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On the extension of the $\to''_@$ intuitionistic fuzzy implication

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Abstract: In the paper a class of parametric intuitionistic fuzzy implications is introduced. The class is an extension of the $\to''_@$ implication, introduced by L. Atanassova. Fulfillment of some axioms and properties, together with Modus Ponens inference rule, are investigated. Negation induced by implication is presented.

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1 Introduction

In the intuitionistic fuzzy logic (IFL) the truth-value of variable x is given by ordered pair $\langle a, b\rangle$, where $a, b, a + b \in [0, 1]$. The numbers a and b are interpreted as the degrees of validity and non-validity of x. We denote the truth-value of x by $V(x)$.

The variable with truth-value *true* in the classical logic we denote by $\underline{1}$ and the variable *false* by $\underline{0}$. For these variables, it also holds $V(\underline{1}) = \langle 1, 0\rangle$ and $V(\underline{0}) = \langle 0, 1\rangle$.

We call the variable x an Intuitionistic Fuzzy Tautology (shortly: IFT), if and only if when for $V(x) = \langle a, b\rangle$ it holds: $a \geq b$ and, similarly, an Intuitionistic Fuzzy co-Tautology (IFcT), *iff* it holds: $a \leq b$.

In the paper [3], Lilija Atanassova introduced the new intuitionistic fuzzy implication $\to''_@$. It is the modification of the $\to_@$ implication introduced first in [2] and generalized in [4, 5].

The $\to_@$ and $\to''_@$ implications are defined by formulas:

$$V(x \to_@ y) = \langle \frac{b+c}{2}, \frac{a+d}{2} \rangle,$$

$$V(x \to''_@ y) = \langle \frac{b+c+\max\{b,c\}}{3}, \frac{a+d+\min\{a,d\}}{3} \rangle.$$

The $\to''_@$ implication can be extended to a class of parametric intuitionistic fuzzy implications.

Assume that the logical connective $\Rightarrow$ can be called a *Intuitionistic Fuzzy Implication* if it fulfills the conditions (i1 IFL)–(i5 IFL), given (see [4]) in the form:

(i1 IFL) if $V(x_1) \preceq V(x_2)$, then $V(x_1 \Rightarrow y) \succeq V(x_2 \Rightarrow y)$,

(i2 IFL) if $V(y_1) \preceq V(y_2)$, then $V(x \Rightarrow y_1) \preceq V(x \Rightarrow y_2)$,

(i3 IFL) $\underline{0} \Rightarrow y$ is an IFT,

(i4 IFL) $x \Rightarrow \underline{1}$ is an IFT,

(i5 IFL) $\underline{1} \Rightarrow \underline{0}$ is an IFcT,

where for $V(x) = \langle a, b\rangle$ and $V(y) = \langle c, d\rangle$ it is $V(x) \preceq V(y)$, if and only if $a \le c$ and $b \ge d$.

We propose to call the implication fulfilling the conditions (i1 IFL)–(i5 IFL) a *Weak Intuitionistic Fuzzy Implication*, because the properties (i3 IFL), (i4 IFL) and (i5 IFL) are also given in the *strong* form $V(\underline{0} \Rightarrow y) = V(x \Rightarrow \underline{1}) = \langle 1, 0\rangle$, and $V(\underline{1} \Rightarrow \underline{0}) = \langle 0, 1\rangle$.

2 Main results

We introduce now a new parametric class of intuitionistic fuzzy implications. Let $V(x) = \langle a, b\rangle$ and $V(y) = \langle c, d\rangle$.

Theorem 1. The intuitionistic logical connective $\to^k_@$ with truth-value:

$$V(x \to^k_@ y) = \langle \frac{b + c + k\max\{b, c\}}{k+2}, \frac{a + d + k\min\{a, d\}}{k+2} \rangle,$$

where $k \in \Re$, $k \ge 0$ is an intuitionistic fuzzy implication fulfilling (i1 IFL)–(i5 IFL).

