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On level operators for temporal intuitionistic fuzzy sets

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Abstract: In 1965, Fuzzy Set Theory was defined by Zadeh as an extension of crisp sets, [8]. K. T. Atanassov generalized fuzzy sets in to Intuitionistic Fuzzy Sets in 1983, [1]. He defined some operations and operators on intuitionistic fuzzy sets, like modal operators, level operators etc. In 1991, Atanassov introduced Temporal Intuitionistic Fuzzy Sets. Temporal intuitionistic fuzzy set is an extension of intuitionistic fuzzy set by "time-moments", [2]. After this extension author shows that all operations and operators on the intuitionistic fuzzy sets can be defined for the temporal intuitionistic fuzzy sets. In 2009, Parvathi and Geetha defined some level operators, max-min implication operators and $P_{\alpha,\beta}$, $Q_{\alpha,\beta}$ operators on temporal intuitionistic fuzzy sets, [7]. In this study we will introduce $N_B(A)$ and $N_B^*(A)$ operators on temporal intuitionistic fuzzy sets with extension to new universal and we will examine some properties of these operators.
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1 Introduction

The concept of fuzzy sets was introduced by Zadeh in [8] as an extension of crisp sets by expanding the truth value set to the real unit interval $[0, 1]$. Let X be a set. The function $\mu : X \to [0, 1]$ is called a fuzzy set over $X(FS(X))$. For $x \in X$, $\mu(x)$ is the membership degree of x and the non-membership degree is $1 - \mu(x)$.

Intuitionistic fuzzy sets have been introduced by Atanassov in [1] as an extension of fuzzy sets. If X is a universal then a intuitionistic fuzzy set A, the membership and non-membership degree for each $x \in X$ respectively, $\mu_A(x)(\mu_A : X \to [0, 1])$ and $\nu_A(x)(\nu_A{:}X \to [0, 1])$ such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. The class of intuitionistic fuzzy sets on X is denoted by $IFS(X)$. While the sum of membership degree and non-membership degree is 1 on FS, this sum is less than 1 on IFS. In 1991, [3], Atanassov introduced Temporal Intuitionistic Fuzzy Sets. Temporal intuitionistic fuzzy set is an extension of intuitionistic fuzzy set by "time-moments".

Definition 1. *Let L = [0, 1] then*

$$L^* = \{(x_1, x_2) \in [0, 1]^2 : x_1 + x_2 \leq 1\}$$

is a lattice with $(x_1, x_2) \leq (y_1, y_2) :\Longleftrightarrow$ "$x_1 \leq y_1$ *and* $x_2 \geq y_2$".

For $(x_1, y_1), (x_2, y_2) \in L^*$,the operators $\wedge$ and $\vee$ on $(L^*, \leq)$ are defined as follows:

$$(x_1, y_1) \wedge (x_2, y_2) = (\min(x_1, x_2), \max(y_1, y_2))$$

$$(x_1, y_1) \vee (x_2, y_2) = (\max(x_1, x_2), \min(y_1, y_2))$$

For each $J \subseteq L^*$

$$\sup J = (\sup\{x \in [0, 1] | (y \in [0, 1])((x, y) \in J)\}, \inf\{y \in [0, 1] | (x \in [0, 1])((x, y) \in J)\})$$

and

$$\inf J = (\inf\{x \in [0, 1] | (y \in [0, 1])((x, y) \in J)\}, \sup\{y \in [0, 1] | (x \in [0, 1])((x, y) \in J)\}).$$

Definition 2 ([1])**.** *An intuitionistic fuzzy set (shortly IFS) on a universe X is an object of the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$$

where $\mu_A(x)(\in [0, 1])$ is called the "degree of membership of x in A", $\nu_A(x)(\in [0, 1])$is called the " degree of non-membership of x in A", and where μ_A and ν_A satisfy the following condition: $(x \in X)\ \ (\mu_A(x) + \nu_A(x) \leq 1)$.

