A generalisation of operators on intuitionistic fuzzy sets using triangular norms and conorms

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Abstract

Several operators on intuitionistic fuzzy sets, such as union, intersection, sum and product, have been defined using common triangular norms and conorms. In this paper we introduce and analyse the properties of a generalised union and a generalised intersection of intuitionistic fuzzy sets using a general t-norm and t-conorm. In particular we will investigate properties such as commutativity, associativity, distributivity, idempotency, Morgan-laws, and absorption laws.

Keywords : intuitionistic fuzzy set, generalised union, generalised intersection

1 Definitions

Intuitionistic fuzzy sets (IFSs, for short) constitute a generalisation of the notion of a fuzzy set (FS, for short) and were introduced by K. T. Atanassov in 1983 in [**?**]. While fuzzy sets give the degree of membership of an element in a given set, intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership. As for fuzzy sets, the degree of membership is a real number between 0 and 1. This is also the case for the degree of non-membership, and furthermore the sum of these two degrees is not greater than 1. In [**?**] intuitionistic fuzzy sets are defined as follows :

Definition 1 *An intuitionistic fuzzy set in a universe E is an object of the form*

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in E\},$$

where $\mu_A(x)$ $(\in [0,1])$ *is called the "degree of membership of* x *in* A*",*
$\nu_A(x)$ $(\in [0,1])$ *is called the "degree of non-membership of* x *in* A*",*
and where μ_A *and* ν_A *satisfy the following condition :*

$$(\forall x \in E)(\mu_A(x) + \nu_A(x) \leq 1).$$

Every fuzzy set can be identified with an intuitionistic fuzzy set for which the degree of non-membership equals one minus the degree of membership.

On intuitionistic fuzzy sets analogous operators as on fuzzy sets can be defined. These operators are backwards compatible with fuzzy sets in the sense that, if applied to FSs, the fuzzy operators and their intuitionistic fuzzy counterparts give the same FS as a result. For instance, the union of two IFSs can be defined using the max-operation for the degree of membership and the min-operation for the degree of non-membership, and the result is still an IFS. Other common operators over FSs can directly be extended to IFSs, and the result of the operation is again an IFS. The set-theoretical properties that these operators establish for fuzzy sets generally still hold in the case of intuitionistic fuzzy sets. In [**?**] these operators are described and their properties are investigated.

In this paper we will discuss a generalisation of these properties using triangular norms and conorms. Triangular norms and conorms are defined as follows.

Definition 2 *A triangular norm (t-norm, for short)* T *is a* $[0,1]^2 - [0,1]$ *map which satisfies :*

(T.1) $(\forall x \in [0,1])\,(T(x,1) = x)$,

(T.2) $(\forall (x,y) \in [0,1]^2)\,(T(x,y) = T(y,x))$,

(T.3) $(\forall (x,y,z) \in [0,1]^3)\,(T(x,T(y,z)) = T(T(x,y),z))$,

(T.4) $(\forall (x_1,y_1,x_2,y_2) \in [0,1]^4)$
$(x_1 \leq y_1 \wedge x_2 \leq y_2 T(x_1,x_2) \leq T(y_1,y_2))$.

Definition 3 *A triangular conorm (t-conorm, for short)* S *is a* $[0,1]^2 - [0,1]$ *map which satisfies :*

(S.1) $(\forall x \in [0,1])\,(S(x,0) = x)$,

(S.2) $(\forall(x,y)\in[0,1]^2)\,(S(x,y)=S(y,x))$,

(S.3) $(\forall(x,y,z)\in[0,1]^3)\,(S(x,S(y,z))=S(S(x,y),z))$,

(S.4) $(\forall(x_1,y_1,x_2,y_2)\in[0,1]^4)$
$(x_1\le y_1\wedge x_2\le y_2 S(x_1,x_2)\le S(y_1,y_2))$.

