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On intuitionistic L-fuzzy socle of modules

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Abstract: In this paper we try to study the intuitionistic L-fuzzy aspects of socle of modules over rings. We demonstrate some properties of a socle of intuitionistic L-fuzzy submodules and their relations with intuitionistic L-fuzzy essential submodules and a family of intuitionistic L-fuzzy complemented submodules of a module. Some related results are also established.
Keywords: Intuitionistic L-fuzzy submodule, Intuitionistic L-fuzzy simple submodule, Intuitionistic L-fuzzy essential submodule, Socle of an intuitionistic L-fuzzy submodule.
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1 Introduction

Let M be a unitary module over a commutative ring R with zero element θ. Recall that a submodule K of an R-module M is called an essential submodule of M denoted by $K \trianglelefteq_e M$, if for every submodule N of M, $K \cap N = \{\theta\}$ implies that $N = \{\theta\}$. Equivalently, $K \cap N \neq \{\theta\}$ for all non-zero submodules N of M. A submodule K of a module M is called complement for a submodule N of M if it is maximal with respect to the property that $K \cap N = \{\theta\}$. The socle of an R-module M is denoted by $\mathrm{Soc}(M)$ and is defined as the sum of all simple submodules of M, i.e., the socle of M is the largest submodule of M generated by simple modules. For more information about essential submodules, complement of a submodule and socle of a module, we refer to [1, 8, 16].

Atanassov and Stoeva [3] generalized the notion of L-fuzzy subsets given by Goguen [6] to an intuitionistic L-fuzzy subset, where L is any complete lattice with a complete order reversing involution N. Wang and He in [15] and Deschrijver and Kerre in [5] studied the relationship between intuitionistic fuzzy sets and L-fuzzy sets and some extensions of fuzzy set theory. Palaniappan and others in [10] studied intuitionistic L-fuzzy subgroups. Meena and Thomas in [9] discussed the notion of intuitionistic L-fuzzy subrings. Sharma et al. [7, 11, 12] discussed intuitionistic L-fuzzy submodules, intuitionistic L-fuzzy prime and primary submodule of a module. The notions like intuitionistic L-fuzzy essential submodule, intuitionistic L-fuzzy complement of a submodule and intuitionistic L-fuzzy simple submodule were studied by the author et al. in [13] and [14].

In this paper, our attempt is to investigate the intuitionistic L-fuzzy aspects of a socle of a module. Using the concepts of intuitionistic L-fuzzy essentiality and relative complement defined by the author et al. in [13], it is proved that if A is an intuitionistic L-fuzzy submodule of M such that $A = \mathrm{Soc}(A)$, then A has no proper intuitionistic L-fuzzy essential submodules. Further, if $A = \mathrm{Soc}(A)$ and $IF_L(A)$ (the family of intuitionistic L-fuzzy submodules of A) is complemented, then for any $C \in IF_L(A)$, $IF_L(C)$ is also intuitionistic L-fuzzy complemented. It is shown that if E is the intersection of all intuitionistic L-fuzzy essential submodules of A, where A is an intuitionistic L-fuzzy submodule of M, then every non-zero intuitionistic L-fuzzy submodule of E contains a simple intuitionistic L-fuzzy submodule of E. It leads us to the result that $\mathrm{Soc}(A) = E$. Apart of these results we have also evaluated the socle of a direct sum of intuitionistic L-fuzzy submodules.

2 Preliminaries

Throughout this paper, R is a commutative ring with identity, M a unitary R-module and L stands for a complete lattice with least element 0 and greatest element 1, θ denotes the zero element of M. The lattice L is called regular if $a \wedge b \neq 0$ for every $a \neq 0, b \neq 0$ and $a \vee b \neq 1$ for every $a \neq 1, b \neq 1$ (see [4]).