The class of IFSs on a universal X will be denoted $IFS(X)$.

Remark 1. *Sets* $\mathbf{0}$ *and* X *are defined at intuitionistic fuzzy set theory as follows:* $\mathbf{0} = \{\langle x, 0, 1\rangle | x \in X\}$ *and* $X = \{\langle x, 1, 0\rangle | x \in X\}$.

Definition 3 ([1])**.** *An IFS A is said to be contained in an IFS B (denoted by $A \sqsubseteq_X B$) if and only if,*

$$\mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x), \forall x \in X$$

The intersection and the union of two IFSs A and B on X is defined by

$$A \sqcap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle | x \in X\}$$

$$A \sqcup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle | x \in X\}$$

Definition 4 ([2])**.** *Let $A \in IFS$ and let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$ then the above set is called the complement of A*

$$CompA = \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in X\}.$$

Definition 5 ([3]). *Let X be a universal and T be non-empty set and elements of T called "time-moments". A temporal intuitionistic fuzzy set is an object of the form;*

$$A(T) = \{\langle x, \mu_A(x,t), \nu_A(x,t)\rangle \,|(x,t) \in X \times T\}$$

where

a. $A \subset X$ *is a fixed set.*

b. $\mu_A(x,t) + \nu_A(x,t) \leq 1$ *for every* $(x,t) \in X \times T$

c. $\mu_A(x,t)$ *and* $\nu_A(x,t)$ *are the degrees of membership and non-membership, respectively, of the element* $x \in X$ *at the time-moment* $t \in T$ *.*

The intersection, union and complement of temporal intuitionistic fuzzy set were defined in [7] as following,

Definition 6 ([7]). *Consider two TIFSs,*

$$A(T') = \{\langle (x,t), \mu_A(x,t), \nu_A(x,t)\rangle \,|(x,t) \in X \times T'\}$$

and

$$B(T'') = \{\langle (x,t), \mu_B(x,t), \nu_B(x,t)\rangle \,|(x,t) \in X \times T''\}.$$

Let us denote by $T^{\cup}$ *the union* $T' \cup T''$*. The basic operations namely intersection, union and complement are defined as follows:*

$$A(T') \sqcap B(T'') = \{\Big\langle (x,t), \min(\overline{\mu_A(x,t)}, \overline{\mu_B(x,t)}), \max(\overline{\nu_A(x,t)}, \overline{\nu_B(x,t)})\Big\rangle \,|(x,t) \in X \times T^{\cup}\}$$

$$A(T') \sqcup B(T'') = \{\Big\langle (x,t), \max(\overline{\mu_A(x,t)}, \overline{\mu_B(x,t)}), \min(\overline{\nu_A(x,t)}, \overline{\nu_B(x,t)})\Big\rangle \,|(x,t) \in X \times T^{\cup}\}$$

$$Comp(A(T')) = \{\langle (x,t), \nu_A(x,t), \mu_A(x,t)\rangle \,|(x,t) \in X \times T'\}$$

where

$$\overline{\mu_A(x,t)} = \begin{cases} \mu_A(x,t) & \textit{if } t \in T' \\ 0 & \textit{if } t \in T'' - T' \end{cases} \quad \overline{\mu_B(x,t)} = \begin{cases} \mu_B(x,t) & \textit{if } t \in T'' \\ 0 & \textit{if } t \in T' - T'' \end{cases}$$

$$\overline{\nu_A(x,t)} = \begin{cases} \nu_A(x,t) & \textit{if } t \in T' \\ 1 & \textit{if } t \in T'' - T' \end{cases} \quad \overline{\nu_B(x,t)} = \begin{cases} \nu_A(x,t) & \textit{if } t \in T' \\ 1 & \textit{if } t \in T'' - T' \end{cases}$$

We can express this definition for three situations as following;