Some examples of t-norms and t-conorms are :

1. min is a t-norm, max is a t-conorm;

2. Z is a norm and Z^* is a conorm, where Z and Z^* are defined by Z(x,y) $=\begin{cases}\min(x,y) & if\max(x,y)=1,\\ 0 & if\max(x,y)<1;\end{cases}$ $Z^*(x,y)=\begin{cases}\max(x,y) & if\min(x,y)=0,\\ 1 & if 0<\min(x,y).\end{cases}$ Furthermore, for each norm T and conorm S the following inequalities hold : $Z\le T\le\min$ and $\max\le S\le Z^*$.

3. $\cap_W$ is a norm and $+_b$ is a conorm, where

$$(\forall(x,y)\in[0,1]^2)\,(x\cap_W y=\max(0,x+y-1))$$
$$(\forall(x,y)\in[0,1]^2)\,(x+_b y=\min(1,x+y))$$

4. $\cdot$ is a norm and $\hat{+}$ is a conorm, where

$$(\forall(x,y)\in[0,1]^2)\,(x\cdot y=xy)$$
$$(\forall(x,y)\in[0,1]^2)\,(x\,\hat{+}\,y=x+y-xy)$$

Now we can generalise the operators on intuitionistic fuzzy sets.

Definition 4 *For two intuitionistic fuzzy sets A and B in E, we define the generalised intersection and union as :* $A\cap_{T,S}B=\{(x,T(\mu_A(x),\mu_B(x)),S(\nu_A(x),\nu_B(x))\mid x\in E\}$,$A\cup_{S,T}B=\{(x,S(\mu_A(x),\mu_B(x)),T(\nu_A(x),\nu_B(x))\mid x\in E\}$,*where T denotes a t-norm and S a t-conorm.*

For instance, the Atanassov-intersection of two intuitionistic fuzzy sets A and B can be obtained by $A\cap_{\min,\max}B$, the sum by $A\cup_{\hat{+},\cdot}B$, ...

The generalised intersection $A\cap_{T,S}B$ will be an intuitionistic fuzzy set in E if $T(\mu_A(x),\mu_B(y))+S(\nu_A(x),\nu_B(y))\le 1$. Since A and B are IFSs, we have $\nu_A(x)\le 1-\mu_A(x)$ and $\nu_B(y)\le 1-\mu_B(y)$, and because of (S.4), $S(1-\mu_A(x),1-\mu_B(y))\ge S(\nu_A(x),\nu_B(y))$, which is equivalent to

$S^*(\mu_A(x), \mu_B(y)) = 1 - S(1 - \mu_A(x), 1 - \mu_B(y)) \leq 1 - S(\nu_A(x), \nu_B(y))$, where equality holds if A and B are fuzzy sets. So, if $(\forall(x, y) \in E^2)\,(T(\mu_A(x), \mu_B(y)) \leq S^*(\mu_A(x), \mu_B(y)))$,then $A \cap_{T,S} B$ is an intuitionistic fuzzy set in E. Analogously, the generalised union $A \cup_{S,T} B$ will be an intuitionistic fuzzy set when $(\forall(x, y) \in E^2)\,(S(\mu_A(x), \mu_B(y)) \leq T^*(\mu_A(x), \mu_B(y)))$,where $T^*(x, y) = 1 - T(1 - x, 1 - y)$, the so-called t-conorm dual to the t-norm T.

2 Properties of $\cap_{T,S}$ and $\cup_{S,T}$

We will now investigate the lattice order-theoretical properties of these operators. For a certain universe E we will denote by E the set $\{(x, 1, 0) \mid x \in E\}$, and by the set $\{(x, 0, 1) \mid x \in E\}$. We note that there exists an ordering on intuitionistic fuzzy sets defined by $A \subseteq B \iff (\forall x \in E)\,(\mu_A(x) \leq \mu_B(x) \wedge \nu_A(x) \geq \nu_B(x))$. Then the following properties hold for A, B, C and D intuitionistic fuzzy sets in E.