Proof. Preliminary note:

$$\langle \frac{b + c + k\max\{b, c\}}{k+2}, \frac{a + d + k\min\{a, d\}}{k+2} \rangle$$

holds IFS conditions because

1^0) $0 \le \dfrac{b + c + k\max\{b, c\}}{k+2} \le \dfrac{\max\{b,c\} + \max\{b,c\} + k\max\{b, c\}}{k+2} \le \dfrac{(k+2)\max\{b,c\}}{k+2} \le 1,$

2^0) $0 \le \dfrac{a + d + k\min\{a, d\}}{k+2} \le 1,$

3^0) $0 \le \dfrac{b + c + k\max\{b, c\}}{k+2} + \dfrac{a + d + k\min\{a, d\}}{k+2} \le$

$$\le \frac{a+b+c+d+k(\max\{b,c\}+\min\{a,d\})}{k+2} \le \frac{2+k(\max\{b,c\}+\min\{a,d\})}{k+2} \le$$

$$\le \frac{2+k(1-\min\{a,d\}+\min\{a,d\})}{k+2} = 1.$$

Conditions (i1 IFL)–(i5 IFL):

(i1 IFL) If $\langle a_1, b_1\rangle = V(x_1) \preceq V(x_2) = \langle a_2, b_2\rangle$, therefore $a_1 \le a_2$ and $b_1 \ge b_2$, so

$$\frac{b_1 + c + k\max\{b_1, c\}}{k+2} \ge \frac{b_2 + c + k\max\{b_2, c\}}{k+2} \text{ and } \frac{a_1 + d + k\min\{a_1, d\}}{k+2} \le \frac{a_2 + d + k\min\{a_2, d\}}{k+2}$$

and consequently $V(x_1 \to_{@}^{k} y) \succeq V(x_2 \to_{@}^{k} y)$.

(i2 IFL) If $\langle c_1, d_1\rangle = V(y_1) \preceq V(y_2) = \langle c_2, d_2\rangle$, therefore $c_1 \le c_2$ and $d_1 \ge d_2$, so

$$\frac{b + c_1 + k\max\{b, c_1\}}{k+2} \le \frac{b + c_2 + k\max\{b, c_2\}}{k+2} \text{ and } \frac{a + d_1 + k\min\{a, d_1\}}{k+2} \ge \frac{a + d_2 + k\min\{a, d_2\}}{k+2}$$

and consequently $V(x \to_{@}^{k} y_1) \preceq V(x \to_{@}^{k} y_2)$.

(i3 IFL) It is, by definition, $V(\underline{0} \to_{@}^{k} y) = \langle \frac{1 + c + k\max\{1, c\}}{k+2}, \frac{d + k\min\{0, d\}}{k+2} \rangle$. Because $\frac{1+c+k}{k+2} \ge \frac{d}{k+2}$ is equivalent to $1 + c + k \ge d$, which holds, therefore $\underline{0} \to_{@}^{k} y$ is an IFT.

(i4 IFL) It is $V(x \to_{@}^{k} \underline{1}) = \langle \frac{b + 1 + k\max\{b, 1\}}{k+2}, \frac{a + k\min\{a, 0\}}{k+2} \rangle$. Because $\frac{b+1+k}{k+2} \ge \frac{a}{k+2}$ is equivalent to $b + 1 + k \ge a$, and this holds, therefore $x \to_{@}^{k} \underline{1}$ is an IFT.

(i5 IFL) It is $V(\underline{1} \to_{@}^{k} \underline{0}) = \langle 0, 1\rangle$, therefore $\underline{1} \to_{@}^{k} \underline{0}$ is an IFcT. □

The special cases of the $\to_{@}^{k}$ (for $k = 0$ and $k = 1$) were given first by L. Atanassova [2, 3].

It is easy to check that the implication $\to_{@}^{k}$ does not satisfy the classical (two-valued) logic axioms. Namely

$$V(\underline{0} \to_{@}^{k} \underline{0}) = V(\underline{1} \to_{@}^{k} \underline{1}) = \langle \frac{k+1}{k+2}, \frac{1}{k+2} \rangle \ne V(\underline{1}),\ V(\underline{1} \to_{@}^{k} \underline{0}) = V(\underline{0}),$$

and $V(\underline{0} \to_{@}^{k} \underline{1}) = V(\underline{1})$. But we notice that $\underline{0} \to_{@}^{k} \underline{0}$ and $\underline{1} \to_{@}^{k} \underline{1}$ are IFTs for any k.