1. If $T' \subset T''$ then,

$$(A(T') \sqcup B(T''))(x,t) = \begin{cases} \max(A(x,t), B(x,t)) & t \in T' \\ B(x,t) & t \in T' - T'' \end{cases}$$

and

$$(A(T') \sqcap B(T''))(x,t) = \begin{cases} \min(A(x,t), B(x,t)) & t \in T' \\ (0,1) & t \in T' - T'' \end{cases}$$

2. If $T' \cap T'' = \varnothing$ then,

$$(A(T') \sqcup B(T''))(x,t) = \begin{cases} A(x,t) & t \in T' \\ B(x,t) & t \in T'' \end{cases}$$

and

$$(A(T') \sqcap B(T''))(x,t) = \begin{cases} (0,1) & t \in T' \\ (0,1) & t \in T'' \end{cases}$$

3. If $T' \cap T'' \neq \varnothing, T' \nsubseteq T''(T'' \nsubseteq T')$ then,

$$(A(T') \sqcup B(T''))(x,t) = \begin{cases} B(x,t) & t \in T'' - T' \\ \max(A(x,t), B(x,t)) & T' \cap T'' \\ A(x,t) & t \in T' - T'' \end{cases}$$

and

$$(A(T') \sqcap B(T''))(x,t) = \begin{cases} (0,1) & t \in T'' - T' \\ \min(A(x,t), B(x,t)) & T' \cap T'' \\ (0,1) & t \in T' - T'' \end{cases}$$

If given intuitionistic fuzzy sets are defined in different universals then it is not possible to talk about membership and non-membership degree of the element not defined in related universal. Because, if $a \in X$ then its membership degree can be 0. If $a \notin X$ then we can not say anything about membership degree of element a. So, for intuitionisitc fuzzy sets which have different universals, operations of sets undefinable. In that case, we should expand the universal set. For this situation, the image of an intuitionistic fuzzy set is important. If the membership and non-membership degree of any element is not equal to $(0,1)$ then to determine the membership and non-membership degree as $(0,1)$ for such type elements is unsuitable. So if we use the $\inf$ or $\sup$ for these elements, it will be more suitable. After this discussion, for these intuitionistic fuzzy sets we can define the following extensions.

Definition 7 ([6]). *Let $A \in IFS(X)$ and $B \in IFS(Y)$. If $X \subset Y$ then,*

1. $\alpha_B A(x) = \begin{cases} \inf B, & x \in Y - X \\ A(x), & x \in X \end{cases}$

2. $\alpha^B A(x) = \begin{cases} \sup B, & x \in Y - X \\ A(x), & x \in X \end{cases}$

3. $\gamma_B A(x) = \begin{cases} \inf A, & x \in Y - X \\ B(x), & x \in X \end{cases}$

4. $\gamma^B A(x) = \begin{cases} \sup A, & x \in Y - X \\ B(x), & x \in X \end{cases}$

Definition 8 ([6]). *Let, $A \in IFS(X)$ and $B \in IFS(Y)$, if $X \cap Y = \varnothing$ then,*

1. $\theta_B\, A(x) = \begin{cases} A(x), & x \in X \\ \inf B, & x \in Y \end{cases}$

2. $\theta^B A(x) = \begin{cases} A(x), & x \in X \\ \sup B, & x \in Y \end{cases}$

3. $\delta_B\, A(x) = \begin{cases} \inf A, & x \in X \\ B(x), & x \in Y \end{cases}$

4. $\delta^B A(x) = \begin{cases} \sup A, & x \in X \\ B(x), & x \in Y \end{cases}$

Definition 9 ([6]). *Let $A \in IFS(X)$ and $B \in IFS(Y)$. If $X \cap Y \neq \varnothing$, $X \nsubseteq Y$ and $Y \nsubseteq X$ then*

1. $\rho_B\, A(x) = \begin{cases} B(x), & x \in Y - X \\ \inf B, & x \in X \cap Y \\ A(x), & x \in X - Y \end{cases}$