(1) $E =$

(1') $= E$

(2) $(A) = A$

(3) $(A \cup_{S,T} B) = A \cap_{T,S} B$ Since $A \cup_{S,T} B = \{(x, S(\mu_A(x), \mu_B(x)), T(\nu_A(x), \nu_B(x)) \mid x \in E\}$, the left hand of the equality is equal to $\{(x, T(\nu_A(x), \nu_B(x)), S(\mu_A(x), \mu_B(x))) \mid x \in E\}$. Since $A = \{(x, \nu_A(x), \mu_A(x)) \mid x \in E\}$ and $B = \{(x, \nu_B(x), \mu_B(x)) \mid x \in E\}$, the right hand of the equality also equals $\{(x, T(\nu_A(x), \nu_B(x)), S(\mu_A(x), \mu_B(x))) \mid x \in E\}$.

(3') $(A \cap_{T,S} B) = A \cup_{S,T} B$

(4) $A \cup_{S,T} A \supseteq A$ (weakened idempotency law) The left hand of the inequality is equal to $\{(x, S(\mu_A(x), \mu_A(x)), T(\nu_A(x), \nu_A(x)) \mid x \in E\}$. Since for all $(a, b) \in [0, 1]^2$ there holds $\max(a, b) \leq S(a, b)$, we obtain $\mu_A(x) = \max(\mu_A(x), \mu_A(x)) \leq S(\mu_A(x), \mu_A(x))$. For all $(a, b) \in E$ there holds $T(a, b) \leq \min(a, b)$, so that we obtain $T(\mu_A(x), \mu_A(x)) \leq \min(\mu_A(x), \mu_A(x)) = \mu_A(x)$. From these two inequalities follows the property.

Suppose that equality holds, then $(\forall x \in E)\,(S(\mu_A(x), \mu_A(x)) = \mu_A(x) \wedge T(\nu_A(x), \nu_A(x)) = \nu_A(x))$. Suppose that for arbitrary $x, y \in E$, $\mu_A(x) \leq \mu_A(y)$, then $\mu_A(y) = \max(\mu_A(x), \mu_A(y)) \leq S(\mu_A(x), \mu_A(y)) \leq$

$S(\mu_A(y), \mu_A(y)) = \mu_A(y)$. We obtain $S(\mu_A(x), \mu_A(y) = \max(\mu_A(x), \mu_A(y)), \forall x, y \in E$. Similarly it follows that $T_{/\nu_A(E)\times\nu_A(E)} = \min$. Hence the union is idempotent if and only if $S_{/\mu_A(E)\times\mu_A(E)} = \max \quad \wedge \quad T_{/\nu_A(E)\times\nu_A(E)} = \min$.

(4') $A \cap_{T,S} A \subseteq A$

(5) $A \cup_{S,T} B = B \cup_{S,T} A$
$A \cap_{T,S} B = B \cap_{T,S} A$ This follows immediately from the commutativity of T and S.

(6) $(A \cup_{S,T} B) \cup_{S,T} C = A \cup_{S,T} (B \cup_{S,T} C)$ Follows easily from the associativity of T and S.

(6') $(A \cap_{T,S} B) \cap_{T,S} C = A \cap_{T,S} (B \cap_{T,S} C)$ (weakened absorption law)

(7) $A \cup_{S,T} (A \cap_{T',S'} B) \supseteq A$ An element of the set in the left hand of the inequality has the form (x, S$(\mu_A(x), T'(\mu_A(x), \mu_B(x))), T(\nu_A(x), S'(\nu_A(x), \nu_B(x))))$Since for all $(a, b) \in [0, 1]^2$ holds that $S(a, b) \geq \max(a, b) \geq a$ and $T(a, b) \leq \min(a, b) \leq a$, we obtain $S(\mu_A(x), T'(\mu_A(x), \mu_B(x))) \geq \mu_A(x)$ and $T(\nu_A(x), S'(\nu_A(x), \nu_B(x))) \leq \nu_A(x)$. So we obtain $A \cup_{S,T} (A \cap_{T',S'} B) \supseteq A$, which we had to prove.

