Notes on Intuitionistic Fuzzy Sets
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An in-depth exploration of intuitionistic fuzzy T_0 in the context of bitopology

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Abstract: Intuitionistic fuzzy topological space and bitopological space have been introduced by using the concepts of intuitionistic fuzzy sets which are the generalizations of interval valued fuzzy sets. This paper commences by presenting the notion of intuitionistic fuzzy T_0 in the context of bitopological spaces ($IFB - T_0$). Subsequently, we explore various connections and relationships between these concepts. Then, we find out the relation between intuitionistic T_0 and IFB-T_0 spaces. Further, we investigated continuity between two IFB spaces.

Keywords: Intuitionistic fuzzy sets, Bitopological spaces, T_0 spaces, Fuzzy topological spaces, IFB-T_0 spaces, Continuity, Separation axioms, Quasi-coincidence.
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1 Introduction

In the year of 1963, J.C. Kelly [18] introduced the fundamental concept of bitopological spaces by using two spaces. Just two years later, in the year 1965, L.A. Zadeh [27] brought forth the groundbreaking concept of fuzzy sets, which inaugurated a new era of mathematical abstraction, enabling the representation of uncertainty and imprecision. Building upon Zadeh's pioneering work, just few years later, Chang [12] and Lowen [19] embarked on a journey to develop the theory of fuzzy topological space by using the concept fuzzy sets of L. A. Zadeh [27]. The next breakthrough came in the year 1983 when Krassimir T. Atanassov [9] introduced intuitionistic fuzzy sets, providing a more nuanced framework for handling uncertainty by introducing level of membership and non-membership. Further enriching this landscape, Kandil with El-Shafee [17] presented the idea of fuzzy pairwise T_0 space in the context of bitopological spaces. Abu Sufiya et al. [1] and Nouh [24] extended these ideas by introducing fuzzy pairwise T_0 separation axioms, adding an essential layer of structure to the theory. In the late 20th century, Çoker [13] introduced the concept of intuitionistic sets and intuitionistic points. Following this, in a subsequent work, Çoker [14] delved into the exploration of fuzzy subspaces within the realm of intuitionistic fuzzy topological spaces. Later, Çoker [15, 16], and he, together with his colleague, Bayhan [10, 11] harnessed the power of IF sets to introduce IFTS, this research integrates two distinct methodologies for representing uncertainty, namely probabilistic and fuzzy logic approaches, thereby enhancing the applicability of our analysis. Few years ago, Estiaq Ahmed et al. [2] introduced intuitionistic fuzzy T_0 by using the context of intuitionistic sets having level of membership and membership. In recent years, S.S. Miah, M.R. Amin and M. Jahan [21] work on FT_0 TS and they observed the mappings of fuzzy T_0 in the context of quasi-coincidence. Further, S.S. Miah, M.R. Amin, with M. Shahjalal [22] continued their efforts to explore and identify properties of separation axioms [4, 5, 7], within the context of topology [3, 8] and bitopology [6, 20, 23] using a quasi-coincident approach in different aspects of fuzziness. Building upon their earlier work, the authors of this paper explored the property of IFT_0 within the framework of bitopology by using two topologies.

2 Preliminary results and basic notions

In order to derive the principal findings, it is essential to present several definitions and concepts.

Definition 2.1 ([13]). *An intuitionistic set A is an object having the form $A = (x, A_1, A_2)$ where A_1 and A_2 are subsets of X satisfying $A_1 \cap A_2 = \varnothing$. The set A_1 is called the set of members, while A_2 is called the set of non-members of A. Throughout this paper, we use the simpler notation $M = (\mu_M, \nu_M)$ for an intuitionistic set.*

Remark 2.1. *Every subset A on a nonempty set X may obviously be regarded as an intuitionistic set having the form $A' = (A, A^C)$, where $A^C = X \setminus A$ is the complement of A in X.*

Definition 2.2 ([9])**.** *Let the intuitionistic sets A and B on X be of the forms $A = (A_1, A_2)$ and $B = (B_1, B_2)$, respectively. Furthermore, let $\{A_j : j \in J\}$ be an arbitrary family of intuitionistic sets in X, where $A_j = (A_j^{(1)}, A_j^{(2)})$. Then,*

(a) $A \subseteq B$ *if and only if* $A_1 \subseteq B_1$ *and* $A_2 \supseteq B_2$.

