$max_{\mathcal{L}}\left((S_I)\int_A \widetilde{f}_1 dv, (S_I)\int_A \widetilde{f}_2 dv\right) \leq_{\mathcal{L}} (S_I)\int_A max_{\mathcal{L}}\left(\widetilde{f}_1, \widetilde{f}_2\right)dv,$$

where

$$max_{\mathcal{L}}\left(\widetilde{f}_1, \widetilde{f}_2\right)(x) = \left(\max\left(g_1(x), g_2(x)\right), \min\left(h_1(x), h_2(x)\right)\right)$$

for $\widetilde{f}_1 = (g_1, h_1), \widetilde{f}_2 = (g_2, h_2)$;

(vii)

$$(S_I)\int_A min_{\mathcal{L}}\left(\widetilde{f}_1,\widetilde{f}_2\right)dv \leq_{\mathcal{L}} min_{\mathcal{L}}\left((S_I)\int_A \widetilde{f}_1 dv,(S_I)\int_A \widetilde{f}_2 dv\right),$$

where

$$min_{\mathcal{L}}\left(\widetilde{f}_1,\widetilde{f}_2\right)(x) = (\min(g_1(x),g_2(x)),\max(h_1(x),h_2(x)))$$

for $\widetilde{f}_1 = (g_1,h_1), \widetilde{f}_2 = (g_2,h_2)$;

(viii)

$$max_{\mathcal{L}}\left((S_I)\int_A \widetilde{f}dv,(S_I)\int_B \widetilde{f}dv\right) \leq_{\mathcal{L}} (S_I)\int_{A\cup B}\widetilde{f}dv;$$

(ix)

$$(S_I)\int_{A\cap B}\widetilde{f}dv \leq_{\mathcal{L}} min_{\mathcal{L}}\left((S_I)\int_A \widetilde{f}dv,(S_I)\int_B \widetilde{f}dv\right).$$

Proof. It is an immediate consequence of Theorem 4.1 and Theorem 1.4. As example, we prove *(iii)*.

If $\widetilde{f} = (g,h)$ then

$$\begin{aligned}
&(S_I)\int_X f\mathcal{T}_P\widetilde{\chi}_A dv\\
&= \left((S)\int_X g\chi_A dv_1,\left((S)\int_X (h+\chi_A^c - h\chi_A^c)^c dv_2^c\right)^c\right)\\
&= \left((S)\int_X g\chi_A dv_1,\left((S)\int_X h^c\chi_A dv_2^c\right)^c\right)\\
&= \left((S)\int_A g dv_1,\left((S)\int_A h^c dv_2^c\right)^c\right)\\
&= (S_I)\int_A \widetilde{f}dv
\end{aligned}$$

taking into account the property *(iii)* in Theorem 1.4. ■

Theorem 4.3 *Let $\widetilde{f}$ be an $\mathcal{A}$-measurable intuitionistic fuzzy-valued function and $v:\mathcal{A}\longrightarrow \mathcal{L}, v=(v_1,v_2)$ be a continuous intuitionistic fuzzy-valued fuzzy measure. Then*

$$\left((S_I)\int_X \widetilde{f}^* d\overline{v}\right)^* = (S_I)\int_X \widetilde{f}dv,$$

where $\overline{v}$ is the dual of v, that is $\overline{v}:\mathcal{A}\longrightarrow\mathcal{L}$ is given by

$$\overline{v}(A) = (v_2(A^c),v_1(A^c)),$$

for every $A\in\mathcal{A}$, $\widetilde{f}^* = (h,g)$ if $\widetilde{f}=(g,h)$ and $(x_1,x_2)^* = (x_2,x_1)$.