(7') $A \cap_{T,S} (A \cup_{S',T'} B) \subseteq A$ (weakened absorption law)

(8) $A \subseteq A \cup_{S,T} B$
$B \subseteq A \cup_{S,T} B$

(8') $A \supseteq A \cap_{T,S} B$
$B \supseteq A \cap_{T,S} B$

(9) $A \cup_{S,T} = A$ Because of (T.1) and (S.1) we obtain the following equalities

$$\begin{aligned} & A \cup_{S,T} \\ &= \{(x, S(\mu_A(x), 0), T(\nu_A(x), 1)) \mid x \in E\} \\ &= \{(x, \mu_A(x), \nu_A(x)) \mid x \in E\} \\ &= A \end{aligned}$$

(9') $A \cap_{T,S} E = A$

(10) $A \cup_{S,T} E = E$

(10') $A \cap_{T,S} =$

(11) $A \cup_{\max,\min} B \subseteq A \cup_{S,T} B \subseteq A \cup_{Z^*,Z} B$,
$A \cap_{\min,\max} B \supseteq A \cap_{T,S} B \supseteq A \cup_{Z,Z^*} B$. This follows immediately from the fact that $Z \leq T \leq \min$ and $\max \leq S \leq Z^*$, for every norm T and conorm S.

(12) $A \subseteq BB \subseteq A$

(13) $A \subseteq B \wedge C \subseteq DA \cup_{S,T} C \subseteq B \cup_{S,T} D$ From $\mu_A(x) \leq \mu_B(x)$, $\mu_C(x) \leq \mu_D(x)$ and (S.4) follows that $S(\mu_A(x), \mu_C(x)) \leq S(\mu_B(x), \mu_D(x))$, and similarly from $\nu_A(x) \geq \nu_B(x)$, $\nu_C(x) \geq \nu_D(x)$ and (T.4) follows $T(\nu_A(x), \nu_C(x)) \geq T(\nu_B(x), \nu_D(x))$.

(13') $A \subseteq B \wedge C \subseteq DA \cap_{T,S} C \subseteq B \cap_{T,S} D$

(14) $A \subseteq B \iff A \cup_{S,T} B = B$ if and only if $S_{/\mu_A(E)\times\mu_B(E)} = \max$ and $T_{/\nu_A(E)\times\nu_B(E)} = \min$. If $A \cup_{S,T} B = B$, then $S(\mu_A(x), \mu_B(x)) = \mu_B(x)$ and from $\mu_B(x) \leq \max(\mu_A(x), \mu_B(x)) \leq S(\mu_A(x), \mu_B(x))$ it follows $\mu_B(x) = \max(\mu_A(x), \mu_B(x)) = S(\mu_A(x), \mu_B(x))$. Thus $\mu_A(x) \leq \mu_B(x)$, and this holds for all $x \in E$. We also have that $S_{/\mu_A(E)\times\mu_B(E)} = \max$. Analogously, from $T(\nu_A(x), \nu_B(x)) = \nu_B(x)$ and from $\nu_B(x) \geq \min(\nu_A(x), \nu_B(x)) \geq T(\nu_A(x), \nu_B(x))$ it follows that $\nu_B(x) = \min(\nu_A(x), \nu_B(x)) = T(\nu_A(x), \nu_B(x))$, and thus $\nu_A(x) \geq \nu_B(x)$, for all $x \in E$, and $T_{/\nu_A(E)\times\nu_B(E)} = \min$.

