Notes on Intuitionistic Fuzzy Sets
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A remark on the operations "+" and "." between intuitionistic fuzzy pairs

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Abstract: In this paper we investigate the operations "+" and "." defined for intuitionistic fuzzy sets, and, hence, for intuitionsitic fuzzy pairs. We provide an alternative point of view regarding its definition which renders some of its properties evident. We also discuss the class of intuitionistic pairs obtainable from a fixed intuitionistic fuzzy pair via the operation "+" or the operation ".".
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1 Introduction

Intuitionistic fuzzy pairs are a useful tool in intuitionistic fuzzy set theory [2, 3]. They can be understood as one-element universes (omitted in the notation), with given membership and non-membership degrees, which, of course, obey the usual constraint. Their formal definition is the following:

Definition 1 (cf. [1]). An intuitionistic fuzzy pair (IFP) is an ordered couple of real non-negative numbers $\langle a, b\rangle$, which satisfy:

$$a + b \leq 1. \tag{1}$$

Among the IFPs there exists a partial ordering. It is partial, since not all IFPs are comparable in terms of it. Two notable exceptions are the pairs $\langle 0, 1\rangle$ and $\langle 1, 0\rangle$ that denote respectively *Falsity* and *Truth* in intuitionistic fuzzy sets theory.

Definition 2 (cf. [1]). Given two IFPs $x = \langle a, b\rangle$, $y = \langle c, d\rangle$, we say that x is greater than y, and write $x \geq y$, iff:

$$\begin{cases} a \geq c \\ 1 - b \geq 1 - d \end{cases} \tag{2}$$

If we have neither, $x \geq y$, nor $y \geq x$, we will denote this fact by $x \not\sim y$.

Remark 1. For any IFP $\langle a, b\rangle$, we always have $\langle 1, 0\rangle \geq \langle a, b\rangle$ and $\langle a, b\rangle \geq \langle 0, 1\rangle$. When $ab \neq 0$, we have $\langle a, b\rangle \not\sim \langle 0, 0\rangle$.

We now turn our attention to the operation "+".

Definition 3 (cf. [1]). For any IFPs $\langle a, b\rangle$, and $\langle c, d\rangle$, we define

$$\langle a, b\rangle + \langle c, d\rangle = \langle a + c - ac, bd\rangle. \tag{3}$$

Remark 2. The result of the above operation is always an IFP which is greater than both $\langle a, b\rangle$ and $\langle c, d\rangle$, even when $\langle a, b\rangle \not\sim \langle c, d\rangle$.

We now turn our attention to the operation ".".

Definition 4 (cf. [1]). For any IFPs $\langle a, b\rangle$, and $\langle c, d\rangle$, we define

$$\langle a, b\rangle.\langle c, d\rangle = \langle ac, b + d - bd\rangle. \tag{4}$$

Remark 3. The result of the above operation is always an IFP which is lesser than both $\langle a, b\rangle$ and $\langle c, d\rangle$, even when $\langle a, b\rangle \not\sim \langle c, d\rangle$.

2 Main results

First, we start with an alternative formulation of Definition 3.

Definition 5. For any IFPs $\langle a, b\rangle$, and $\langle c, d\rangle$, we have

$$\langle a, b\rangle + \langle c, d\rangle = \langle 1 - (1 - a)(1 - c), bd\rangle. \tag{5}$$

The usefulness of Definition 5 lies in the fact that we can now recurrently write (an also clearly see where associativity of operation "+" comes from) the result of consecutive application of the operation over many IFPs. It is quite easy to see (following from (5)) that:

$$\langle a_1, b_1\rangle + \langle a_2, b_2\rangle + \ldots + \langle a_n, b_n\rangle = \left\langle 1 - \prod_{i=1}^{n}(1-a_i), \prod_{i=1}^{n} b_i \right\rangle. \qquad (6)$$

Remark 4. We have for any IFPs $\langle a, b\rangle$ and $\langle c, d\rangle$:

$$\langle 1, 0\rangle + \langle a, b\rangle = \langle 1, 0\rangle,$$
$$\langle 0, 1\rangle + \langle a, b\rangle = \langle a, b\rangle,$$
$$\langle 0, 0\rangle + \langle a, b\rangle = \langle a, 0\rangle,$$
$$\langle a, 0\rangle + \langle c, d\rangle = \langle 1-(1-a)(1-c), 0\rangle.$$

From the above it is clear that starting from $\langle 0, 1\rangle$ we can obtain any IFP through the operation "+". In some sense, $\langle 0, 1\rangle$ acts like an analogue of zero for the real numbers. On the other hand, it is clear that starting from $\langle 1, 0\rangle$ we can only obtain $\langle 1, 0\rangle$ through "+". In some sense this IFP acts as an analogue of ∞.