Definition 2.1. [7] *Let $(L, \leq)$ be a complete lattice with an evaluative order reversing operation $N : L \to L$. Let X be a non-empty set. An intuitionistic L-fuzzy set A in X is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where $\mu_A : X \to L$ and $\nu_A : X \to L$ define respectively the degree of membership and the degree of non-membership for every $x \in X$ satisfying $\mu_A(x) \leq N(\nu_A(x))$. A complete order reversing involution is a mapping $N : L \to L$ such that*

(i) $N(0) = 1$ *and* $N(1) = 0$*;*

(ii) If $\alpha \leq \beta$*, then* $N(\beta) \leq N(\alpha)$*;*

(iii) $N(N(\alpha)) = \alpha$ *;*

(iv) $N(\vee_{i=1}^{n}\alpha_i) = \wedge_{i=1}^{n}N(\alpha_i)$ *and* $N(\wedge_{i=1}^{n}\alpha_i) = \vee_{i=1}^{n}N(\alpha_i)$.

We also denote an intuitionistic L-fuzzy set simply by $ILFS$ and the set of all $ILFS's$ on X by $ILFS(X)$.

Remark 2.2. *When $\mu_A(x) = N(\nu_A(x))$, for all $x \in X$, then A is called L-fuzzy set. We use the notion $A = (\mu_A, \nu_A)$ to denote the intuitionistic L-fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$.*

For $A, B \in ILFS(X)$ we say that $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$. Also, $A \subset B$ if and only if $A \subseteq B$ and $A \neq B$.

For $A \in ILFS(X)$ and $\alpha, \beta \in L$ with $\alpha \leq N(\beta)$, in analogy with the operator $A_{(\alpha,\beta)}$ defined by Atanassov for intuitionistic fuzzy sets in [2], we define here

$$A_{(\alpha,\beta)} = \{x \in X : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}.$$

Then $A_{(\alpha,\beta)}$ is called the (α,β)-cut set of A. The support of an $ILFS$ A is denoted by A^* and is defined as

$$A^* = \{x \in X : \mu_A(x) > 0, \nu_A(x) < 1\}.$$

Definition 2.3. [11] *Let $A = (\mu_A, \nu_A)$ be an ILFS of X and $Y \subseteq X$. Then the intuitionistic L-fuzzy characteristic function $\chi_Y = (\mu_{\chi_Y}, \nu_{\chi_Y})$ on Y is defined as*

$$\mu_{\chi_Y}(y) = \begin{cases} 1, & \textit{if } y \in Y \\ 0, & \textit{otherwise} \end{cases}; \quad \nu_{\chi_Y}(y) = \begin{cases} 0, & \textit{if } y \in Y \\ 1, & \textit{otherwise}. \end{cases}$$

Definition 2.4. [7, 11] *Let $A \in ILFS(M)$. Then A is called an intuitionistic L-fuzzy module ($ILFM$) of M if for all $x, y \in M, r \in R$, the following statements are satisfied:*

(i) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$;

(ii) $\mu_A(rx) \geq \mu_A(x)$;

(iii) $\mu_A(\theta) = 1$;

(iv) $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$;

(v) $\nu_A(rx) \leq \nu_A(x)$;

(vi) $\nu_A(\theta) = 0$.

The collection of all intuitionistic L-fuzzy modules of M is denoted by $IF_L(M)$. If $A, B \in IF_L(M)$ such that $B \subseteq A$, then B is called an intuitionistic L-fuzzy submodule of A. If L is regular and $A, B \in IF_L(M)$, then A^*, B^* are submodules of M. Further we see that $(A + B)^* = A^* + B^*$ and $(A \cap B)^* = A^* \cap B^*$ (see [7]). Also, $A^* = \{\theta\}$ if and only if $A = \chi_{\{\theta\}}$ (see [13]). If $A, B \in IF_L(M)$, then the direct sum of A and B is $A + B$ provided $A \cap B = \chi_{\{\theta\}}$, and this is denoted by $A \oplus B$. If $A, B, C \in IF_L(M)$ be such that $C = A \oplus B$, then A, B are called intuitionistic L-fuzzy direct summands of C.