Conversely, the reverse implication holds when $S_{/\mu_A(E)\times\mu_B(E)} = \max$ and $T_{/\nu_A(E)\times\nu_B(E)} = \min$, since then from $\mu_A(x) \leq \mu_B(x)$ and from $\nu_A(x) \geq \nu_B(x)$ it follows $\mu_B(x) = \max(\mu_A(x), \mu_B(x))$ and $\nu_A(x) = \min(\nu_A(x), \nu_B(x))$. Suppose that the reverse implication also holds for $S_{/\mu_A(E)\times\mu_B(E)} \neq \max$ or $T_{/\nu_A(E)\times\nu_B(E)} \neq \min$. In the case of $S_{/\mu_A(E)\times\mu_B(E)} \neq \max$ there exists an $x \in E$ such that $S(\mu_A(x), \mu_B(x)) > \max(\mu_A(x), \mu_B(x))$. If $(\forall x \in E)\,(\mu_A(x) \leq \mu_B(x) \quad S(\mu_A(x), \mu_B(x)) = \mu_B(x) = \max(\mu_A(x), \mu_B(x)))$, then we have a contradiction. Analogously, the assumption that $T_{/\nu_A(E)\times\nu_B(E)} \neq \min$ leads to a contradiction. So we can conclude that the reverse implication holds if and only if $S_{/\mu_A(E)\times\mu_B(E)} = \max$ and $T_{/\nu_A(E)\times\nu_B(E)} = \min$.

(14') $A \subseteq B \iff A \cap_{T,S} B = A$ if and only if $T_{/\mu_A(E)\times\mu_B(E)} = \min$ and $S_{/\nu_A(E)\times\nu_B(E)} = \max$.

(15) If $S_{/\mu_A(E)\times\mu_B(E)} \leq +_b$, then the following holds :

$$A \cup_{S,T} B = E A \subseteq B$$
$$A \cap_{T,S} B = A \subseteq B$$

If $S(\mu_A(x), \mu_B(x)) \leq \min(1, \mu_A(x) + \mu_B(x))$ then

$$\begin{aligned} & S(\mu_A(x), \mu_B(x)) = 1 \\ & \quad \min(1, \mu_A(x) + \mu_B(x)) = 1 \\ & \quad \mu_A(x) + \mu_B(x) \geq 1 \\ & \quad \begin{cases} \mu_A(x) \geq 1 - \mu_B(x) \geq \nu_B(x) \\ \nu_A(x) \leq 1 - \mu_A(x) \leq \mu_B(x) \end{cases} \\ & \quad A \subseteq B \end{aligned}$$

The second statement is proved in the same way.

(15') If $T_{/\mu_A(E)\times\mu_B(E)} \leq \cap_W$, then the following holds :

$$\subset A \cap_{T,S} B A \subset B$$
$$A \cup_{S,T} B \subset E A \subset B$$

If $T(\mu_A(x), \mu_B(x)) \leq \max(1, \mu_A(x) + \mu_B(x) - 1)$ then

$$\begin{aligned} & T(\mu_A(x), \mu_B(x)) > 0 \\ & \quad \max(0, \mu_A(x) + \mu_B(x) - 1) > 0 \\ & \quad \mu_A(x) + \mu_B(x) - 1 > 0 \\ & \quad \begin{cases} \mu_A(x) > 1 - \mu_B(x) \geq \nu_B(x) \\ \nu_A(x) \leq 1 - \mu_A(x) < \mu_B(x) \end{cases} \\ & \quad A \subset B \end{aligned}$$

The second statement is proved in the same way.

(16) If $S = \max$ and $T = \min$, then $A \subseteq C \wedge B \subseteq C A \cup_{S,T} B \subseteq C$.

From $\mu_A(x) \leq \mu_C(x)$ and $\mu_B(x) \leq \mu_C(x)$ it does not follow $S(\mu_A(x), \mu_B(x)) \leq \mu_C(x)$. Consider for instance $S = Z^*$, $0 < \mu_A(x) \leq \mu_C(x) < 1$ and

$0 < \mu_B(x) \le \mu_C(x)$, then $S(\mu_A(x), \mu_B(x)) = 1 \not\le \mu_C(x)$. But if $S_{/\mu_A(E)\times\mu_B(E)} = \max$ then it follows from the given conditions that $S(\mu_A(x), \mu_B(x)) = \max(\mu_A(x), \mu_B(x)) \le \mu_C(x)$.

