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Another definition of symmetrical difference over intuitionistic fuzzy sets

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Abstract: In the present paper, a new way to define symmetrical difference generated by union, intersection and negation over intuitionistic fuzzy sets are introduced, and its related properties are also investigated.
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1 Introduction

Since the seminal work of Zadeh [13], the fuzzy set theory characterized by a membership function between 0 and 1 serves as a useful tool to treat imprecision and uncertainty. In 1986, Atanassov [2] introduced the notion of intuitionistic fuzzy sets (IFS) as an extension of fuzzy set, in which not only the degree of membership is given, but also the degree of non-membership degree. Since then, the IFS theory has achieved great success in various areas such as approximate reasoning [7], pattern recognition [11], decision-making [12], medical diagnosis [8], etc.

Another primary extensions of the conventional fuzzy set theory is interval-valued fuzzy set (IVFS) conceived also by Zadeh [14]. It has been proved by Deschrijver and Kerre [9] that IVFS theory is equivalent to IFS theory. The concept of vague set presented by Gau [10] is another extension of ordinary fuzzy set. But Bustince and Burillo [6] showed that vague sets are intuitionistic fuzzy sets.

With the development of IFS theory, a number of operations (denoted by $\cap, \cup, \bar{}\ , -, \mapsto$) and relations (such as $\subseteq, =$) over intuitionistic fuzzy sets have been proposed as well. Antonov [1] defined the mathematical symmetrical difference operation $\div$ over IFSs through three basic operations (union, intersection, negation), and some interesting properties are studied.

This paper is organized as follows: In Section 2, we recall basic concepts of intuitionistic fuzzy sets and operations over intuitionistic fuzzy sets. In Section 3, a new definition of symmetrical difference over intuitionistic fuzzy sets is introduced, while its basic properties are given in Section 4.

2 Preliminaries

In this section, we briefly recall some basic definitions relating to IFSs and operations and relations over IFSs.

Definition 1. [2–5] Let a (crisp) set *E* be fixed and let $A \subseteq E$ be a fixed set. An Atanassov's intuitionistic fuzzy set (IFS, for short) A^* in *E* is an object of the following form

$$A^* = \left\{ \langle x, \mu_A(x), \upsilon_A(x) \rangle \middle| x \in E \right\},$$

where functions $\mu_A(x): E \to [0,1]$ and $\upsilon_A(x): E \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in E$ to the set *A*, respectively, and for every $x \in E$

$$0 \le \mu_A(x) + \upsilon_A(x) \le 1.$$

The function $\pi_A(x): E \to [0,1]$, which is given by

$$\pi_A(x) = 1 - \mu_A(x) - \upsilon_A(x)$$

defines the degree of uncertainty (non-determinacy) of the membership of the element $x \in E$ to the set *A*.

For simplicity, below we write *A* instead of A^*.

Let, for every IFS *A*,

$$\overline{A} = \left\{ \langle x, \upsilon_A(x), \mu_A(x) \rangle \middle| x \in E \right\},$$

$$\Box A = \left\{ \langle x, \mu_A(x) \rangle \middle| x \in E \right\} = \left\{ \langle x, \mu_A(x), 1 - \mu_A(x) \rangle \middle| x \in E \right\},$$

$$\Diamond A = \left\{ \langle x, 1 - \upsilon_A(x) \rangle \middle| x \in E \right\} = \left\{ \langle x, 1 - \upsilon_A(x), \upsilon_A(x) \rangle \middle| x \in E \right\}.$$

The negation of *A* denoted $\overline{A}$ just interchanges the membership and non-membership components. The other two operations $\Box A$ and $\Diamond A$, which transform each IFS to FS, are similar to operations 'necessity' and 'possibility' defined in model logic.