2. $\rho^B\, A(x) = \begin{cases} B(x), & x \in Y - X \\ \sup B, & x \in X \cap Y \\ A(x), & x \in X - Y \end{cases}$

3. $\sigma_B\, A(x) = \begin{cases} B(x), & x \in Y - X \\ \inf A, & x \in X \cap Y \\ A(x), & x \in X - Y \end{cases}$

4. $\sigma^B\, A(x) = \begin{cases} B(x), & x \in Y - X \\ \sup A, & x \in X \cap Y \\ A(x), & x \in X - Y \end{cases}$

If "$X \subset Y$", "$X \cap Y = \varnothing$" or "$X \cap Y \neq \varnothing$, $X \nsubseteq Y$ and $Y \nsubseteq X$" and $A \in IFS(X)$ then with these definitions we obtain that $A \in IFS(Y)$.

So, we can see that if $T' \subset T''$ and then $A(T') \sqcup B(T'')$ and $(\alpha_{B(T'')} A(T')) \sqcup B(T'')$ coincide, if $T' \cap T'' = \varnothing$ then $A(T') \sqcup B(T'')$ and $(\theta_{B(T'')} A(T')) \sqcup (\theta_{A(T')} B(T''))$ coincide.

Example 1. *Let X be a universal, $T_1 = \{t | t = 6k, k = 0, 1, ..., 100\}$,*

$T_2 = \{t |\, t = 2k, k = 0, 1, ..., 100\}$. Since $T_1 \subset T_2$,

$$A(T_1)(x) = \begin{cases} (\frac{1}{2}, \frac{1}{4}) & t = 4k, k \in \mathbb{N} \\ (\frac{1}{9}, \frac{3}{5}) & t = 4k + 2, k \in \mathbb{N} \end{cases},$$

$$B(T_2)(x) = \begin{cases} (\frac{2}{3}, \frac{1}{6}) & t = 3k, k \in \mathbb{N} \\ (\frac{3}{8}, \frac{1}{5}) & t = 3k + 1, k \in \mathbb{N} \\ (\frac{7}{11}, \frac{2}{9}) & t = 3k + 2, k \in \mathbb{N} \end{cases}$$

$$\alpha^{B(T_2)}A(x) = \begin{cases} (\frac{1}{2}, \frac{1}{4}) & t = 12k, k \in \mathbb{N} \\ (\frac{1}{9}, \frac{3}{5}) & t = 12k+6, k \in \mathbb{N} \\ (\frac{2}{3}, \frac{1}{6}) & t = 3k+1, k \in \mathbb{N} \\ (\frac{2}{3}, \frac{1}{6}) & t = 3k+2, k \in \mathbb{N} \end{cases}$$

and

$$(\alpha^{B(T_2)}A(T_1) \sqcup B(T_2))(x) = \begin{cases} (\frac{1}{2}, \frac{1}{4}) & t = 12k, k \in \mathbb{N} \\ (\frac{2}{3}, \frac{1}{6}) & t = 12k+6, k \in \mathbb{N} \\ (\frac{2}{3}, \frac{1}{6}) & t = 3k+1, k \in \mathbb{N} \\ (\frac{2}{3}, \frac{1}{6}) & t = 3k+2, k \in \mathbb{N} \end{cases}$$

Example 2. *Let X be a universal,* $T_1 = \{t|\ t = 2k, k = 0, 1, ..., 100\}$,

$T_2 = \{t|\ t = 2k+1, k = 0, 1, ..., 100\}$. Since $T_1 \cap T_2 = \varnothing$,

$$A(T_1)(x) = \begin{cases} (\frac{1}{4}, \frac{1}{2}) & t = 3k, k \in \mathbb{N} \\ (\frac{1}{5}, \frac{1}{3}) & t = 3k+1, k \in \mathbb{N} \\ (\frac{1}{2}, \frac{1}{4}) & t = 3k+2, k \in \mathbb{N} \end{cases},$$