Proposition 1. From the above remarks it is clear that for a fixed IFP $\langle a, b\rangle$ and for any $\langle c, d\rangle$

$$\langle a, b\rangle + \langle c, d\rangle = \langle u, v\rangle$$

is impossible iff:
1) $a > u$,
2) $a \le u$ but $1-b > 1-v$,
3) $a \le u$, $1-b \le 1-v$, but $b(1-u) < (1-a)v$.

Proof. Condition 1) combines the cases when $a > u$ and

$$\langle a, b\rangle \begin{cases} \geq \langle u, v\rangle \\ \not> \langle u, v\rangle \end{cases},$$

i.e., $\langle u, v\rangle$ is not a valid result of the "+" operation. Condition 2) corresponds to the case when $a \le u$ but

$$\langle a, b\rangle \not> \langle u, v\rangle,$$

i.e., $\langle u, v\rangle$ is not a valid result of the "+" operation.

Condition 3) describes the seemingly possible case when $\langle u, v\rangle \geq \langle a, b\rangle$. In this case we should have:

$$1-a \geq \max(1-u, b) \geq v.$$

Using (5) and solving for c and d we should have:

$$c = 1 - \frac{1-u}{1-a}; \;\; d = \frac{v}{b}.$$

If we have $c > 1-d$, we see that $\langle c, d\rangle$ is not an IFP, i.e. $\langle u, v\rangle$ could not have been obtained by "+" operation. The condition $b(1-u) < (1-a)v$ is exactly the equivalent condition to $c > 1-d$. □

Moving to the operation "." and proceeding in a similar manner, we provide an alternative formulation of Definition 4.

Definition 6. For any IFPs $\langle a, b\rangle$, and $\langle c, d\rangle$, we have

$$\langle a, b\rangle.\langle c, d\rangle = \langle ac, 1-(1-b)(1-d)\rangle. \tag{7}$$

The usefulness of Definition 6 lies in the fact that we can now recurrently write (and also clearly see where associativity of operation "." comes from) the result of the consecutive application of the operation over many IFPs. It is quite easy to see (following from (7)) that:

$$\langle a_1, b_1\rangle.\langle a_2, b_2\rangle.\cdots.\langle a_n, b_n\rangle = \left\langle \prod_{i=1}^{n} a_i, 1-\prod_{i=1}^{n}(1-b_i)\right\rangle. \tag{8}$$

Remark 5. We have for any IFPs $\langle a, b\rangle$ and $\langle c, d\rangle$:

$$\begin{aligned}\langle 1, 0\rangle.\langle a, b\rangle &= \langle a, b\rangle,\\ \langle 0, 1\rangle.\langle a, b\rangle &= \langle 0, 1\rangle,\\ \langle 0, 0\rangle.\langle a, b\rangle &= \langle 0, b\rangle,\\ \langle 0, b\rangle.\langle c, d\rangle &= \langle 0, 1-(1-b)(1-d)\rangle.\end{aligned}$$

From the above it is clear that starting from $\langle 1, 0\rangle$ we can obtain any IFP through the operation ".". In some sense, $\langle 1, 0\rangle$ acts like an analogue of one for the real numbers. On the other hand, it is clear that starting from $\langle 0, 1\rangle$ we can only obtain $\langle 0, 1\rangle$ through ".". In some sense, this IFP acts as an analogue of zero.

Proposition 2. From the above remarks it is clear that for a fixed IFP $\langle a, b\rangle$ and for any $\langle c, d\rangle$

$$\langle a, b\rangle.\langle c, d\rangle = \langle u, v\rangle$$

is impossible iff:
1) $a < u$,
2) $a \geq u$ but $1-b < 1-v$,
3) $a \geq u$, $1-b \geq 1-v$, but $u(1-b) > (1-v)a$.

Proof. Condition 1) combines the cases when $a < u$ and

$$\langle a, b\rangle \begin{cases} \leq \langle u, v\rangle \\ \not\sim \langle u, v\rangle \end{cases},$$

i.e., $\langle u, v\rangle$ is not a valid result of the "." operation. Condition 2) corresponds to the case when $a \geq u$ but

$$\langle a, b\rangle \not\sim \langle u, v\rangle,$$

i.e., $\langle u, v\rangle$ is not a valid result of the "." operation.