Similarly from $\nu_A(x) \ge \nu_C(x)$ and $\nu_B(x) \ge \nu_C(x)$ it does not follow in general that $T(\nu_A(x), \nu_B(x)) \ge \nu_C(x)$. But it does if $T_{/\nu_A(E)\times\nu_B(E)} = \min$. We conclude that the assertion holds if $S = \max$ and $T = \min$.

(16') $C \subseteq A \wedge C \subseteq BC \subseteq A \cap_{T,S} B$ if $T = \min$ and $S = \max$.

(17) $k_{0,\frac{1}{2}} \subseteq A \cup_{S,T} A$, where $k_{0,\frac{1}{2}} = \{(x, 0, \frac{1}{2}) \mid x \in E\}$ (weakened law of excluded middle).

An element of the set in the right hand of the inequality has the form $(x, S(\mu_A(x), \nu_A(x)), T(\nu_A(x), \mu_A(x)))$. Now we have $T(\nu_A(x), \mu_A(x)) \le \min(\nu_A(x), \mu_A(x)) \le \min(\nu_A(x), 1 - \nu_A(x)) \le \frac{1}{2}$, since $(\forall x \in E)\,(\mu_A(x) + \nu_A(x) \le 1)$, and since either $\nu_A(x)$ or $1 - \nu_A(x)$ is smaller than $\frac{1}{2}$ (because $\nu_A(x) \le 1$). Obviously $S(\mu_A(x), \nu_A(x)) \ge 0$ holds.

(17') $A \cap_{T,S} A \subseteq k_{\frac{1}{2},0}$, where $k_{\frac{1}{2},0} = \{(x, \frac{1}{2}, 0) \mid x \in E\}$ (weakened law of contradiction).

An element of the set in the left hand of the inequality has the form $(x, T(\mu_A(x), \nu_A(x)), S(\nu_A(x), \mu_A(x)))$. Now we have $T(\mu_A(x), \nu_A(x)) \le \min(\mu_A(x), \nu_A(x)) \le \min(\mu_A(x), 1 - \mu_A(x)) \le \frac{1}{2}$, since $(\forall x \in E)\,(\mu_A(x) + \nu_A(x) \le 1)$, and since either $\mu_A(x)$ or $1 - \mu_A(x)$ is smaller than $\frac{1}{2}$ (because $\mu_A(x) \le 1$). Obviously $S(\mu_A(x), \nu_A(x)) \ge 0$ holds.

(18) In general we cannot obtain one of the following inclusions :

$$A \cap_{T,S} (B \cap_{T',S'} C) \subseteq (A \cap_{T,S} B) \cap_{T',S'} (A \cap_{T,S} C),$$
$$A \cap_{T,S} (B \cup_{S',T'} C) \subseteq (A \cap_{T,S} B) \cup_{S',T'} (A \cap_{T,S} C),$$
$$A \cap_{T,S} (B \cap_{T',S'} C) \supseteq (A \cap_{T,S} B) \cap_{T',S'} (A \cap_{T,S} C),$$
$$A \cap_{T,S} (B \cup_{S',T'} C) \supseteq (A \cap_{T,S} B) \cup_{S',T'} (A \cap_{T,S} C),$$
$$A \cup_{S,T} (B \cap_{T',S'} C) \subseteq (A \cup_{S,T} B) \cap_{T',S'} (A \cup_{S,T} C),$$
$$A \cup_{S,T} (B \cup_{S',T'} C) \subseteq (A \cup_{S,T} B) \cup_{S',T'} (A \cup_{S,T} C),$$
$$A \cup_{S,T} (B \cap_{T',S'} C) \supseteq (A \cup_{S,T} B) \cap_{T',S'} (A \cup_{S,T} C),$$
$$A \cup_{S,T} (B \cup_{S',T'} C) \supseteq (A \cup_{S,T} B) \cup_{S',T'} (A \cup_{S,T} C),$$

