Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–5132, Online ISSN 2367–8283
2025, Volume 31, Number 1, 9–14
DOI: 10.7546/nifs.2025.31.1.9-14

A new intuitionistic fuzzy extended modal operator

Krassimir Atanassov

Department of Bioinformatics and Mathematical Modelling,
Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 105, Sofia-1113, Bulgaria
e-mail: krat@bas.bg

Received: 22 December 2024 **Revised:** 12 February 2025
Accepted: 24 February 2025 **Online First:** 3 March 2025

Abstract: A new intuitionistic fuzzy extended modal operator from a first type is introduced and some of its properties are discussed. It is shown that it represents all hitherto existing intuitionistic fuzzy extended modal operators from a first type. Some open problems are formulated.
Keywords: Intuitionistic fuzzy extended modal operator, Intuitionistic fuzzy set.
2020 Mathematics Subject Classification: 03E72.

1 Introduction

Exactly ten years ago, in paper [3], I wrote that "operator $X_{a,b,c,d,e,f}$ is the highest extension of the modal operators, defined over Intuitionistic Fuzzy Sets". After writing paper [4], it was clear that it is possible to construct a new operator, extending operator $X_{a,b,c,d,e,f}$. In the present paper, a definition of such operator is given and some of its properties are studied.

2 Preliminaries

When some intuitionistic fuzzy set (IFS, see [1, 2]):

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\}$$

over the universe E is given, over it the following intuitionistic fuzzy extending modal operators from the first type were defined in [4] (B, C being IFSs over the same universe):

$$F_B(A) = \{\langle x, \mu_A(x) + \mu_B(x).\pi_A(x), \nu_A(x) + \nu_B(x).\pi_A(x)\rangle \mid x \in E\},$$
$$G_{B,C}(A) = \{\langle x, \mu_B(x).\mu_A(x), \nu_B(x).\nu_A(x)\rangle \mid x \in E\},$$
$$H_{B,C}(A) = \{\langle x, \mu_B(x).\mu_A(x), \nu_A(x) + \nu_C(x).\pi_A(x)\rangle \mid x \in E\},$$
$$H^*_{B,C}(A) = \{\langle x, \mu_B(x).\mu_A(x), \nu_A(x) + \nu_C(x).(1 - \mu_B(x).\mu_A(x) - \nu_A(x))\rangle \mid x \in E\},$$
$$J_{B,C}(A) = \{\langle x, \mu_A(x) + \mu_B(x).\pi_A(x), \nu_C(x).\nu_A(x)\rangle \mid x \in E\},$$
$$J^*_{B,C}(A) = \{\langle x, \mu_A(x) + \mu_B(x).(1 - \mu_A(x) - \nu_C(x).\nu_A(x)), \nu_C(x).\nu_A(x)\rangle \mid x \in E\},$$

where
$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x).$$

We must mention that in [4], the operator $D_B(A)$ is not defined as a particular case of the operator $F_{A,B}(A)$. Its form is

$$D_B(A) = \{\langle x, \mu_A(x) + \mu_B(x)\pi_A(x), \nu_A(x) + (1 - \mu_B(x))\pi_A(x)\rangle | x \in E\}$$

or, if we know that the set B is an ordinary fuzzy set, i.e., $\nu_B(x) = 1 - \mu_A(x)$, then it can have the form

$$D_B(A) = \{\langle x, \mu_A(x) + \mu_B(x)\pi_A(x), \nu_A(x) + \nu_B(x)\pi_A(x)\rangle \mid x \in E\}.$$

Let us define
$$O^* = \{\langle x, 0, 1\rangle \mid x \in E\},$$
$$U* = \{\langle x, 0, 0\rangle \mid x \in E\},$$
$$E^* = \{\langle x, 1, 0\rangle \mid x \in E\}.$$

3 Main results

Let us have the IFSs A, B, C, D, F, G over a given universe E, defined so that for each $x \in E$,

$$\mu_B(x) + \nu_C(x) - \nu_C(x)\nu_G(x) \leq 1, \tag{1}$$
$$\mu_C(x) + \nu_F(x) - \mu_C(x)\mu_D(x) \leq 1. \tag{2}$$

Obviously, both terms in (1) and (2) are greater than or equal to 0.