Let *A* and *B* be two IFSs given as

$$A = \left\{ \langle x, \mu_A(x), \upsilon_A(x) \rangle \middle| x \in E \right\}$$

and

$$B = \left\{ \langle x, \mu_B(x), \upsilon_B(x) \rangle \middle| x \in E \right\},$$

the following operations and relations can be defined:

$$A \cap B = \left\{ \langle x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\upsilon_A(x), \upsilon_B(x) \rangle \middle| x \in E \right\};$$

$$A \cup B = \left\{ \langle x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\upsilon_A(x), \upsilon_B(x) \rangle \middle| x \in E \right\};$$

$$A \subseteq B \text{ iff } (\forall x \in E)\ (\mu_A(x) \le \mu_B(x) \,\&\, \upsilon_A(x) \ge \upsilon_B(x));$$

$$A = B \text{ iff } (\forall x \in E)\ (\mu_A(x) = \mu_B(x) \,\&\, \upsilon_A(x) = \upsilon_B(x)).$$

We certainly have, by the notations above, $A \subseteq B$ iff $A \cap B = A$ iff $A \cup B = B$.

Difference and implication over ordinary sets are defined through union, intersection and negation over ordinary sets. Likewise, the corresponding operations over IFS can be defined as follows:

$$A-B=A\cap\overline{B}=\left\{\langle x,\min\{\mu_A(x),\upsilon_B(x)\},\max\{\upsilon_A(x),\mu_B(x)\rangle\middle|x\in E\right\},$$
$$A\mapsto B=\overline{A}\cup B=\left\{\langle x,\max\{\upsilon_A(x),\mu_B(x)\},\min\{\mu_A(x),\upsilon_B(x)\rangle\middle|x\in E\right\}.$$

Let us define the empty IFS, the totally uncertain IFS, and the unit IFS [4] respectively by:

$$O^*=\left\{\langle x,0,1\rangle\middle|x\in E\right\},$$
$$E^*=\left\{\langle x,1,0\rangle\middle|x\in E\right\},$$
$$U^*=\left\{\langle x,0,0\rangle\middle|x\in E\right\}.$$

Definition 2. [3] A set of (α,β)-cut, generated by an IFS A, where $\alpha,\beta\in[0,1]$ are fixed numbers such that $\alpha+\beta\le 1$ is defined as

$$A_{\alpha,\beta}=\left\{\langle x,\mu_A(x),\upsilon_A(x)\rangle\middle|x\in E\ \&\ \mu_A(x)\ge\alpha\ \&\ \upsilon_A(x)\le\beta\right\}.$$

3 Another definition of symmetrical difference over intuitionistic fuzzy sets

In this section, a new way different from Antonov's is presented to define the symmetrical difference operation over intuitionistic fuzzy sets.

In classical sets theory, symmetrical difference could be expressed at least two ways as follows: $A\triangle B=(A-B)\cup(B-A)$ and $A\triangle B=(A\cup B)-(A\cap B)$.

By employing the first equation, Antonov introduced the symmetrical difference operation, denoted by $\div$, in the following form:

$$A\div B=\left\{\left\langle x,\max\left\{\min\{\mu_A(x),\upsilon_B(x)\},\min\{\upsilon_A(x),\mu_B(x)\}\right\},\min\left\{\max\{\upsilon_A(x),\mu_B(x)\},\max\{\mu_A(x),\upsilon_B(x)\}\right\}\right\rangle\middle|x\in E\right\}$$

for two IFSs *A* and *B*.

Similarly, by using the second equality, another way to define symmetrical difference operation can be shown as:

$$A\triangle B=\left\{\left\langle x,\min\left\{\max\{\mu_A(x),\mu_B(x)\},\max\{\upsilon_A(x),\upsilon_B(x)\}\right\},\max\left\{\min\{\mu_A(x),\mu_B(x)\},\min\{\upsilon_A(x),\upsilon_B(x)\}\right\}\right\rangle\middle|x\in E\right\}$$

for two IFSs *A* and *B*.