(b) $A = B$ *if and only if* $A \subseteq B$ *and* $B \subseteq A$.

(c) $\overline{A} = (A_2, A_1)$ *denotes the complement of* A.

(d) $\bigcap A_j = \left(\bigcap A_j^{(1)}, \bigcup A_j^{(2)}\right)$.

(e) $\bigcup A_j = \left(\bigcup A_j^{(1)}, \bigcap A_j^{(2)}\right)$.

(f) $\phi_\sim = (\phi, X)$ *and* $X_\sim = (X, \phi)$.

Definition 2.3 ([16])**.** *An intuitionistic topology on a set X is a family τ of intuitionistic sets in X satisfying the following axioms:*

(1) $\phi_\sim, X_\sim \in \tau$.

(2) $G_1 \cap G_2 \in \tau$ *for any* $G_1, G_2 \in \tau$.

(3) $\bigcup G_i \in \tau$ *for any arbitrary family* $G_i \in \tau$.

In this case, the pair (X, τ) is called an intuitionistic topological space, and any intuitionistic set in τ is known as an intuitionistic open set in X.

Definition 2.4 ([9])**.** *Let X be a nonempty set and I be the unit interval $[0, 1]$. An intuitionistic fuzzy set A (IFS in short) in X is an object having the form $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$, where $\mu_A : X \longrightarrow I$ and $\nu_A : X \longrightarrow I$ denote the degree of membership and the degree of non-membership, respectively, and $\mu_A(x)+\nu_A(x) \leq 1$. Let $I(X)$ denote the set of all intuitionistic fuzzy sets in X. Obviously, every fuzzy set μ_A in X is an intuitionistic fuzzy set of the form $(\mu_A, 1 - \mu_A)$. Throughout this paper, we use the simpler notation $M = (\mu_M(x), \nu_M(x))$ instead of $M = \{(x, \mu_M(x), \nu_M(x)) \mid x \in X\}$.*

Definition 2.5 ([9])**.** *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy sets in X. Then,*

(1) $A \subseteq B$ *if and only if* $\mu_A \leq \mu_B$ *and* $\nu_A \geq \nu_B$.

(2) $A = B$ *if and only if* $A \subseteq B$ *and* $B \subseteq A$.

(3) $A^C = (\nu_A, \mu_A)$.

(4) $A \cap B = (\mu_A \cap \mu_B, \nu_A \cup \nu_B)$.

(5) $A \cup B = (\mu_A \cup \mu_B, \nu_A \cup \nu_B)$.

(6) $0_\sim = (0^\sim, 1^\sim)$ *and* $1_\sim = (1^\sim, 0^\sim)$.

Definition 2.6 ([15]). *An intuitionistic fuzzy topology (IFT in short) on X is a family t of IFS in X which satisfies the following properties:*

(1) $0_\sim, 1_\sim \in t$.

(2) If $A_1, A_2 \in t$, then $A_1 \cap A_2 \in t$.

(3) If $A_i \in t$ for each i, then $\bigcup A_i \in t$.

The pair (X, t) is an IFTS. Then any member of t is called an intuitionistic fuzzy open set (IFOS in short) in X. The complement of an IFOS in X is called an intuitionistic fuzzy closed set (IFCS in short) in X.

Definition 2.7 ([26]). *A fuzzy topological space (X, t) is called FT_0 if and only if, for any pair $x, y \in X$ with $x \neq y$, there exists $u \in t$ such that $u(x) = 1$, $u(y) = 0$, or $v(y) = 1$, $v(x) = 0$.*

Definition 2.8 ([25]). *A fuzzy bitopological space (X, s, t) is called FPT_0 if and only if, for any pair $x, y \in X$ with $x \neq y$, there exists $u \in (s \cup t)$ such that $u(x) = 1$, $u(y) = 0$, or $v(y) = 1$, $v(x) = 0$.*