Let us define the operator

$$\begin{aligned} X_{B,C,D,F,G}(A) &= \{\langle x, \mu_B(x)\mu_A(x) + \mu_C(x)(1 - \mu_A(x) - \mu_D(x)\nu_A(x)), \\ &\qquad \nu_F(x)\nu_A(x) + \nu_C(x)(1 - \nu_G(x)\mu_A(x) - \nu_A(x))\rangle \mid x \in E\}. \end{aligned}$$

First, we must prove the following assertion.

Theorem 1. *The definition of the operator $X_{B,C,D,F,G}$ is correct.*

Proof. Let us have the IFSs A, B, C, D, E, F, G that satisfy the conditions (1) and (2).

We check sequentially:

$$\begin{aligned}
&\mu_B(x)\mu_A(x) + \mu_C(x)(1 - \mu_A(x) - \mu_D(x)\nu_A(x))\\
&\geq \mu_B(x)\mu_A(x) + \mu_C(x)(1 - \mu_A(x) - \nu_A(x)) \geq 0;
\end{aligned}$$

$$\begin{aligned}
&\mu_B(x)\mu_A(x) + \mu_C(x)(1 - \mu_A(x) - \mu_D(x)\nu_A(x))\\
&= \mu_B(x)\mu_A(x) + \mu_C(x) - \mu_C(x)\mu_A(x) - \mu_C(x)\mu_D(x)\nu_A(x)\\
&\leq \mu_B(x)\mu_A(x) + \mu_C(x) - \mu_C(x)\mu_A(x)\\
&\leq \mu_B(x)\mu_A(x) + \mu_C(x)(1 - \mu_A(x))\\
&\leq \mu_A(x) + 1 - \mu_A(x) = 1,
\end{aligned}$$

$$\begin{aligned}
&\nu_F(x)\nu_A(x) + \nu_C(x)(1 - \nu_G(x)\mu_A(x) - \nu_A(x))\\
&\geq \nu_F(x)\nu_A(x) + \nu_C(x)(1 - \mu_A(x) - \nu_A(x)) \geq 0;
\end{aligned}$$

$$\begin{aligned}
&\nu_F(x)\nu_A(x) + \nu_C(x)(1 - \nu_G(x)\mu_A(x) - \nu_A(x))\\
&= \nu_F(x)\nu_A(x) + \nu_C(x) - \nu_C(x)\nu_G(x)\mu_A(x) - \nu_C(x)\nu_A(x)\\
&\leq \nu_F(x)\nu_A(x) + \nu_C(x) - \nu_C(x)\nu_A(x)\\
&\leq \nu_A(x) + \nu_C(x)(1 - \nu_A(x))\\
&\leq \nu_A(x) + 1 - \nu_A(x) = 1.
\end{aligned}$$

Below, we use (1) and (2) and first and third inequalities from above and see:

$$\begin{aligned}
0 \;\leq\;& \mu_B(x)\mu_A(x) + \mu_C(x)(1 - \mu_A(x) - \mu_D(x)\nu_A(x))\\
&+\nu_F(x)\nu_A(x) + \nu_C(x)(1 - \nu_G(x)\mu_A(x) - \nu_A(x))\\
=\;& \mu_A(x)(\mu_B(x) - \mu_C(x) - \nu_C(x)\nu_G(x)) + \nu_A(x)(\nu_F(x) - \nu_C(x) - \mu_C(x)\mu_D(x))\\
&+\mu_C(x) + \nu_C(x))\\
\leq\;& \mu_A(x)(1 - \nu_C(x) + \nu_C(x)\nu_G(x) - \mu_C(x) - \nu_C(x)\nu_G(x))\\
&+\nu_A(x)(1 - \mu_C(x) + \mu_C(x)\mu_D(x) - \nu_C(x) - \mu_C(x)\mu_D(x)) + \mu_C(x) + \nu_C(x)\\
=\;& \mu_A(x)(1 - \mu_C(x) - \nu_C(x)) + \nu_A(x)(1 - \mu_C(x) - \nu_C(x)) + \mu_C(x) + \nu_C(x)\\
=\;& (\mu_A(x) + \nu_A(x))(1 - \mu_C(x) - \nu_C(x)) + \mu_C(x) + \nu_C(x)\\
\leq\;& 1 - \mu_C(x) - \nu_C(x) + \mu_C(x) + \nu_C(x) = 1
\end{aligned}$$