Consider for the second and the fourth inequality the case where $T = Z$. Then for an arbitrary $x \in E$ such that $0 < \mu_A(x) < 1$ and such that $\mu_B(x) = \mu_C(x) = 1$, we obtain $Z(\mu_A(x),\ S'(\mu_B(x),\ \mu_C(x))) = Z(\mu_A(x), S'(1,1)) = \mu_A(x)$ and $S'(Z(\mu_A(x), \mu_B(x)), Z(\mu_A(x), \mu_C(x))) = S'(Z(\mu_A(x), 1), Z(\mu_A(x), 1)) = S'(\mu_A(x), \mu_A(x))$. Since $(\forall a \in [0,1])$ $(a \leq S'(a,a))$ and equality only holds when $S' = \max$, we obtain $Z(\mu_A(x),\ S'(1,1)) \leq S'(Z(\mu_A(x), 1),\ Z(\mu_A(x), 1))$, where the equality only holds when $S' = \max$.
For arbitrary $x \in E$ such that $0 < \mu_A(x) < \mu_B(x) < \mu_C(x) < 1$ and $S'(\mu_B(x), \mu_C(x)) = 1$, we find that $Z(\mu_A(x), S'(\mu_B(x),\ \mu_C(x))) = \mu_A(x)$ and $S'(Z(\mu_A(x), \mu_B(x)), Z(\mu_A(x), \mu_C(x))) = 0$. Under the given conditions, we thus find that $Z(\mu_A(x), S'(\mu_B(x), \mu_C(x))) > S'(Z(\mu_A(x), \mu_B(x)),\ Z(\mu_A(x), \mu_C(x)))$.
This excludes the second and the fourth inequality in the most general case.

Analogous examples show that the other inequalities don't hold either in the general case.

(19)
$$A = 0pt\alpha \in]0,1]\beta \in [0,1[\bigcup(\alpha,\beta)A_{\alpha}^{\beta} \quad = 0pt\alpha \in (\mu_A) \setminus \{0\}\beta \in ($$
$$A = 0pt\alpha \in [0,1[\beta \in]0,1]\bigcup(\alpha,\beta)A_{\overline{\alpha}}^{\overline{\beta}} \quad \supseteq 0pt\alpha \in (\mu_A) \setminus \{1\}\beta \in ($$

where
$A_{\alpha}^{\beta} = \{x \mid x \in E \wedge \mu_A(x) \geq \alpha \wedge \nu_A(x) \leq \beta\}, \forall\alpha \in]0,1], \forall\beta \in [0,1[;$
$A_{\overline{\alpha}}^{\overline{\beta}} = \{x \mid x \in E \wedge \mu_A(x) > \alpha \wedge \nu_A(x) < \beta\}, \forall\alpha \in [0,1[, \forall\beta \in]0,1];$
$(\alpha,\beta)A_{\alpha}^{\beta} = \{(x, \mu_{(\alpha,\beta)A_{\alpha}^{\beta}}(x), \nu_{(\alpha,\beta)A_{\alpha}^{\beta}}(x)) \mid x \in E\}$ and
$(\alpha,\beta)A_{\overline{\alpha}}^{\overline{\beta}} = \{(x, \mu_{(\alpha,\beta)A_{\overline{\alpha}}^{\overline{\beta}}}(x), \nu_{(\alpha,\beta)A_{\overline{\alpha}}^{\overline{\beta}}}(x)) \mid x \in E\}$
with