$$B(T_2)(x) = \begin{cases} (\frac{1}{9}, \frac{3}{7}) & t = 3k, k \in \mathbb{N} \\ (\frac{2}{5}, \frac{4}{7}) & t = 3k+1, k \in \mathbb{N} \\ (\frac{2}{11}, \frac{7}{9}) & t = 3k+2, k \in \mathbb{N} \end{cases}$$

$$\theta_{B(T_2)}A(T_1)(x) = \begin{cases} (\frac{1}{4}, \frac{1}{2}) & t = 3k, k \in \mathbb{N} \text{ and } t \in T_1 \\ (\frac{1}{5}, \frac{1}{3}) & t = 3k+1, k \in \mathbb{N} \text{ and } t \in T_1 \\ (\frac{1}{2}, \frac{1}{4}) & t = 3k+2, k \in \mathbb{N} \text{ and } t \in T_1 \\ (\frac{1}{9}, \frac{7}{9}) & t = 2k+1, k \in \mathbb{N} \end{cases},$$

$$\theta_{A(T_1)}B(T_2)(x) = \begin{cases} (\frac{1}{9}, \frac{3}{7}) & t = 3k, k \in \mathbb{N} \text{ and } t \in T_2 \\ (\frac{2}{5}, \frac{4}{7}) & t = 3k+1, k \in \mathbb{N} \text{ and } t \in T_2 \\ (\frac{2}{11}, \frac{7}{9}) & t = 3k+2, k \in \mathbb{N} \text{ and } t \in T_2 \\ (\frac{1}{5}, \frac{1}{2}) & t = 2k, k \in \mathbb{N} \end{cases}$$

and

$$(\theta_{B(T_2)}A(T_1)) \sqcup (\theta_{A(T_1)}B(T_2))(x) = \begin{cases} (\frac{1}{4}, \frac{1}{2}) & t = 3k, k \in \mathbb{N} \text{ and } t \in T_1 \\ (\frac{1}{5}, \frac{1}{3}) & t = 3k+1, k \in \mathbb{N} \text{ and } t \in T_1 \\ (\frac{1}{2}, \frac{1}{4}) & t = 3k+2, k \in \mathbb{N} \text{ and } t \in T_1 \\ (\frac{1}{9}, \frac{3}{7}) & t = 3k, k \in \mathbb{N} \text{ and } t \in T_2 \\ (\frac{2}{5}, \frac{4}{7}) & t = 3k+1, k \in \mathbb{N} \text{ and } t \in T_2 \\ (\frac{2}{11}, \frac{7}{9}) & t = 3k+2, k \in \mathbb{N} \text{ and } t \in T_2 \end{cases}$$

Example 3. *Let X be a universal,* $T_1 = \{t|t = 3k, k = 0, 1, ..., 100\}$

$T_2 = \{t|\ t = 2k+1, k = 0, 1, ..., 100\}$.
Since $T_1 \cap T_2 \neq \varnothing, T_1 \nsubseteq T_2$ and $T_2 \nsubseteq T_1$,

$$A(T_1)(x) = \begin{cases} (\frac{2}{11}, \frac{4}{9}) & t = 6k, k \in \mathbb{N} \\ (\frac{6}{11}, \frac{2}{9}) & t = 6k+3, k \in \mathbb{N} \end{cases},$$

$$B(T_2)(x) = \begin{cases} (\frac{3}{5}, \frac{2}{7}) & t = 6k+1, k \in \mathbb{N} \\ (\frac{4}{11}, \frac{4}{7}) & t = 6k+3, k \in \mathbb{N} \\ (\frac{1}{4}, \frac{3}{8}) & t = 6k+5, k \in \mathbb{N} \end{cases}$$