Definition 2.9 ([2]). *An intuitionistic space (ITS in short), (X, τ) is called intuitionistic T_0-space (I-T_0 space) if for all $x, y \in X$, $x \neq y$, there exists $C = (C_1, C_2) \in \tau$ such that ($x \in C_1$ and $y \in C_2$) or ($y \in C_1$ and $x \in C_2$).*

3 Intuitionistic fuzzy bitopological T_0 spaces

Definition 3.1. *An intuitionistic fuzzy bitopological space (X, s, t) is called,*

(a) IFB-T_0 (i) if for all $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists $M = (\mu_M, \nu_M) \in s \cup t$ such that,

$$\mu_M(x_1) = 1, \quad \nu_M(x_1) = 0;$$
$$\mu_M(x_2) = 0, \quad \nu_M(x_2) = 1$$

or,

$$\mu_M(x_2) = 1, \quad \nu_M(x_2) = 0;$$
$$\mu_M(x_1) = 0, \quad \nu_M(x_1) = 1$$

(b) IFB-T_0 (ii) if for all $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists $M = (\mu_M, \nu_M) \in s \cup t$ such that,

$$\mu_M(x_1) = 1, \quad \nu_M(x_1) = 0;$$
$$\mu_M(x_2) = 0, \quad \nu_M(x_2) > 0$$

or,

$$\mu_M(x_2) = 1, \quad \nu_M(x_2) = 0;$$
$$\mu_M(x_1) = 0, \quad \nu_M(x_1) > 0$$

(c) IFB-T_0 (iii) if for all $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists $M = (\mu_M, \nu_M) \in s \cup t$ such that,

$$\mu_M(x_1) > 0, \quad \nu_M(x_1) = 0;$$
$$\mu_M(x_2) = 0, \quad \nu_M(x_2) = 1$$

or,

$$\mu_M(x_2) > 0, \quad \nu_M(x_2) = 0;$$
$$\mu_M(x_1) = 0, \quad \nu_M(x_1) = 1$$

(d) IFB-T_0 (iv) if for all $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists $M = (\mu_M, \nu_M) \in s \cup t$ such that,

$$\mu_M(x_1) > 0, \quad \nu_M(x_1) = 0;$$
$$\mu_M(x_2) = 0, \quad \nu_M(x_2) > 0$$

or,

$$\mu_M(x_2) > 0, \quad \nu_M(x_2) = 0;$$
$$\mu_M(x_1) = 0, \quad \nu_M(x_1) > 0$$

Definition 3.2. *Let us consider $\alpha \in (0, 1)$. Then, an intuitionistic fuzzy bitopological space (X, s, t) is called,*

(a) α-IFB-T_0 (i) if for all $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists $M = (\mu_M, \nu_M) \in s \cup t$ such that,

$$\mu_M(x_1) = 1, \quad \nu_M(x_1) = 0;$$
$$\mu_M(x_2) = 0, \quad \nu_M(x_2) \geq \alpha$$

or,

$$\mu_M(x_2) = 1, \quad \nu_M(x_2) = 0;$$
$$\mu_M(x_1) = 0, \quad \nu_M(x_1) \geq \alpha$$

(b) α-IFB-T_0 (ii) if for all $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists $M = (\mu_M, \nu_M) \in s \cup t$ such that,

$$\mu_M(x_1) \geq \alpha, \quad \nu_M(x_1) = 0;$$
$$\mu_M(x_2) = 0, \quad \nu_M(x_2) \geq \alpha$$

or,

$$\mu_M(x_2) \geq \alpha, \quad \nu_M(x_2) = 0;$$
$$\mu_M(x_1) = 0, \quad \nu_M(x_1) \geq \alpha$$

(c) α-IFB-T_0 (iii) if for all $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists $M = (\mu_M, \nu_M) \in s \cup t$ such that,

$$\mu_M(x_1) > 0, \quad \nu_M(x_1) = 0;$$
$$\mu_M(x_2) = 0, \quad \nu_M(x_2) \geq \alpha$$

or,

$$\mu_M(x_2) > 0, \quad \nu_M(x_2) = 0;$$
$$\mu_M(x_1) = 0, \quad \nu_M(x_1) \geq \alpha$$