Definition 2.5. [13] *Let M be an R-module and $A, C \in IF_L(M)$ be such that $\chi_{\{\theta\}} \neq C \subseteq A$. Then C is called an intuitionistic L-fuzzy essential submodule of A if $C \cap B \neq \chi_{\{\theta\}}$, $\forall B \in IF_L(M)$ such that $\chi_{\{\theta\}} \neq B \subseteq A$. We denote this by writing $C \trianglelefteq_e A$.*

In particular, when $A = \chi_M$. Then C is called an intuitionistic L-fuzzy essential submodule of M, written as $C \trianglelefteq_e \chi_M$ or $C \trianglelefteq_e M$, if $C \cap B \neq \chi_{\{\theta\}}$, $\forall B \neq \chi_{\{\theta\}} \in IF_L(M)$.

Proposition 2.6. [13] *Let M be an R-module and $A, C \in IF_L(M)$ be such that $C \trianglelefteq_e A$. Then $C^* \trianglelefteq_e A^*$, but the converse is true when L is regular.*

Theorem 2.7. [13] *Let $A, B, C \in IF_L(M)$ be such that $C \subseteq B \subseteq A$. Then $C \trianglelefteq_e A$ if and only if $C \trianglelefteq_e B$ and $B \trianglelefteq_e A$.*

Theorem 2.8. [13] *Let $C_1, C_2, A_1, A_2 \in IF_L(M)$. If $C_1 \trianglelefteq_e A_1$ and $C_2 \trianglelefteq_e A_2$, then $C_1 \cap C_2 \trianglelefteq_e A_1 \cap A_2$.*

Corollary 2.9. [13] *Let $C_1, C_2, A \in IF_L(M)$. If $C_1 \trianglelefteq_e A$ and $C_2 \trianglelefteq_e A$, then $C_1 \cap C_2 \trianglelefteq_e A$.*

Theorem 2.10. [13] *Let L be a regular lattice and $C_1, C_2, A_1, A_2 \in IF_L(M)$. If $C_i \trianglelefteq_e A_i$, $i = 1, 2$. If $C_1 \cap C_2 = \chi_{\{\theta\}}$, then $A_1 \cap A_2 = \chi_{\{\theta\}}$ and $C_1 \oplus C_2 \trianglelefteq_e A_1 \oplus A_2$.*

Corollary 2.11. [13] *Let L be a regular lattice and $C_1, C_2, A \in IF_L(M)$. If $C_i \trianglelefteq_e A$, $i = 1, 2$. If $C_1 \cap C_2 = \chi_{\{\theta\}}$, then $C_1 \oplus C_2 \trianglelefteq_e A$.*

Theorem 2.12. [13] *Let L be a regular lattice and $C, A \in IF_L(M)$ where $C \subseteq A$. Let $f : N \to M$ be a module homomorphism such that $f(B) \subseteq A$ where $B \in IF_L(N)$. If $C \trianglelefteq_e A$, then $f^{-1}(C) \trianglelefteq_e B$.*

Definition 2.13. [13] *Let M be an R-module and $A, B, C \in IF_L(M)$ be such that $B \subseteq A$. Then C is called an intuitionistic L-fuzzy complement of B in A if $C \subseteq A$ and C is maximal with the property that $B \cap C = \chi_{\{\theta\}}$. We write C is complement of B in A.*

Theorem 2.14. [13] *Let L be a regular lattice and M be an R-module. If C is complement of B in A, then C^* is complement of B^* in A^*.*

Remark 2.15. [13] *The converse of the above theorem is not true. If for any $A, B, C \in IF_L(M)$ the submodule C^* is complement of B^* in A^*, then C need not be complement of B in A.*