Note that $A\triangle B$ is an IFS, in fact, it suffice to verify that the sum of the membership and non-membership of $A\triangle B$ is not greater than 1.

$$\min\left\{\max\{\mu_A(x),\mu_B(x)\},\max\{\upsilon_A(x),\upsilon_B(x)\}\right\}+\max\left\{\min\{\mu_A(x),\mu_B(x)\},\min\{\upsilon_A(x),\upsilon_B(x)\}\right\}$$
$$\le\min\left\{\max\{\mu_A(x),\mu_B(x)\},\max\{\upsilon_A(x),\upsilon_B(x)\}\right\}+\max\left\{\min\{1-\upsilon_A(x),1-\upsilon_B(x)\},\min\{1-\mu_A(x),1-\mu_B(x)\}\right\}$$
$$=\min\left\{\max\{\mu_A(x),\mu_B(x)\},\max\{\upsilon_A(x),\upsilon_B(x)\}\right\}+\left(1-\min\left\{\max\{\upsilon_A(x),\upsilon_B(x)\},\max\{\mu_A(x),\mu_B(x)\}\right\}\right)$$
$$=1.$$

From the fact that $\min\left\{\max\{\mu_A(x),\mu_B(x)\},\max\{\upsilon_A(x),\upsilon_B(x)\}\right\}\ge\max\left\{\min\{\mu_A(x),\mu_B(x)\},\right.$ $\left.\min\{\upsilon_A(x),\upsilon_B(x)\right\}$, we have $\mu_{A\triangle B}(x)\ge\upsilon_{A\triangle B}(x)$ for all $x\in E$. Consequently, we conclude that $A\triangle B$ is an intuitionistic fuzzy tautological set [5].

The following equation reveals the inclusion relationship between $A\div B$ and $A\triangle B$:

$$A\div B\subseteq A\triangle B.$$

In fact, we have

$$A\triangle B=(A\bigcup B)-(A\bigcap B)=(A\bigcup B)\bigcap(\overline{A}\bigcup\overline{B})=(A\bigcap\overline{A})\bigcup(A\bigcap\overline{B})\bigcup(B\bigcap\overline{A})\bigcup(B\bigcap\overline{B})$$
$$\supseteq(A\bigcap\overline{B})\bigcup(B\bigcap\overline{A})=A\div B.$$

Specially, if both A and B degenerate into fuzzy sets, then the membership function of $A\triangle B$ is:

$$\mu_{A\triangle B}(x)=\min\left\{\max\{\mu_A(x),\mu_B(x)\},1-\min\{\mu_A(x),\mu_B(x)\}\right\}.$$

For special IFSs E^*,O^*,U^*, we have: for each IFS A,

$$A\triangle E^*=\overline{A};$$
$$A\triangle O^*=A;$$
$$A\triangle U^*=\Box(A\bigcap\overline{A})\bigcup O^*=\left\{\left\langle x,\min\{\mu_A(x),\upsilon_A(x)\},0\right\rangle\middle|x\in E\right\}.$$

4 Properties of symmetrical difference over IFS

In this section, some basic properties of symmetrical difference operation are established. Let $IFS(E)$ denote the family of all IFSs in the universe E.

By the definition in Section 3, immediately, we have the following theorem.

Theorem 1. For $\forall A,B\in IFS(E)$, the following properties are valid:

(1) $A\triangle B=B\triangle A$,

(2) $A\triangle A=A\bigcap\overline{A}$.

Lemma 1. [4, 5] For $\forall A,B\in IFS(E)$, the following assertions are valid:

(1) $\overline{\overline{A}}=A$,

(2) $\overline{A\bigcap B}=\overline{A}\bigcup\overline{B}$,

(3) $\overline{A\bigcup B}=\overline{A}\bigcap\overline{B}$.

Theorem 2. For $\forall A,B\in IFS(E)$, we have

(1) $A\triangle\overline{A}=A\bigcup\overline{A}$,

(2) $A\triangle B=\overline{A}\triangle\overline{B}$,

(3) $\overline{A\triangle B}=(A\bigcap B)\bigcup(\overline{A}\bigcap\overline{B})$,

(4) $A\triangle B=\overline{A}\mapsto B\bigcap A\mapsto\overline{B}$.

Proof: (1) $A\triangle\overline{A}=(A\bigcup\overline{A})-(A\bigcap\overline{A})=(A\bigcup\overline{A})\bigcap\overline{(A\bigcap\overline{A})}=(A\bigcup\overline{A})\bigcap(\overline{A}\bigcup A)=A\bigcup\overline{A}$.