Theorem 3.1. *Given an intuitionistic fuzzy bitopological space denoted as (X, s, t), then the subsequent implications are observed:*

(a) $$\text{IFB-T}_0(i) \implies \text{IFB-T}_0(ii)$$

(b) $$\text{IFB-T}_0(i) \implies \text{IFB-T}_0(iii)$$

(c) $$\text{IFB-T}_0(i) \implies \text{IFB-T}_0(iv)$$

(d) $$\text{IFB-T}_0(ii) \implies \text{IFB-T}_0(iv)$$

(e) $$\text{IFB-T}_0(iii) \implies \text{IFB-T}_0(iv)$$

Proof. Let us prove (c).
Given that (X, s, t) is an IFB-T_0 (i), we have to prove that (X, s, t) is an IFB-T_0 (iv). Since (X, s, t) is an IFB-T_0 (i), for all $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists $M = (\mu_M, \nu_M) \in s \cup t$, then

$$\begin{aligned} &\mu_M(x_1) = 1, \quad \nu_M(x_1) = 0; \quad \mu_M(x_2) = 0, \quad \nu_M(x_2) = 1, \\ \implies &\mu_M(x_1) > 0, \quad \nu_M(x_1) = 0; \quad \mu_M(x_2) = 0, \quad \nu_M(x_2) > 0, \end{aligned}$$

which is an IFB-T_0 (iv). Hence, IFB-T_0 (i) $\implies$ IFB-T_0 (iv).

We have established the proof for (c), and it follows that we can also establish proofs for (a), (b), (d), and (e) in a similar approach. □

Theorem 3.2. *Given an intuitionistic fuzzy bitopological space denoted as (X, s, t), then the subsequent implications are observed:*

(a) $$\alpha\text{-IFB-T}_0(i) \implies \alpha\text{-IFB-T}_0(ii)$$

(b) $$\alpha\text{-IFB-T}_0(i) \implies \alpha\text{-IFB-T}_0(iii)$$

(c) $$\alpha\text{-IFB-T}_0(ii) \implies \alpha\text{-IFB-T}_0(iii)$$

Proof. Let us prove (b).
Let $\alpha \in (0, 1)$. Suppose that (X, s, t) is an α-IFB-T_0 (i), we have to prove that (X, s, t) is an α-IFB-T_0 (iii). Since (X, s, t) is an α-IFB-T_0 (i), for all $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists $M = (\mu_M, \nu_M) \in s \cup t$ such that,

$$\begin{aligned} &\mu_M(x_1) = 1, \quad \nu_M(x_1) = 0; \quad \mu_M(x_2) = 0, \quad \nu_M(x_2) \geq \alpha, \\ \implies &\mu_M(x_1) \geq \alpha, \quad \nu_M(x_1) = 0; \quad \mu_M(x_2) = 0, \quad \nu_M(x_2) \geq \alpha, \end{aligned}$$

which is an α-IFB-T_0 (iii). Therefore, α-IFB-T_0 (i) $\implies$ α-IFB-T_0 (iii).

We have established the proof for (b), and it follows that we can also establish proofs for (a) and (c) in a similar manner. □

Theorem 3.3. *Given an intuitionistic fuzzy bitopological space denoted as (X, s, t) and $0 < \alpha \leq \beta < 1$, then the subsequent implications are observed:*

(a) $$\beta\text{-IFB-T}_0(i) \implies \alpha\text{-IFB-T}_0(i)$$

(b) $$\beta\text{-IFB-T}_0(ii) \implies \alpha\text{-IFB-T}_0(ii)$$

(c) $$\beta\text{-IFB-T}_0(iii) \implies \alpha\text{-IFB-T}_0(iii)$$

Proof. Let us prove (b).
Suppose that the IFBS (X, s, t) is an β-IFB-T$_0$ (ii). Then we shall prove that (X, s, t) is an α-IFB-T$_0$ (ii). Since (X, s, t) is β-IFB-T$_0$ (ii), then for all $x_1, x_2 \in X$, $x_1 \neq x_2$ with $\beta \in (0, 1)$ there exists $M = (\mu_M, \nu_M) \in s \cup t$ such that,