Definition 2.16. [13] *Let $A, B \in IF_L(M)$. Then B is said to be a strictly proper intuitionistic L-fuzzy submodule of A if $B \subseteq A$ and $B \neq \chi_{\{\theta\}}$ and $A|_{B^*} = B$ and $B^* \subseteq A^*$. Also B is said to be a proper intuitionistic L-fuzzy submodule of A if $B \subseteq A$, $B \neq \chi_{\{\theta\}}$ and $B^* \subseteq A^*$.*

Definition 2.17. [14] *$A \in IF_L(M)$ is said to be an intuitionistic L-fuzzy simple module if A has no proper intuitionistic L-fuzzy submodules.*

Theorem 2.18. [14] *Let L be a regular lattice and M be a module over ring R. Then M is simple if and only if χ_M is an intuitionistic L-fuzzy simple module.*

Lemma 2.19. *Let L be a regular lattice and $B, C, D \in IF_L(M)$ such that $B \subseteq C$. Then*

$$C \cap (D + B) = (C \cap D) + B.$$

Proof. Let $A = C \cap (D + B)$. Then

$$A^* = [C \cap (D + B)]^* = C^* \cap (D^* + B^*).$$

Since $B \subseteq C$, so $B^* \subseteq C^*$. Now by the modular law

$$C^* \cap (D^* + B^*) = (C^* \cap D^*) + B^*.$$

Thus

$$A^* = (C^* \cap D^*) + B^* = (C \cap D)^* + B^*.$$

Also, we get $A = (C \cap D) + B$. Thus, the result follows. □

3 Socle of an intuitionistic L-fuzzy submodule

In this section we study the concept of a socle of an intuitionistic L-fuzzy submodule of a module and we analyse some of its properties.

Definition 3.1. *If $A \in IF_L(M)$, then the socle of A, denoted by $\mathrm{Soc}(A)$, is defined as the sum of all intuitionistic L-fuzzy simple submodules of A. Thus $\mathrm{Soc}(A) = \sum B_i$, where B_i is an intuitionistic L-fuzzy simple submodule of A. If A has no intuitionistic L-fuzzy simple submodule, then $\mathrm{Soc}(A) = \chi_{\{\theta\}}$.*

Theorem 3.2. *If $A = \mathrm{Soc}(A)$, then A has no proper intuitionistic L-fuzzy essential submodules.*

Proof. Given $A = \mathrm{Soc}(A) = \sum B_i$, where B_i are intuitionistic L-fuzzy simple submodules of A, let C be an intuitionistic L-fuzzy essential submodule of A. Then there exist intuitionistic L-fuzzy submodules $B_i^{'}$ of A such that $C \cap B_i = B_i^{'}$ and $B_i^{'} \neq \chi_{\{\theta\}}$.

Now

$$\mu_{B_i^{'}}(x) = \mu_{C \cap B_i}(x) = \mu_C(x) \wedge \mu_{B_i}(x) \leq \mu_C(x)$$

and

$$\nu_{B_i^{'}}(x) = \nu_{C \cap B_i}(x) = \nu_C(x) \vee \nu_{B_i}(x) \geq \nu_C(x)$$

imply that $B_i^{'} \subseteq C$. Similarly, $B_i^{'} \subseteq B_i$. Now B_i is an intuitionistic L-fuzzy simple submodule of A, so $B_i^{'} \subseteq B_i$ implies $B_i^{'} = B_i$. Also $B_i^{'} \subseteq C$ implies that C contains all intuitionistic L-fuzzy simple submodules of A. Thus $\mathrm{Soc}(A) \subseteq C$. This gives $A \subseteq C$. Thus $A = C$, i.e., A is an essential submodule of itself. Hence A has no proper intuitionistic L-fuzzy essential submodules. □

Theorem 3.3. *If $A \in IF_L(M)$ and E is the intersection of all intuitionistic L-fuzzy essential submodules of A, then $\mathrm{Soc}(A) \subseteq E$.*

Proof. Let $E = \cap\{B_i : B_i \trianglelefteq_e A\}$. Suppose $B_i, C \in IF_L(M)$ be such that C is an intuitionistic L-fuzzy simple submodule of A and $B_i \trianglelefteq_e A$. Then $B_i \cap C \neq \chi_{\{\theta\}}$. Also $B_i \cap C \subseteq C$ and C being an intuitionistic L-fuzzy simple submodule of A, we have $B_i \cap C = C$.