$$\rho_{B(T_2)}A(T_1)(x) = \begin{cases} (\frac{2}{11}, \frac{4}{9}) & t = 6k, k \in \mathbb{N} \\ (\frac{3}{5}, \frac{2}{7}) & t = 6k+1, k \in \mathbb{N} \\ (\frac{1}{4}, \frac{4}{7}) & t = 6k+3, k \in \mathbb{N} \\ (\frac{1}{4}, \frac{3}{8}) & t = 6k+5, k \in \mathbb{N} \end{cases},$$

$$\rho_{A(T_1)}B(T_2)(x) = \begin{cases} (\frac{2}{11}, \frac{4}{9}) & t = 6k, k \in \mathbb{N} \\ (\frac{3}{5}, \frac{2}{7}) & t = 6k+1, k \in \mathbb{N} \\ (\frac{2}{11}, \frac{4}{9}) & t = 6k+3, k \in \mathbb{N} \\ (\frac{1}{4}, \frac{3}{8}) & t = 6k+5, k \in \mathbb{N} \end{cases}$$

and

$$(\rho_{B(T_2)}A(T_1)) \sqcup \rho_{A(T_1)}B(T_2))(x) = \begin{cases} (\frac{2}{11}, \frac{4}{9}) & t = 6k, k \in \mathbb{N} \\ (\frac{3}{5}, \frac{2}{7}) & t = 6k+1, k \in \mathbb{N} \\ (\frac{1}{4}, \frac{4}{9}) & t = 6k+3, k \in \mathbb{N} \\ (\frac{1}{4}, \frac{3}{8}) & t = 6k+5, k \in \mathbb{N} \end{cases}$$

Now, we will examine $N_B(A)$ and $N_B^*(A)$ level operators on temporal intuitionistic fuzzy sets.The definition of these level operators on intuitionistic fuzzy sets is following;

Definition 10 ([5]). *Let X be a universal and $A, B \in IFS(X)$. Then;*

1. $N_B(A) = \{< x, \mu_A(x), \nu_A(x) > | \mu_A(x) \geq \mu_B(x) \; \& \; \nu_A(x) \leq \nu_B(x), x \in X\}$
2. $N_B^*(A) = \{< x, \mu_A(x), \nu_A(x) > | \mu_A(x) \leq \mu_B(x) \; \& \; \nu_A(x) \geq \nu_B(x), x \in X\}$

Definition 11. *Let two TIFSs $A(T')$,$B(T'')$.Then;*

1. $N_B(A) = \left\{ \begin{array}{c} ((x), \mu_A(x,t), v_A(x,t)) \mid \mu_A(x,t) \geq \mu_B(x,t) \; \& \\ \nu_A(x,t) \leq \nu_B(x,t), (x,t) \in X \times (T' \cup T'') \end{array} \right\}$
2. $N_B^*(A) = \left\{ \begin{array}{c} ((x), \mu_A(x,t), v_A(x,t)) \mid \mu_A(x,t) \leq \mu_B(x,t) \; \& \\ \nu_A(x,t) \geq \nu_B(x,t), (x,t) \in X \times (T' \cup T'') \end{array} \right\}$

It is clear from definition that in order to determine $N_B(A)$ and $N_B^*(A)$ level operators, the A, B intuitionistic fuzzy sets must have same universals. But it is seen from the following theorem that we can determine level operators for intuitionistic fuzzy sets that have different universals,too.

Theorem 1. *Let two TIFSs $A(T')$,$B(T'')$ and $T' \subset T''$. If we extend universals as*

$$C = \alpha_{B(T'')}(A(T'))$$

and

$$C' = \alpha^{B(T'')}(A(T'))$$

then

1. $\alpha_{N_{B}(C')}(N_B(C)) \sqsubseteq N_B(C')$

2. $N_{B}(C') \sqsubseteq \alpha^{N_B(C')}(N_{B}(C))$

Proof. (1) For $C = \alpha_{B(T'')}(A(T'))$ and $C' = \alpha^{B(T'')}(A(T'))$ we can say that there are new universals $Y \times T^*$ and $Y' \times T^{**}$ encompassed by the $X \times T''$.