Proof. We denote $\overline{v} = (\overline{v}_1, \overline{v}_2)$ that is $\overline{v}_1(A) = v_2(A^c)$ and $\overline{v}_2(A) = v_1(A^c)$. As above, $v_2^c(A) = 1 - v_2(A), \overline{v}_2^c(A) = 1 - \overline{v}_2(A)$. The property of Sugeno integral given in Theorem 1.5 imply

$$(S)\int_X hd\overline{v}_1 = \left((S)\int_X h^c dv_2^c\right)^c$$

and

$$(S)\int_X gdv_1 = \left((S)\int_X g^c d\overline{v}_2^c\right)^c$$

therefore (Theorem 4.1 is used)

$$\begin{aligned}
&\left((S_I)\int_X \widetilde{f}^* d\overline{v}\right)^* \\
&= \left((S)\int_X hd\overline{v}_1, \left((S)\int_X g^c d\overline{v}_2^c\right)^c\right)^* \\
&= \left(\left((S)\int_X g^c d\overline{v}_2^c\right)^c, (S)\int_X hd\overline{v}_1\right) \\
&= \left((S)\int_X gdv_1, \left((S)\int_X h^c dv_2^c\right)^c\right) \\
&= (S_I)\int_X \widetilde{f}dv.
\end{aligned}$$

■

The previous theorem help us to solve the problem of duality of the Sugeno integral with respect to intuitionistic fuzzy-valued fuzzy measures.

Definition 4.4 *An intuitionistic fuzzy-valued fuzzy measure $v = (v_1, v_2)$ on σ-algebra $\mathcal{A}$ is called self-dual if $v = \overline{v}$, that is $v_1(A) = v_2(A^c)$ and $v_2(A) = v_1(A^c)$, for every $A \in \mathcal{A}$.*

With the above notations we obtain

Corollary 4.5 *If the continuous intuitionistic fuzzy-valued fuzzy measure v is self-dual then the Sugeno integral with respect to v has the property of duality, that is*

$$\left((S_I)\int_X \widetilde{f}dv\right)^* = (S_I)\int_X \widetilde{f}^* dv,$$

for every $\mathcal{A}$-measurable intuitionistic fuzzy-valued function $\widetilde{f}$.

Let us give two examples of calculus of Sugeno integral of an intuitionistic fuzzy-valued function with respect to an intuitonistic fuzzy-valued fuzzy measure.

Example 4.6 *Let $X = \{x, y, z\}, \mathcal{A} = \mathcal{P}(X)$ and $v_1, v_2 : \mathcal{A} \to [0, 1]$ defined by*

$$v_1(A) = \begin{cases} \frac{cardA}{3}, & \text{if } A \neq \{x, y\} \\ 1, & \text{if } A = \{x, y\}, \end{cases}$$

and

$$v_2(A) = \begin{cases} 1 - \frac{cardA}{3}, & \text{if } A \neq \{x,y\} \text{ and } A \neq \{y,z\} \\ 0, & \text{if } A = \{x,y\} \text{ or } A = \{y,z\}. \end{cases}$$

It is easy to prove $v : \mathcal{A} \to \mathcal{L}, v = (v_1, v_2)$ *is an intuitionistic fuzzy-valued fuzzy measure. Let* $\widetilde{f} : X \to \mathcal{L}, \widetilde{f} = (g,h)$*, where* $g, h : X \to [0,1]$ *are defined by*

$$g(x) = h(x) = \frac{1}{4},$$
$$g(y) = h(z) = \frac{1}{3},$$
$$g(z) = h(y) = \frac{2}{3}.$$

Then

$$G_a = \begin{cases} \{x,y,z\}, & \text{if } a \in \left[0, \frac{1}{4}\right] \\ \{y,z\}, & \text{if } a \in \left]\frac{1}{4}, \frac{1}{3}\right] \\ \{z\}, & \text{if } a \in \left]\frac{1}{3}, \frac{2}{3}\right] \\ \emptyset, & \text{if } a \in \left]\frac{2}{3}, 1\right] \end{cases}$$

and

$$H^a = \begin{cases} \{x,y,z\}, & \text{if } a \in \left[\frac{2}{3}, 1\right] \\ \{x,z\}, & \text{if } a \in \left[\frac{1}{3}, \frac{2}{3}\right[\\ \{x\}, & \text{if } a \in \left[\frac{1}{4}, \frac{1}{3}\right[\\ \emptyset, & \text{if } a \in \left[0, \frac{1}{4}\right[. \end{cases}$$