Now

$$\mu_C(x) = \mu_{B_i \cap C}(x) = \mu_{B_i}(x) \wedge \mu_C(x) \leq \mu_{B_i}(x)$$

and

$$\nu_C(x) = \nu_{B_i \cap C}(x) = \nu_{B_i}(x) \vee \nu_C(x) \geq \nu_{B_i}(x).$$

Thus $C \subseteq B_i$. This implies that if C is an intuitionistic L-fuzzy simple submodule of A, then C is contained in every intuitionistic L-fuzzy essential submodule B_i of A. Hence $\mathrm{Soc}(A) \subseteq E$. □

Theorem 3.4. *Let L be a regular lattice and $A \in IF_L(M)$. Let E be the intersection of all intuitionistic L-fuzzy essential submodules of A. If every non-zero intuitionistic L-fuzzy submodule of E is a direct summand of E, then every non-zero intuitionistic L-fuzzy submodule of E contains an intuitionistic L-fuzzy simple submodule of A.*

Proof. Let $E = \cap\{B_i : B_i \trianglelefteq_e A\}$. Suppose $C(\neq \chi_{\{\theta\}}) \in IF_L(M)$ be such that $C \subseteq E$. We consider $\mathfrak{F} = \{F : F \subseteq C, F \in IF_L(M)\}$. By Zorn's Lemma there exists a maximal element B in $\mathfrak{F}$ such that $B \subseteq C$ and $B \in IF_L(M)$. By the given condition $E = B \oplus B^{'}$, for some $B^{'} \in IF_L(M)$. Now

$$C = C \cap E = C \cap (B \oplus B^{'}) = B \oplus (C \cap B^{'})$$

by Lemma (2.19).

If $C \cap B^{'}$ is not an intuitionistic L-fuzzy simple submodule, then it contains a non-zero intuitionistic L-fuzzy submodule D of M. So there exists $D^{'} \in IF_L(M)$ such that $E = D \oplus D^{'}$. Also

$$C \cap B^{'} = (C \cap B^{'}) \cap E = (C \cap B^{'}) \cap (D \oplus D^{'}) = D^{'} \oplus (C \cap B^{'} \cap D).$$

This implies

$$B \oplus (C \cap B^{'}) = B \oplus D^{'} \oplus (C \cap B^{'} \cap D) = B \oplus D.$$

Thus $C \cap B^{'} = D$, which is a contradiction. Therefore $C \cap B^{'}$ is an intuitionistic L-fuzzy simple submodule of A. Thus C contains an intuitionistic L-fuzzy simple submodule $C \cap B^{'}$ of A. This proves the result. □

Theorem 3.5. *Let L be a regular lattice and $A \in IF_L(M)$. If E is the intersection of all intuitionistic L-fuzzy essential submodules of A, then $E \subseteq \mathrm{Soc}(A)$.*

Proof. Firstly, we show that every intuitionistic L-fuzzy submodule of E is a direct summand. Let C be an intuitionistic L-fuzzy submodule of E. Then C is an intuitionistic L-fuzzy submodule of A. So there exists an intuitionistic L-fuzzy submodule B such that B is a complement of C in A, i.e., $C \cap B = \chi_{\{\theta\}}$. Let $C^{'} \in IF_L(M)$ be such that $C^{'} \cap (C \oplus B) = \chi_{\{\theta\}}$.