$N_B(C) \in IFS(Y \times T^*)$ and $N_B(C') \in IFS(Y' \times T^{**}), Y \times T^* \subset Y' \times T^{**}$.

If $(x,t) \in Y \times T^*$ then $\alpha_{N_{B}(C')}(N_B(C))(x,t) = N_B(C)(x,t)$.

So, if $(x,t) \in X \times T'$ then $\alpha_{N_{B}(C')}(N_B(C))(x,t) = A(x,t) = N_B(C')(x,t)$. If $(x,t) \in X \times (T'' - T')$ then $\alpha_{N_{B}(C')}(N_B(C))(x,t) = \inf B \leq \sup B = N_B(C')(x,t)$.

On the other hand, if $(x,t) \in (Y' - Y) \times (T^{**} - T^*)$ then

$$\alpha_{N_{B}(C')}(N_B(C))(x,t) = \inf N_{B}(C')$$

and

$$\inf N_{B}(C') \leq N_{B}(C')(x,t)$$

for all (x,t). So, $\alpha_{N_{B}(C')}(N_B(C)) \leq N_{B}(C')$ □

Theorem 2. *Let two TIFSs $A(T')$,$B(T'')$ and $T' \subset T''$. If we extend universals as*

$$C = \alpha_{B(T'')}(A(T'))$$

and

$$C' = \alpha^{B(T'')}(A(T'))$$

then

1. $\alpha_{N^*_B(C)}(N^*_B(C')) \sqsubseteq N^*_B(C)$

2. $N^*_B(C) \sqsubseteq \alpha^{N^*_B(C)}(N^*_B(C'))$

Proof. (2) For $C = \alpha_{B(T'')}(A(T'))$ and $C' = \alpha^{B(T'')}(A(T'))$ we can say that there are new universals $Y_1 \times T_1$ and $Y_2 \times T_2$ encompassed by the $X \times T''$.

$N^*_B(C) \in IFS(Y_1 \times T_1)$ and $N_B(C') \in IFS(Y_2 \times T_2), Y_2 \times T_2 \subset Y_1 \times T_1$.

If $(x,t) \in Y_2 \times T_2$ then $\alpha^{N^*_B(C)}(N^*_B(C'))(x,t) = N_B(C')(x,t)$.

So, if $(x,t) \in X \times T'$ then $\alpha^{N^*_B(C)}(N^*_B(C'))(x,t) = A(x,t) = N^*_B(C)(x,t)$. If $(x,t) \in X \times (T'' - T')$ then $\alpha^{N^*_B(C)}(N^*_B(C'))(x,t) = \sup B = B(x,t) \geq N^*_B(C)(x,t)$.

On the other hand, if $(x,t) \in (Y_1 - Y_2) \times (T_1 - T_2)$ then

$$\alpha^{N^*_B(C)}(N^*_B(C'))(x,t) = \sup N^*_B(C)$$

and

$$\sup N^*_B(C) \geq N^*_B(C)(x,t)$$

for all (x,t). So, $N^*_B(C) \sqsubseteq \alpha^{N^*_B(C)}(N^*_B(C'))$ □

Theorem 3. *Let two TIFSs $A(T')$,$B(T'')$ and $T' \cap T'' = \varnothing$. If we extend universals as $C = \theta^{B(T'')}(A(T'))$, $C' = \theta^{A(T')}(B(T''))$, $D = \delta_{B(T'')}(A(T'))$ and $D' = \delta_{A(T')}(B(T''))$ then*

1. $\sigma^{N_C(C')}(N_D(D')) \sqsubseteq \rho^{N_C(C')}(N_D(D'))$

2. $\sigma_{N_C(C')}(N_D(D')) \sqsubseteq \rho_{N_C(C')}(N_D(D'))$

Proof. (1) $N_D(D') \in IFS(Y \times T^*)$ and $N_C(C') \in IFS(Y' \times T^{**})$.