$$\mu_{(\alpha,\beta)A_{\alpha}^{\beta}}(x) = \begin{cases} \alpha & if \quad x \in A_{\alpha}^{\beta} \\ 0 & if \quad x \in E \setminus A_{\alpha}^{\beta} \end{cases}$$
$$\nu_{(\alpha,\beta)A_{\alpha}^{\beta}}(x) = \begin{cases} \beta & if \quad x \in A_{\alpha}^{\beta} \\ 1 & if \quad x \in E \setminus A_{\alpha}^{\beta} \end{cases}$$

and

$$\mu_{(\alpha,\beta)A_{\overline{\alpha}}^{\overline{\beta}}}(x) = \begin{cases} \alpha & if \quad x \in A_{\overline{\alpha}}^{\overline{\beta}} \\ 0 & if \quad x \in E \setminus A_{\overline{\alpha}}^{\overline{\beta}} \end{cases}$$

$$\nu_{(\alpha,\beta)A^{\overline{\beta}}_{\overline{\alpha}}}(x) = \begin{cases} \beta & if \quad x \in A^{\overline{\beta}}_{\overline{\alpha}} \\ 1 & if \quad x \in E \setminus A^{\overline{\beta}}_{\overline{\alpha}} \end{cases}$$

$$\begin{aligned} 0pt\alpha \in]0,1]\beta \in [0,1[\bigcup(\alpha,\beta)A^{\beta}_{\alpha} &= 0pt\alpha \in]0,1]\beta \in [0,1[\bigcup\{(x, \mu_{(\alpha,\beta)A^{\beta}_{\alpha}}(x), \nu_{(\alpha,\beta)A^{\beta}_{\alpha}}(x)) \mid x \in E\} \\ &= \{(x, \sup\{\mu_{(\alpha,\beta)A^{\beta}_{\alpha}}(x) \mid \alpha \in]0,1] \wedge \beta \in [0,1[\}, \\ &\quad \inf\{\nu_{(\alpha,\beta)A^{\beta}_{\alpha}}(x) \mid \alpha \in]0,1] \wedge \beta \in [0,1[\}) \mid x \in E\} \end{aligned}$$

Consider for instance the degree of non-membership (the calculations are analogous for the degree of membership). We obtain

$$\begin{aligned} &\inf\{\nu_{(\alpha,\beta)A^{\beta}_{\alpha}}(x) \mid \alpha \in]0,1] \wedge \beta \in [0,1[\} \\ &= \inf\{\nu_{(\alpha,\beta)A^{\beta}_{\alpha}}(x) \mid \alpha \in]0,1] \wedge \beta \in [0,1[\wedge x \in A^{\beta}_{\alpha}\} \\ &= \inf\{\nu_{(\alpha,\beta)A^{\beta}_{\alpha}}(x) \mid \alpha \in]0,1] \wedge \beta \in [0,1[\wedge \mu_A(x) \geq \alpha \wedge \nu_A(x) \leq \beta\} \\ &= \inf\{\beta \mid \alpha \in]0,1] \wedge \beta \in [0,1[\wedge \mu_A(x) \geq \alpha \wedge \nu_A(x) \leq \beta\} \\ &= \nu_A(x) \end{aligned}$$

So we obtain 0pt$\alpha \in]0,1]\beta \in [0,1[\bigcup(\alpha,\beta)A^{\beta}_{\alpha} = \{(x, \mu_A(x), \nu_A(x)) \mid x \in E\} = A$The other proofs are analogous.

3 Conclusion

We see that most of the set-theoretical properties that hold for the special cases of $\cap_{T,S}$ and $\cup_{S,T}$ still hold for intuitionistic fuzzy sets, in some cases under a slightly modified form. However, in the general case, the law of the excluded middle survives only in a very weak form, and the distributivity laws don't hold at all. The characterisation of $A \subseteq B$ by $A \cup_{S,T} B = B$ or by $A \cap_{T,S} B = A$ only holds if $T = \min$ and $S = \max$.

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