$$\mu_M(x_1) \geq \beta, \quad \nu_M(x_1) = 0; \quad \mu_M(x_2) = 0, \quad \nu_M(x_2) \geq \beta,$$
$$\implies \mu_M(x_1) \geq \alpha, \nu_M(x_1) = 0; \mu_M(x_2) = 0, \nu_M(x_2) \geq \alpha, \text{ as } 0 < \alpha \leq \beta < 1,$$

which is an α-IFB-T$_0$ (ii). Therefore, β-IFB-T$_0$ (ii) $\implies$ α-IFB-T$_0$ (ii).

We have established the proof for (b), and it follows that we can also establish proofs for (a) and (c) in a similar manner. □

Theorem 3.4. *Let (X, s, t) be an intuitionistic fuzzy bitopological space, $W \subseteq X$, $s_W = \{M \mid W : M \in s\}$, $t_W = \{N \mid W : N \in t\}$, and $\alpha \in (0, 1)$. Then,*

(a) (X, s, t) *is IFB-T*$_0(i) \implies (W, s_W, t_W)$ *is IFB-T*$_0(i)$

(b) (X, s, t) *is IFB-T*$_0(ii) \implies (W, s_W, t_W)$ *is IFB-T*$_0(ii)$

(c) (X, s, t) *is IFB-T*$_0(iii) \implies (W, s_W, t_W)$ *is IFB-T*$_0(iii)$

(d) (X, s, t) *is IFB-T*$_0(iv) \implies (W, s_W, t_W)$ *is IFB-T*$_0(iv)$

(e) (X, s, t) *is* α*-IFB-T*$_0(i) \implies (W, s_W, t_W)$ *is* α*-IFB-T*$_0(i)$

(f) (X, s, t) *is* α*-IFB-T*$_0(ii) \implies (W, s_W, t_W)$ *is* α*-IFB-T*$_0(ii)$

(g) (X, s, t) *is* α*-IFB-T*$_0(iii) \implies (W, s_W, t_W)$ *is* α*-IFB-T*$_0(iii)$

Proof. Let us prove (c).
Suppose that (X, s, t) is the intuitionistic fuzzy bitopological space and is also α-IFB-T$_0$ (ii). We shall prove that (W, s_W, t_W) is an α-IFB-T$_0$ (ii). Let $x_1, x_2 \in W$ with $x_1 \neq x_2$ then $x_1, x_2 \in X$ with $x_1 \neq x_2$ as $W \subseteq X$. Since (X, s, t) is an α-IFB-T$_0$ (ii), then there exists $P = (\mu_P, \nu_P) \in s \cup t$ then,

$$\mu_P(x_1) \geq \alpha, \quad \nu_P(x_1) = 0; \quad \mu_P(y) = 0, \quad \nu_P(y) \geq \alpha,$$
$$\implies \mu_{(P|U)}(x_1) \geq \alpha, \quad \nu_{(P|U)}(x_1) = 0; \quad \mu_{(P|U)}(x_2) = 0, \quad \nu_{(P|U)}(x_2) \geq \alpha.$$

Since, $P|U = (\mu_{(P|U)}, \nu_{(P|U)}) \in s_U \cup t_U$. Hence the IFBS (W, s_W, t_W) is an α-IFB-T$_0$ (ii).

Having successfully demonstrated the proof for (f), it logically follows that we can similarly establish proofs for (a), (b), (c), (d), (e), and (g). □

Definition 3.3. *An intuitionistic bitopological space (X, τ_1, τ_2) is classified as an intuitionistic BT_0-space (IB-T_0 space) if for all $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists $M = (M_1, M_2) \in \tau_1 \cup \tau_2$ such that $(x_1 \in M_1$ and $x_2 \in M_2)$ or $(x_2 \in M_1$ and $x_1 \in M_2)$.*

Theorem 3.5. *Let (X, τ_1, τ_2) be an IBS and let (X, t_1, t_2) be the IFBS, then the subsequent implications are observed*