(2) $\overline{A}\triangle\overline{B}=(\overline{A}\bigcup\overline{B})-(\overline{A}\bigcap\overline{B})=\overline{(\overline{A}\bigcap\overline{B})}\bigcap\overline{(A\bigcap B)}=(A\bigcup B)-(A\bigcap B)=A\triangle B$.

(3) $(A\bigcap B)\bigcup(\overline{A}\bigcap\overline{B})=(A\bigcap B)\bigcup\overline{(A\bigcup B)}=\overline{(A\bigcup B)\bigcap\overline{(A\bigcap B)}}=\overline{A\triangle B}$.

(4) $\overline{A}\mapsto B\bigcap A\mapsto\overline{B}=(\overline{\overline{A}}\bigcup B)\bigcap(\overline{A}\bigcup\overline{B})=(A\bigcup B)\bigcap\overline{(A\bigcap B)}=(A\bigcup B)-(A\bigcap B)=A\triangle B$.

Theorem 3. For $\forall A,B\in IFS(E)$, the following holds:

$\Box(A\triangle B)\subseteq\Box A\triangle\Box B$,

$\Diamond(A\triangle B)\supseteq\Diamond A\triangle\Diamond B$.

Proof: $\Box(A \triangle B) = \left\{\left\langle x, \min\left\{\max\{\mu_A(x), \mu_B(x)\}, \max\{\upsilon_A(x), \upsilon_B(x)\}\right\}\right\rangle \middle| x \in E\right\}$.

$$\Box A \triangle \Box B = \left\{\left\langle x, \mu_A(x)\right\rangle \middle| x \in E\right\} \triangle \left\{\left\langle x, \mu_B(x)\right\rangle \middle| x \in E\right\}$$
$$= \left\{\left\langle x, \min\left\{\max\{\mu_A(x), \mu_B(x)\}, 1-\min\{\mu_A(x), \mu_B(x)\}\right\}\right\rangle \middle| x \in E\right\}$$
$$= \left\{\left\langle x, \min\left\{\max\{\mu_A(x), \mu_B(x)\}, \max\{1-\mu_A(x), 1-\mu_B(x)\}\right\}\right\rangle \middle| x \in E\right\}$$
$$\geq \left\{\left\langle x, \min\left\{\max\{\mu_A(x), \mu_B(x)\}, \max\{\upsilon_A(x), \upsilon_B(x)\}\right\}\right\rangle \middle| x \in E\right\}.$$

The proof of the next assertion is analogous.

Definition 3. A set of strong-(α, β)-cut, generated by an IFS A, where $\alpha, \beta \in [0,1]$ are fixed numbers such that $\alpha + \beta \leq 1$ is defined as

$$A_{\underline{\underline{\alpha,\beta}}} = \left\{\left\langle x, \mu_A(x), \upsilon_A(x)\right\rangle \middle| x \in E \,\&\, \min\{\mu_A(x), \upsilon_A(x)\} \geq \alpha \,\&\, \max\{\mu_A(x), \upsilon_A(x)\} \leq \beta\right\}.$$

Apparently, we have

$$A_{\underline{\underline{0,1}}} = A \text{ and } A_{\underline{\underline{\alpha,\beta}}} \subseteq A_{\alpha,\beta}.$$

Here we shall introduce two new notations, related to above mentioned one ($\alpha \in [0,1]$ is a fixed number):

$$A_{\underline{\underline{\alpha}}} = \left\{\left\langle x, \mu_A(x), \upsilon_A(x)\right\rangle \middle| x \in E \,\&\, \min\{\mu_A(x), \upsilon_A(x)\} \geq \alpha\right\},$$
$$A^{\overline{\overline{\alpha}}} = \left\{\left\langle x, \mu_A(x), \upsilon_A(x)\right\rangle \middle| x \in E \,\&\, \max\{\mu_A(x), \upsilon_A(x)\} \leq \alpha\right\}.$$

For every IFS A and for every $\alpha, \beta \in [0,1]$:

$$A_{\underline{\underline{\alpha,\beta}}} = A_{\underline{\underline{\alpha}}} \cap A^{\overline{\overline{\beta}}}.$$