Condition 3) describes the seemingly possible case when $\langle u, v\rangle \leq \langle a, b\rangle$. In this case we should have:

$$1 - b \geq \max(1 - v, a) \geq u.$$

Using (7) and solving for c and d we should have:

$$c = \frac{u}{a}; \quad d = 1 - \frac{1 - v}{1 - b}.$$

If we have $c > 1 - d$, we see that $\langle c, d\rangle$ is not an IFP, i.e. $\langle u, v\rangle$ could not have been obtained by "." operation. The condition $u(1 - b) > (1 - v)a$ is exactly the equivalent condition to $c > 1 - d$. □

We must note that for the operations "+" and "." one must take caution, as in the general case:

$$\langle x, y\rangle = (\langle a, b\rangle + \langle c, d\rangle) \,.\, \langle e, f\rangle \neq (\langle a, b\rangle \,.\, \langle e, f\rangle) + (\langle c, d\rangle \,.\, \langle e, f\rangle) = \langle z, t\rangle \qquad (9)$$

Proposition 3. Considering (9), it is always fulfilled:

$$\langle x, y\rangle \leq \langle z, t\rangle \,. \qquad (10)$$

Proof. First, we will prove that that $x \leq z$, and after that, $1 - y \leq 1 - t$, which is exactly Definition 2. Using Definition 3 and Definition 4, we obtain that

$$x = (a + c - ac)e = ae + ce - ace.$$

In the same manner, we obtain that

$$z = ae + ce - ace^2.$$

Hence, we have

$$z - x = ace(1 - e) \geq 0,$$

since $a, c, e \in [0, 1]$. The last proves that $x \leq z$.

In similar manner, we check that:

$$1 - y = (1 - bd)(1 - f)$$

and (using Definition 6 and Definition 3)

$$1 - t = (1 - b)(1 - f) + (1 - d)(1 - f) - (1 - b)(1 - d)(1 - f)^2,$$

hence,

$$\begin{aligned} 1 - t - (1 - y) &= (1 - f)(1 - d - b + bd - (1 - b)(1 - d)(1 - f)) \\ &= (1 - f)f(1 - b)(1 - d) \geq 0, \end{aligned}$$

since $b, d, f \in [0, 1]$.

This completes the proof. □

As we have clearly established, it is not always possible to obtain a fixed IFP, starting from an arbitrary one and using only one of the operations "+" or ".".

We have, thus far, demonstrated that neither of the two operations by itself is sufficient to obtain from a starting IFP $\langle a,b\rangle$ another IFP $\langle u,v\rangle$, which is not comparable to it. However, when both operations are used in combination it is sometimes possible to obtain such a pair as is illustrated by the following example:

$$(\langle 0.6,0.3\rangle + \langle 0.5,0.2\rangle).\langle 0.2,0.1\rangle = \langle 0.16,0.154\rangle,$$

and $\langle 0.16,0.154\rangle \not\sim \langle 0.6,0.3\rangle$.

Remark 6. We have used parentheses, to denote the order in which operations are taken, since:

$$(\langle 0.6,0.3\rangle + \langle 0.5,0.2\rangle).\langle 0.2,0.1\rangle \neq \langle 0.6,0.3\rangle + (\langle 0.5,0.2\rangle.\langle 0.2,0.1\rangle) = \langle 0.64,0.084\rangle.$$

Proposition 4. It is always possible to obtain a given IFP $\langle x,y\rangle$, starting from a fixed IFP $\langle a,b\rangle$ and using the "+" and "." operations.

Proof. The following steps always work:

$$(\langle a,b\rangle \,.\, \langle 0,1\rangle) + \langle x,y\rangle = \langle x,y\rangle\,.$$

Alternatively,

$$(\langle a,b\rangle + \langle 0,1\rangle)\,.\,\langle x,y\rangle = \langle x,y\rangle\,.$$

This completes the proof. □

Remark 7. Of course, the more interesting question is, given a starting IFP $\langle a,b\rangle$, such that $ab \neq 0$, and a target IFP $\langle x,y\rangle$, such that $xy \neq 0$, can we always reach it not using pairs which contain 0 as a component?

We will just note here that it is always possible to obtain $\langle b,a\rangle$ from $\langle a,b\rangle$ using the following reasoning (here, and further without loss of generality we assume $a < b$ or $a > b$, as $a = b$ means we already have the desired pair):

1) If $a < b$,

$$\langle a,b\rangle + \left\langle \frac{b-a}{1-a}, \frac{a}{b}\right\rangle = \langle b,a\rangle\,.$$

$\left\langle \frac{b-a}{1-a}, \frac{a}{b}\right\rangle$ is an IFP, since both its components are in the $[0,1]$ and the inequality for their sum being less than 1 is equivalent to $a+b \leq 1$.

2) If $a > b$,

$$\langle a,b\rangle \,.\, \left\langle \frac{b}{a}, \frac{a-b}{1-b}\right\rangle = \langle b,a\rangle\,.$$

3 Conclusion

We have presented alternative definitions of the operations "+" and "." we have stated in explicit form the class of all IFPs $\langle u,v\rangle$ which cannot be obtained from a given IFP $\langle a,b\rangle$ through these operations. We have also established some additional properties of the composition of these operations.

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