(a) $\quad$ *IB-T*$_0 \iff$ *IFB-T*$_0(i)$

(b) $\quad$ *IB-T*$_0 \implies$ *IFB-T*$_0(ii)$

(c) $\quad$ *IB-T*$_0 \implies$ *IFB-T*$_0(iii)$

(d) $\quad$ *IB-T*$_0 \implies$ *IFB-T*$_0(iv)$

Proof. Let us prove (b).
Suppose that (X, τ_1, τ_2) is an IB-T$_0$ space. We shall prove (X, t_1, t_2) is an IFB-T$_0$ (ii). Since (X, τ_1, τ_2) is an IB-T$_0$, then for all $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists $M = (M_1, M_2) \in \tau_1 \cup \tau_2$ such that,

$$x_1 \in M_1 \text{ and } x_2 \in M_2 \implies 1_{M_1}(x_1) = 1, \quad 1_{M_2}(x_2) = 1$$

Let $1_{M_1} = \mu_M, 1_{M_2} = \nu_M$, then $\mu_M(x) = 1, \nu_M(x) = 0; \mu_M(x_2) = 0, \nu_M(x_2) = 1$. This implies $\mu_M(x_1) = 1, \nu_M(x_1) = 0; \mu_M(x_2) = 0, \nu_M(x_2) > 0$.
Since $(\mu_M, \nu_M) \in t_1 \cup t_2$, (X, t_1, t_2) is IFB-T$_0$ (ii). Hence, IB-T$_0 \implies$ IFB-T$_0$ (ii).

We have established the proof for (b), and it follows that we can also establish proofs for (a), (c), and (d) in a similar manner. □

Theorem 3.6. *Let (X, τ_1, τ_2) be an IBS, and let (X, t_1, t_2) be the IFBS. Then the subsequent implications are observed:*

(a) $\quad$ *IB-T*$_0 \implies \alpha -$ *IFB-T*$_0(i)$

(b) $\quad$ *IB-T*$_0 \implies \alpha -$ *IFB-T*$_0(ii)$

(c) $\quad$ *IB-T*$_0 \implies \alpha -$ *IFB-T*$_0(iii)$

Proof. Let us prove (c).
Consider $\alpha \in (0, 1)$. Suppose that (X, τ_1, τ_2) is an IB-T$_0$ space. We have to show that (X, t_1, t_2) is an α-IFB-T$_0$ (iii). Since (X, τ_1, τ_2) is IB-T$_0$, then for all $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists $M = (M_1, M_2) \in \tau_1 \cup \tau_2$ such that,

$$x_1 \in M_1 \text{ and } x_2 \in M_2 \implies 1_{M_1}(x_1) = 1, \quad 1_{M_2}(x_2) = 1$$

Let $1_{M_1} = \mu_M, 1_{M_2} = \nu_M$, then $\mu_M(x_1) = 1, \nu_M(x_1) = 0; \mu_M(x_2) = 0, \nu_M(x_2) = 1$. This implies $\mu_M(x_1) > 0, \nu_M(x_1) = 0; \mu_M(x_2) = 0, \nu_M(x_2) \geq \alpha$ for any $\alpha \in (0, 1)$.
Since $(\mu_M, \nu_M) \in t_1 \cup t_2$, (X, t_1, t_2) is an α-IFB-T$_0$ (iii). Therefore, IB-T$_0 \implies \alpha$-IFB-T$_0$ (iii).

We have established the proof for (c), and we can also establish proofs for (a) and (b) in a similar approach. □

Theorem 3.7. *Let (W, s_1, s_2) and (Z, t_1, t_1) be two intuitionistic fuzzy bitopological spaces, and $f : W \implies Z$ be one-one, onto, continuous mapping, and $\alpha \in (0, 1)$. Then the subsequent implications are observed:*

(a) (W, s_1, s_2) *is IFB-T*$_0(i) \Longleftrightarrow (Z, t_1, t_2)$ *is IFB-T*$_0(i)$

(b) (W, s_1, s_2) *is IFB-T*$_0(ii) \Longleftrightarrow (Z, t_1, t_2)$ *is IFB-T*$_0(ii)$

(c) (W, s_1, s_2) *is IFB-T*$_0(iii) \Longleftrightarrow (Z, t_1, t_2)$ *is IFB-T*$_0(iii)$