Let

$$E_1 = \left\{A_{\underline{\underline{\alpha,\beta}}} \middle| A \subseteq E \,\&\, \alpha, \beta, \alpha+\beta \in [0,1]\right\}$$
$$E_2 = \left\{A_{\underline{\underline{\alpha}}} \middle| A \subseteq E \,\&\, \alpha \in [0,1]\right\},$$
$$E_3 = \left\{A^{\overline{\overline{\alpha}}} \middle| A \subseteq E \,\&\, \alpha \in [0,1]\right\}.$$

If $B, C \in E_i$, then $B \cap C \in E_i$, for $i = 1, 2, 3$.

Theorem 4. For $\forall A, B \in IFS(E)$, we have:

$$(A \triangle B)_{\alpha,\beta} \supseteq A_{\underline{\underline{\alpha,\beta}}} \triangle B_{\underline{\underline{\alpha,\beta}}}.$$

Proof:

$$\min\{\mu_A(x), \upsilon_A(x)\} \geq \alpha, \max\{\mu_A(x), \upsilon_A(x)\} \leq \beta, \min\{\mu_B(x), \upsilon_B(x)\} \geq \alpha, \max\{\mu_B(x), \upsilon_B(x)\} \leq \beta$$
$$\Rightarrow \max\{\mu_A(x), \mu_B(x)\} \geq \alpha, \max\{\upsilon_A(x), \upsilon_B(x)\} \geq \alpha, \min\{\mu_A(x), \mu_B(x)\} \leq \beta, \min\{\upsilon_A(x), \upsilon_B(x)\} \leq \beta$$
$$\Leftrightarrow \min\left\{\max\{\mu_A(x), \mu_B(x)\}, \max\{\upsilon_A(x), \upsilon_B(x)\}\right\} \geq \alpha, \max\left\{\min\{\mu_A(x), \mu_B(x)\}, \min\{\upsilon_A(x), \upsilon_B(x)\}\right\} \leq \beta.$$

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References

[1] Antonov, A. Symmetrical difference over intuitionistic fuzzy sets, *Notes on Intuitionistic Fuzzy Sets*, Vol. 10, 2004, No. 3, 33–36.

[2] Atanassov, K. Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, Vol. 20, 1986, 87–96.

[3] Atanassov, K. More on intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, Vol. 33, 1989, 37–46.

[4] Atanassov, K. *Intuitionistic Fuzzy Sets: Theory and Applications*, Springer, Heidelberg, 1999.

[5] Atanassov, K. *On Intuitionistic Fuzzy Sets Theory*, Springer, Berlin, 2012.

[6] Bustince, H., Burillo, P. Vague sets are intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, Vol. 79, 1996, 403–405.

[7] Bustince, H., Burillo, P., Mohedano, V. A method for inference in approximate reasoning based on normal intuitionistic fuzzy sets, *Notes on Intuitionistic Fuzzy Sets*, Vol. 1, 1995, No. 1, 51–55.

[8] De, S. K., R., Biswas, A. R. Roy. An application of intuitionistic fuzzy sets in medical diagnosis. *Fuzzy Sets and Systems*, Vol. 117, 2001, 209–213.

[9] Deschrijver, D., E. E. Kerre. On the relationship between some extensions of fuzzy set theory, *Fuzzy Sets and Systems*, Vol. 133, 2003, 227–235.

[10] Gau, W. L., D. J. Buehrer. Vague sets, *IEEE Transactions on Systems, Man and Cybernetics*, Vol. 23,1993, 610–614.

[11] Li, D., C. Chen. New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions, *Pattern Recognition Letters*, Vol. 23, 2002, 221–225.

[12] Xu, Z. Some similarity measures of intuitionistic fuzzy sets and their applications to multiple attribute decision making, *Fuzzy Optimization and Decision Making*, Vol. 6, 2007, 109–121.

[13] Zadeh, L. A. Fuzzy sets, *Information and Control*, Vol. 8, 1965, 338–353.

[14] Zadeh, L. A. The concept of a linguistic variable and its application to approximate reasoning, part I [J]. *Information Sciences*, Vol. 8, 1975, 199–249.