Now $C \subseteq C \oplus B$ implies $C \cap C^{'} = \chi_{\{\theta\}}$. Similarly $B \cap C^{'} = \chi_{\{\theta\}}$. If $C \cap (B \oplus C^{'}) \neq \chi_{\{\theta\}}$, then there exist a non-zero element x in M such that $x \in [C \cap (B \oplus C^{'})]^* = C^* \cap (B^* \oplus C^{'^*})$, i.e., $\mu_C(x) > 0$, $\nu_C(x) < 1$ and $\mu_{B \oplus C^{'}}(x) > 0$, $\nu_{B \oplus C^{'}}(x) < 1$. This implies that there exist unique $y, z \in M$ such that $x = y + z$ and $\mu_B(y) \wedge \mu_{C'}(z) > 0$ and $\nu_B(y) \vee \nu_{C'}(z) < 1$, where $\mu_B(y) > 0, \mu_{C'}(z) > 0$ and $\nu_B(y) < 1, \nu_{C'}(z) < 1$. Thus $x = y + z$ with $x \in C^*$, $y \in B^*$ and $z \in C^{'*}$. Also z is a non-zero element of M, for otherwise it implies that x is a zero element of M.

Now $z = x - y \in C^{'*} \cap (B^* \oplus C^*) = [C' \cap (B \oplus C)]^*$. This shows that $C' \cap (B \oplus C) \neq \chi_{\{\theta\}}$, a contradiction. Thus $C \cap (B \oplus C') = \chi_{\{\theta\}}$. By the maximality of B we have $B \oplus C' = B$.

Now

$$\mu_B(x) = \mu_{B \oplus C'}(x) \geq \mu_B(0) \wedge \mu_{C'}(x) = \mu_{C'}(x)$$

and

$$\nu_B(x) = \nu_{B \oplus C'}(x) \leq \nu_B(0) \vee \nu_{C'}(x) = \nu_{C'}(x).$$

Thus $C' \subseteq B$ and hence $\chi_{\{\theta\}} = C' \cap B = C'$. This proves $C \oplus B \trianglelefteq_e A$. Thus $E \subseteq C \oplus B$. This implies $E = E \cap (C \oplus B) = E \oplus (C \cap B)$, since $C \subseteq E$ and $C \cap (E \cap B) = \chi_{\{\theta\}}$. Thus every intuitionistic L-fuzzy submodule of E is a direct summand.

Let D be the sum of all intuitionistic L-fuzzy simple submodules of E. Then D is a direct summand of E so there exists $D' \in IF_L(M)$ such that $E = D \oplus D'$. If $D' \neq \chi_{\{\theta\}}$, then there exists an intuitionistic L-fuzzy simple submodule G of D'. This gives $G \subseteq D$, a contradiction. Thus $D' = \chi_{\{\theta\}}$. This implies $E = D$. Hence $E \subseteq \mathrm{Soc}(A)$. □

Using Theorem (3.3), Theorem (3.4) and Theorem (3.5), we get the following theorem.

Theorem 3.6. *Let L be a regular lattice and $A \in IF_L(M)$. If E is the intersection of all intuitionistic L-fuzzy essential submodules of A, then* $\mathrm{Soc}(A) = E$.

Theorem 3.7. *Let L be a regular lattice and $f : N \to M$ be a module homomorphism. If $A \in IF_L(M)$ and $B \in IF_L(N)$ such that $f(B) \subseteq A$, then* $f^{-1}(\mathrm{Soc}(A)) \subseteq \mathrm{Soc}(B)$.

Proof. This follows immediately by using Theorem (3.6), Theorem (2.12) and Corollary (2.9). □

Theorem 3.8. *Let L be a regular lattice and $A, A_1, A_2 \in IF_L(M)$ such that $A_1, A_2 \subseteq A$ and $A = A_1 \oplus A_2$. Then* $\mathrm{Soc}(A) = \mathrm{Soc}(A_1) \oplus \mathrm{Soc}(A_2)$.