We obtain

$$\begin{aligned} &\sup_{a\in[0,1]} \min(a, v_1(G_a)) \\ &= \max\left(\min\left(\frac{1}{4}, v_1(\{x,y,z\})\right), \min\left(\frac{1}{3}, v_1(\{y,z\})\right),\right. \\ &\left.\min\left(\frac{2}{3}, v_1(\{z\})\right)\right) \\ &= \frac{1}{3} \end{aligned}$$

and

$$\begin{aligned} &\inf_{a\in[0,1]} \max(a, v_2(H^a)) \\ &= \min\left(\max\left(\frac{1}{3}, v_2(\{x\})\right), \max\left(\frac{2}{3}, v_2(\{x,z\})\right),\right. \\ &\left.\max(1, v_2(\{x,y,z\}))\right) \\ &= \frac{2}{3}, \end{aligned}$$

therefore

$$(S_I)\int_X \widetilde{f}dv = \left(\frac{1}{3},\frac{2}{3}\right).$$

Example 4.7 *Let* $X=[0,1]$, $\mathcal{A}$ *be the class of all Borel sets in* $X, v:\mathcal{A}\to\mathcal{L}, v=(v_1,v_2)$ *the intuitionistic fuzzy-valued fuzzy measure defined by*

$$v_1(A) = m^2(A),$$
$$v_2(A) = 1-m^2(A),$$

where m *is the Lebesgue measure and* $\widetilde{f}:X\to\mathcal{L}, \widetilde{f}(x)=(g(x),h(x))$, *defined by*

$$g(x)=\begin{cases}\frac{1}{3}, & \text{if } x\in\left[0,\frac{3}{4}\right]\\ 1, & \text{if } x\in\left]\frac{3}{4},1\right],\end{cases}$$

$$h(x)=\begin{cases}\frac{1}{4}, & \text{if } x\in\left[0,\frac{2}{3}\right]\\ 0, & \text{if } x\in\left]\frac{2}{3},1\right].\end{cases}$$

Then

$$G_a=\{x\in[0,1];g(x)\geq a\}=\begin{cases}[0,1], & \text{if } a\in\left[0,\frac{1}{3}\right]\\ \left]\frac{3}{4},1\right], & \text{if } a\in\left]\frac{1}{3},1\right]\end{cases}$$

and

$$H^a=\{x\in[0,1];h(x)\leq a\}=\begin{cases}[0,1], & \text{if } a\in\left[\frac{1}{4},1\right]\\ \left[\frac{2}{3},1\right], & \text{if } a\in\left[0,\frac{1}{4}\right[.\end{cases}$$

We have

$$\sup_{a\in[0,1]}\min(a,v_1(G_a))=\max\left(\frac{1}{3},\frac{1}{16}\right)=\frac{1}{3}$$

and

$$\inf_{a\in[0,1]}\max(a,v_2(H^a))=\min\left(\frac{1}{4},\frac{8}{9}\right)=\frac{1}{4}.$$

We obtain

$$(S_I)\int_X \widetilde{f}dv = \left(\frac{1}{3},\frac{1}{4}\right).$$

References

[1] Atanassov, K. T., Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20 (1986), 87-96.

[2] Atanassov, K. T., Intuitionistic Fuzzy Sets: Theory and Applications, Springer-Physica Verlag, Heidelberg, 1999.

[3] Ban, A. I., Intuitionistic fuzzy-valued fuzzy measures, Proceedings of IEEE Second International Conference on Intelligent Systems, Varna, June 2004, 427-429.

[4] Liu, Xuecheng, Study on the dual problems in the theory of fuzzy measure and fuzzy integral, Journal of Fuzzy Mathematics, 2 (1994), 555-563.

[5] Sugeno, M., Theory of fuzzy integrals and its applications, Doctoral Thesis, Tokyo Institute of Technology, 1974.

[6] Wang, Z. and Klir, G., Fuzzy Measure Theory, Plenum Press, 1992.

[7] Zhang, Guangquan and Meng, Xianli, Lattice-valued fuzzy integrals of lattice-valued functions with respect to lattice-valued functions with respect to lattice-valued fuzzy measure, Journal of Fuzzy Mathematics, 1 (1993), 53-68.