(d) (W, s_1, s_2) *is IFB-T*$_0(iv) \Longleftrightarrow (Z, t_1, t_2)$ *is IFB-T*$_0(iv)$

(e) (W, s_1, s_2) *is* $\alpha -$ *IFB-T*$_0(i) \Longleftrightarrow (Z, t_1, t_2)$ *is* $\alpha -$ *IFB-T*$_0(i)$

(f) (W, s_1, s_2) *is* $\alpha -$ *IFB-T*$_0(ii) \Longleftrightarrow (Z, t_1, t_2)$ *is* $\alpha -$ *IFB-T*$_0(ii)$

(g) (W, s_1, s_2) *is* $\alpha -$ *IFB-T*$_0(iii) \Longleftrightarrow (Z, t_1, t_2)$ *is* α*-IFB-T*$_0(iii)$

Proof. Let us prove (b).
Consider the IFBS (X, s_1, s_2) is IFB-T$_0$ (ii), then we have to prove that the IFBS (Y, t_1, t_2) is IFB-T$_0$ (ii). Let $z_1, z_2 \in Z, z_1 \neq z_2$ with $M = (\mu_M, \nu_M) \in t_1 \cup t_2$ such that $\mu_M(z_1) = 1, \nu_M(z_2) > 0$. Since f is onto, then there exist $w_1, w_2 \in W$ such that $w_1 = f^{-1}(z_1)$ and $w_2 = f^{-1}(z_2)$. Since $z_1 \neq z_2$, then $f^{-1}(z_1) \neq f^{-1}(z_2)$. Hence $w_1 \neq w_2$.
We have $(f^{-1}(\mu_M), f^{-1}(\nu_M)) \in s_1 \cup s_2$ as f is IF-continuous. Here, $(f^{-1}(\mu_M))(w_1) = \mu_M(f(w_1))$ $= \mu_M(z_1) = 1$. And, $(f^{-1}(\nu_M))(w_2) = \nu_M(f(w_2)) = \nu_M(z_2) > 0$ as f is one-to-one and onto. Hence $(f(\mu_M), f(\nu_M)) \in t_1 \cup t_2$. Therefore, (Z, t_1, t_2) is IFB-T$_0$ (ii).

Conversely, suppose that fuzzy bitopological space (Z, t_1, t_2) is IFB-T$_0$ (ii). We have to prove that the IFBS (W, s_1, s_2) is IFB-T$_0$ (ii). Let $w_1, w_1 \in W, w_1 \neq w_2$ with $M = (\mu_M, \nu_M) \in s_1 \cup s_2$ such that $\mu_M(w_1) = 1, \nu_M(w_2) > 0$. Since f is one-to-one, then there exist $y_i \in t_1 \cup t_2$ such that $y_i = f(x_i), i = 1, 2$ as f is one-to-one. Now, $(f(\mu_M), f(\nu_M)) \in t_1 \cup t_2$ since f is continuous. Here, $(f(\mu_M))(z_1) = f(\mu_M)f(w_1) = \mu_M(f^{-1}(f(w_1))) = \mu_M(w_1) = 1$. And, $(f(\nu_M))(z_2) = \nu_M(f^{-1}(f(w_2))) = \nu_M(w_2) > 0$ since f is one-to-one and onto. Hence, $(f^{-1}(\mu_M), f^{-1}(\nu_M)) \in s_1 \cup s_2$. Therefore, (W, s_1, s_2) is IFB-T$_0$ (ii).

So, (W, s_1, s_2) is IFB-T$_0$ (ii) $\Leftrightarrow (Z, t_1, t_2)$ is IFB-T$_0$ (ii).

We have established the proof for (b), and we can also establish proofs for (a), (c), (d), (e), (f), and (g) in a similar approach. □

4 Conclusion

In this article, we expanded our exploration of intuitionistic fuzzy T_0 by integrating the principles of bitopology. Well-known properties like good extension, hereditary, and mappings (one-one, onto, fuzzy open, fuzzy continuous), for instance, have been studied. We anticipate that our research will act as a driving force, encouraging further exploration and innovation in the prescribed field.

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