Proof. This follows immediately by using Theorem (3.6) and Corollaries (2.9) and (2.11). □

Theorem 3.9. *Let L be a regular lattice and $A, B, C \in IF_L(M)$ such that $A \subseteq B \subseteq C$. If A is a direct summand of C, then A is also a direct summand of B.*

Proof. Since A is a direct summand of C, there exists $A' \in IF_L(M)$ with $A' \subseteq C$ such that $A + A' = C$ and $A \cap A' = \chi_{\{\theta\}}$. Now $(A + A') \cap B = B$. Then by using Lemma (2.19) we get $A + (A' \cap B) = B$. Also $A \cap (A' \cap B) = \chi_{\{\theta\}}$. This implies that A is also a direct summand of B. □

Definition 3.10. *If $A \in IF_L(M)$ and $IF_L(A) = \{C \subseteq A : C \in IF_L(M)\}$. Then $IF_L(A)$ is intuitionistic L-fuzzy complemented if for all $C \subseteq A, C \in IF_L(M)$ there exists $C' \in IF_L(M)$ such that $C \cap C' = \chi_{\{\theta\}}$ and $C + C' = A$. In other words, $IF_L(A)$ is intuitionistic L-fuzzy complemented if every element of $IF_L(A)$ is a direct summand of A.*

Theorem 3.11. *Let L be a regular lattice and $A \in IF_L(M)$. If $A = \mathrm{Soc}(A)$, then $IF_L(A)$ is intuitionistic L-fuzzy complemented and for any $C \in IF_L(A)$, $IF_L(C)$ is also intuitionistic L-fuzzy complemented.*

Proof. Since $A = \mathrm{Soc}(A)$, then by Theorem (3.2) A has no proper intuitionistic L-fuzzy essential submodule. Let C be any intuitionistic L-fuzzy submodule of A. If B is a relative complement for C in A, then as Theorem (3.5) we get $B \oplus C \trianglelefteq_e A$. But given that A has no proper intuitionistic L-fuzzy essential submodule, so $B \oplus C = A$ and B being a relative complement for C, we get $B \cap C = \chi_{\{\theta\}}$. Hence $IF_L(A)$ is intuitionistic L-fuzzy complemented.

Let $B \in IF_L(M)$ and $B \subseteq C$. Then $B \subseteq A$. As $IF_L(A)$ is intuitionistic L-fuzzy complemented, so there exists $F \in IF_L(M)$ such that $B + F = A$ and $B \cap F = \chi_{\{\theta\}}$.

Now
$$(C \cap F) \cap B = C \cap (F \cap B) = C \cap \chi_{\{\theta\}} = \chi_{\{\theta\}}.$$

Also by Lemma (2.8),

$$(C \cap F) + B = C \cap (F + B) = C \cap A = C.$$

Hence there exists $C \cap F (\subseteq C) \in IF_L(M)$ such that $(C \cap F) \cap B = \chi_{\{\theta\}}$ and $(C \cap F) + B = C$. Thus, $IF_L(C)$ is intuitionistic L-fuzzy complemented. □

4 Conclusion

In this paper, we studied the concept of a socle of an intuitionistic fuzzy submodule of a module in the lattice setting. It is proved that if the socle of an intuitionistic L-fuzzy submodule A of an R-module M is A, then A has no proper intuitionistic L-fuzzy essential submodules. Further, we showed that if E is the intersection of all intuitionistic L-fuzzy essential submodules of A, then every non-zero intuitionistic L-fuzzy submodule of E contains a simple intuitionistic L-fuzzy submodule of E. Using this result, we showed that when L is regular, then $\mathrm{Soc}(A) = E$. Apart from this we have also evaluated the socle of a direct sum of intuitionistic L-fuzzy submodules. Further, we showed that if $A = \mathrm{Soc}(A)$ and $IF_L(A)$ is intuitionistic L-fuzzy complemented, then for any $C \in IF_L(A)$, $IF_L(C)$ is also intuitionistic L-fuzzy complemented. Finally, the investigations on the relations between the socle of intuitionistic L-fuzzy submodules and intuitionistic L-fuzzy prime submodules will provide various exciting results.

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