Since $X \times T' \subset Y \times T^*$ and $X \times T' \subset Y' \times T^{**}$ it is clear that $(Y \times T^*) \cap (Y' \times T^{**}) \neq \varnothing$.
If $(x,t) \in (Y' - Y) \times (T^{**} - T^*)$ then

$$\rho^{N_C(C')}(N_D(D'))(x,t) = N_C(C')(x,t) = \sigma^{N_C(C')}(N_D(D'))(x,t).$$

If $(x,t) \in (Y - Y') \times (T^* - T^{**})$ then

$$\rho^{N_C(C')}(N_D(D'))(x,t) = N_D(D')(x,t) = \sigma^{N_C(C')}(N_D(D'))(x,t).$$

Finally, $(x,t) \in (Y \cap Y') \times (T^* \cap T^{**})$ implies

$$\rho^{N_C(C')}(N_D(D'))(x,t) = \sup N_C(C') \geq \sup N_D(D') = \sigma^{N_C(C')}(N_D(D'))(x,t).$$

Therefore, $\sigma^{N_C(C')}(N_D(D')) \sqsubseteq \rho^{N_C(C')}(N_D(D'))$. □

Theorem 4. *Let two TIFSs $A(T')$,$B(T'')$ and $T' \cap T'' = \varnothing$. If we extend universals as $C = \theta^{B(T'')}(A(T'))$, $C' = \theta^{A(T')}(B(T''))$, $D = \delta_{B(T'')}(A(T'))$ and $D' = \delta_{A(T')}(B(T''))$ then*

1. $\sigma^{N^*_C(C')}(N^*_D(D')) \sqsubseteq \rho^{N^*_C(C')}(N^*_D(D'))$

2. $\sigma_{N^*_C(C')}(N^*_D(D')) \sqsubseteq \rho_{N^*_C(C')}(N^*_D(D'))$

Proof. (2) $N^*_D(D') \in IFS(Y_1 \times T_1)$ and $N^*_C(C') \in IFS(Y_2 \times T_2)$.

Since $X \times T'' \subset Y_1 \times T_1$ and $X \times T'' \subset Y_2 \times T_2$ it is clear that $(Y_1 \times T_1) \cap (Y_2 \times T_2) \neq \varnothing$.
If $(x,t) \in (Y_1 - Y_2) \times (T_1 - T_2)$ then

$$\sigma_{N^*_C(C')}(N^*_D D')(x,t) = N^*_D(D')(x,t) = \rho_{N^*_C(C')}(N^*_D D')(x,t).$$

If $(x,t) \in (Y_2 - Y_1) \times (T_2 - T_1)$ then

$$\sigma_{N^*_C(C')}(N^*_D D')(x,t) = N^*_C(C')(x,t) = \rho_{N^*_C(C')}(N^*_D D')(x,t).$$

$(x,t) \in (Y_2 \cap Y_1) \times (T_2 \cap T_1) \Rightarrow \sigma_{N^*_C(C')}(N^*_D D')(x,t) = \inf N^*_D(D') \leq \inf N^*_C(C')$.
So, $\sigma_{N^*_C(C')}(N^*_D(D')) \sqsubseteq \rho_{N^*_C(C')}(N^*_D(D'))$. □

Theorem 5. *Let two TIFSs $A(T')$,$B(T'')$ and $T' \cap T'' \neq \varnothing, T' \nsubseteq T''(T'' \nsubseteq T')$.If we extend universals as $C = \rho_{B(T'')}(A(T')), C' = \rho_{A(T')}(B(T'')), D = \sigma^{B(T'')}(A(T'))$ and $D\prime = \sigma^{A(T')}(B(T''))$ then*

1. $\rho_{N_C(C')}(N_D(D')) \sqsubseteq \sigma_{N_C(C')}(N_D(D'))$

2. $\rho_{N^*_C(C')}(N^*_D(D')) \sqsubseteq \sigma_{N^*_C(C')}(N^*_D(D'))$

Proof. It is clear. □

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