The implication $\to_{@}^{k}$ is therefore not a simple generalization of the classical implication. We note, that $\lim_{k\to\infty} V(x \to_{@}^{k} y) = \langle \max\{b, c\}, \min\{a, d\}\rangle$, what is known as Kleene–Dienes intuitionistic fuzzy implication (see eg [1], p. 197).

For any $k \ge 0$ holds:

$$V(x \to_{@} y) \preceq V(x \to_{@}^{k} y),$$

It is so because $\frac{b+c}{2} \le \frac{b + c + k\max\{b, c\}}{k+2}$ and $\frac{a+d}{2} \ge \frac{a + d + k\min\{a, d\}}{k+2}$, for any $k \ge 0$.

Atanassova gives in [3] the theorem concerning on the rule of inference called Modus Ponens. The Modus Ponens rule is, in classical logic, the tautology: $(p \wedge (p \Rightarrow q)) \Rightarrow q$. In the IFL-case, we understand the Modus Ponens as follows: if x is an IFT and $(x \Rightarrow y)$ is an IFT then y is an IFT.

Theorem 2. Implication $\to_{@}^{k}$:

a) satisfies Modus Ponens in the IFL-case for $k = 0$,

b) does not satisfy Modus Ponens in the IFL-case for $k > 0$.

Proof.

a) See [3].

b) Proof by counterexample.

b1) Let $k \geq 1$ be a fixed number. For $V(x) = \langle 0.5, 0.4\rangle$ and $V(y) = \langle 0, 0.1\rangle$ we have $V(x \to_{@}^{k} y) = \langle \frac{0.4+0.4k}{k+2}, \frac{0.6+0.1k}{k+2}\rangle$. In this case x is an IFT and $x \to_{@}^{k} y$ is an IFT, but y is not an IFT.

b2) Let $k \in (0, 1)$ be a fixed number. For $V(x) = \langle \frac{k}{2}, \frac{k(k+2)}{4(k+1)}\rangle$ and $V(y) = \langle 0, \frac{k^2}{4(k+1)}\rangle$ we have $\max\{0, \frac{k(k+2)}{4(k+1)}\} = \frac{k(k+2)}{4(k+1)}$ and $\min\{\frac{k}{2}, \frac{k^2}{4(k+1)}\} = \frac{k^2}{4(k+1)}$.

From the definition,

$$V(x \to_{@}^{k} y) = \langle \frac{\frac{k(k+2)}{4(k+1)} + k\cdot\frac{k(k+2)}{4(k+1)}}{k+2}, \frac{\frac{k}{2} + \frac{k^2}{4(k+1)} + k\cdot\frac{k^2}{4(k+1)}}{k+2}\rangle = \langle \frac{k}{4}, \frac{k}{4}\rangle.$$

Because for $k \in (0, 1)$ it is: $\frac{k}{2}, \frac{k(k+2)}{4(k+1)} \in (0, \frac{1}{2})$ and $\frac{k}{2} > \frac{k(k+2)}{4(k+1)}$, then $V(x)$ is an intuitionistic fuzzy value and, moreover, it is an IFT. Similarly, $x \to_{@}^{k} y$ is an intuitionistic fuzzy value and it is an IFT, while y is an intuitionistic fuzzy value, but it is not an IFT.

Finally: It is x an IFT and $x \to_{@}^{k} y$ an IFT, however y is not an IFT. □

One of the fundamental tautologies of classical logic is the relationship between the implication and negation. This relationship says that the truth-value of negation of the variable x is equal to the value of the logical implications of the antecedent x and the consequent *false*. Symbolically, this tautology is written in the form: $N(x) \Leftrightarrow (x \Rightarrow 0)$. Using this relationship we can, for every intuitionistic fuzzy implication, designate a corresponding negation, called a generated (induced) negation.

Theorem 3. Negation $\neg_{@}^{k}$ generated by $\to_{@}^{k}$ is expressed by formula:

$$V(\neg_{@}^{k} x) = \langle \frac{b(k+1)}{k+2}, \frac{a(k+1)+1}{k+2}\rangle$$

The proof is made by definition of $\to_{@}^{k}$.

It is easy to check that the negation $\neg_{@}^{k}$ does not satisfy the classical (two-valued) logic axioms. We note, that $\lim_{k\to\infty} V(\neg_{@}^{k} x) = \langle b, a\rangle$, what is known as the classical IF negation.

Now we denote $\neg_{@}^{k\ 1}(x) = \neg_{@}^{k}(x)$ and $\neg_{@}^{k\ m+1}(x) = \neg_{@}^{k}(\neg_{@}^{k\ m}(x))$ for any $m \in N_{+}$.

Theorem 4. For a natural number $n \geq 1$ the negation $\neg_{@}^{k}$ there hold the relationships:

a) $$V(\neg_{@}^{k\;2n-1}(x)) = \langle b \cdot \frac{(k+1)^{2n-1}}{(k+2)^{2n-1}} + \frac{(k+1)^{2n-3}}{(k+2)^{2n-2}} + \frac{(k+1)^{2n-5}}{(k+2)^{2n-4}} + \ldots + \frac{(k+1)^{1}}{(k+2)^{2}},$$

$$a \cdot \frac{(k+1)^{2n-1}}{(k+2)^{2n-1}} + \frac{(k+1)^{2n-2}}{(k+2)^{2n-1}} + \frac{(k+1)^{2n-4}}{(k+2)^{2n-3}} + \ldots + \frac{1}{(k+2)^{1}} \rangle =$$

$$= \langle b \cdot \left(\frac{k+1}{k+2}\right)^{2n-1} + \frac{1}{k+1} \sum_{p=1}^{n-1} \left(\frac{k+1}{k+2}\right)^{2p},\ a \cdot \left(\frac{k+1}{k+2}\right)^{2n-1} + \frac{1}{k+1} \sum_{p=1}^{n} \left(\frac{k+1}{k+2}\right)^{2p-1} \rangle =$$

$$= \langle b \cdot \left(\frac{k+1}{k+2}\right)^{2n-1} + \frac{k+1}{2k+3} \cdot \frac{(k+2)^{2n-2} - (k+1)^{2n-2}}{(k+2)^{2n-2}},$$

$$a \cdot \left(\frac{k+1}{k+2}\right)^{2n-1} + \frac{k+2}{2k+3} \cdot \frac{(k+2)^{2n} - (k+1)^{2n}}{(k+2)^{2n}} \rangle.$$

b) $$V(\neg_{@}^{k\;2n}(x)) = \langle a \cdot \frac{(k+1)^{2n}}{(k+2)^{2n}} + \frac{(k+1)^{2n-1}}{(k+2)^{2n}} + \frac{(k+1)^{2n-3}}{(k+2)^{2n-2}} + \ldots + \frac{(k+1)}{(k+2)^{2}},$$

$$b \cdot \frac{(k+1)^{2n}}{(k+2)^{2n}} + \frac{(k+1)^{2n-2}}{(k+2)^{2n-1}} + \frac{(k+1)^{2n-4}}{(k+2)^{2n-3}} + \ldots + \frac{1}{(k+2)^{1}} \rangle =$$

$$= \langle a \cdot \left(\frac{k+1}{k+2}\right)^{2n} + \frac{1}{k+1} \sum_{p=1}^{n} \left(\frac{k+1}{k+2}\right)^{2p},\ b \cdot \left(\frac{k+1}{k+2}\right)^{2n} + \frac{1}{k+1} \sum_{p=1}^{n} \left(\frac{k+1}{k+2}\right)^{2p-1} \rangle =$$

$$= \langle a \cdot \left(\frac{k+1}{k+2}\right)^{2n} + \frac{k+1}{2k+3} \cdot \frac{(k+2)^{2n} - (k+1)^{2n}}{(k+2)^{2n}},$$

$$b \cdot \left(\frac{k+1}{k+2}\right)^{2n} + \frac{k+2}{2k+3} \cdot \frac{(k+2)^{2n} - (k+1)^{2n}}{(k+2)^{2n}} \rangle.$$

The proof is based on the principle of mathematical induction.

Remarks:

R1) $\lim_{k\to\infty} V(\neg_{@}^{k\;2n-1}(x)) = \langle b, a\rangle$ and $\lim_{k\to\infty} V(\neg_{@}^{k\;2n}(x)) = \langle a, b\rangle$.

R2) Negation $\neg_{@}^{k}$ is not involutive.

R3) Negation $\neg_{@}^{k}$ satisfies the classical axiom $V(\neg_{@}^{k}(\underline{1})) = V(\underline{0})$, but does not satisfy the axiom $V(\neg_{@}^{k}(\underline{0})) = V(\underline{1})$. Moreover, $V(\neg_{@}^{k}(\underline{0}))$ is never equal to $V(\underline{1})$.

But $V(\neg_{@}^{k}(\underline{0})) = \langle \frac{k+1}{k+2}, \frac{1}{k+2} \rangle$ what means that the $\neg_{@}^{k}(\underline{0})$ is an IFT.

The values $V(\neg_{@}^{k}(\underline{0}))$ and $V(\neg_{@}^{k}(\underline{1}))$ are classical fuzzy truth-values.

R4) For the negation $\neg_{@}^{k}$ it holds: $\lim_{m\to\infty} V(\neg_{@}^{k\;m}(x)) = \langle \frac{k+1}{2k+3}, \frac{k+2}{2k+3} \rangle$.

R5) For any k the value $\lim_{m\to\infty} V(\neg_{@}^{k\;m}(x))$ is a classical fuzzy set.

R6) For any k it holds: $\lim_{m\to\infty} V(\neg_{@}^{k\ m}(x)) \succeq \langle \frac{1}{3}, \frac{2}{3} \rangle$.

R7) $\lim_{k\to\infty} (\lim_{m\to\infty} V(\neg_{@}^{k\ m}(x))) = \langle \frac{1}{2}, \frac{1}{2} \rangle$.

R8) $\lim_{k\to 0} (\lim_{m\to\infty} V(\neg_{@}^{k\ m}(x))) = \lim_{m\to\infty} V(\neg_{@}^{0\ m}(x)) = \langle \frac{1}{2}, \frac{1}{2} \rangle$.

3 Conclusion

In the paper, some classes of fuzzy intuitionistic implications with their basic properties are presented. The implications may be the subject of further research, both in terms of their properties or comparisons with other intuitionistic fuzzy implications, and possible applications.

References

[1] Atanassov K. T. *On Intuitionistic Fuzzy Sets Theory*, Springer, Berlin, 2012.

[2] Atanassova, L. A new intuitionistic fuzzy implication, *Cybernetics and Information Technologies*, Vol. 9, 2009, No. 2, 21–25.
`http://www.cit.iit.bas.bg/CIT_09/v9-2/21-25.pdf`

[3] Atanassova, L. On two modifications of the intuitionistic fuzzy implication $\rightarrow_{@}$, *Notes on Intuitionistic Fuzzy Sets*, Vol. 18, 2012, No. 2, 26–30.
`http://www.ifigenia.org/wiki/ issue:nifs/18/2/26-30`

[4] Dworniczak, P. Some remarks about the L. Atanassova's paper "A new intuitionistic fuzzy implication", *Cybernetics and Information Technologies*, Vol. 10, 2010, No. 3, 3–9.
`http://www.cit.iit.bas.bg/CIT_2010/v10-3/3-9.pdf`

[5] Dworniczak, P., On some two-parametric intuitionistic fuzzy implications, *Notes on Intuitionistic Fuzzy Sets*, Vol. 17, 2011, No. 2, 8–16.
`http://ifigenia.org/wiki/issue:nifs/17/2/8-16`