Therefore, the definition of the operator $X_{B,C,D,F,G}$ is correct. □

Theorem 2. *Let the IFSs A, B, C, D, E, F, G satisfy the conditions (1) and (2). Then*

$$\neg X_{B,C,D,F,G}(\neg A) = X_{\neg F,\neg C,\neg G,\neg B,\neg D}(A).$$

Proof. We obtain sequentially:

$$\begin{aligned}
\neg X_{B,C,D,F,G}(\neg A) &= \neg X_{B,C,D,F,G}(\{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}) \\
&= \neg\{\langle x, \mu_B(x)\nu_A(x) + \mu_C(x)(1 - \nu_A(x) - \mu_D(x)\mu_A(x)), \\
&\qquad \nu_F(x)\mu_A(x) + \nu_C(x)(1 - \nu_G(x)\nu_A(x) - \mu_A(x))\rangle \mid x \in E\} \\
&= \{\langle x, \nu_F(x)\mu_A(x) + \nu_C(x)(1 - \nu_G(x)\nu_A(x) - \mu_A(x)), \\
&\qquad \mu_B(x)\nu_A(x) + \mu_C(x)(1 - \nu_A(x) - \mu_D(x)\mu_A(x))\rangle \mid x \in E\} \\
&= X_{\neg F, \neg C, \neg G, \neg B, \neg D}(A).
\end{aligned}$$

Now, using the conditions (1) and (2), we obtain that:

$$\begin{aligned}
0 &\leq \nu_F(x)\mu_A(x) + \nu_C(x)(1 - \nu_G(x)\nu_A(x) - \mu_A(x)) \\
&\leq \nu_F(x)\mu_A(x) + 1 - \nu_G(x)\nu_A(x) - \mu_A(x) \\
&= 1 - \mu_A(x)(1 - \nu_F(x)) - \nu_G(x)\nu_A(x) \leq 1,
\end{aligned}$$

$$\begin{aligned}
0 &\leq \mu_B(x)\nu_A(x) + \mu_C(x)(1 - \nu_A(x) - \mu_D(x)\mu_A(x)) \\
&\leq 1 - \nu_A(x) + \mu_B(x)\nu_A(x) - \mu_D(x)\mu_A(x) \\
&= 1 - \nu_A(x)(1 - \mu_B(x)) - \mu_D(x)\mu_A(x)) \leq 1,
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq \nu_F(x)\mu_A(x) + \nu_C(x)(1 - \nu_G(x)\nu_A(x) - \mu_A(x)) \\
&\quad + \mu_B(x)\nu_A(x) + \mu_C(x)(1 - \nu_A(x) - \mu_D(x)\mu_A(x)) \\
&= \mu_A(x)(\nu_F(x) - \nu_C(x) - \mu_C(x)\mu_D(x)) \\
&\quad + \nu_A(x)(\mu_B(x) - \mu_C(x) - \nu_C(x)\nu_G(x)) + \mu_C(x) + \nu_C(x) \\
&\leq \mu_A(x)(1 - \mu_C(x) + \mu_C(x)\mu_D(x) - \nu_C(x) - \mu_C(x)\mu_D(x)) \\
&\quad + \nu_A(x)(1 - \mu_C(x) + \nu_C(x)\nu_G(x) - \mu_C(x) - \nu_C(x)\nu_G(x)) + \mu_C(x) + \nu_C(x) \\
&= (\mu_A(x) + \nu_A(x))(1 - \mu_C(x) - \nu_C(x)) + \mu_C(x) + \nu_C(x) \\
&\leq 1.
\end{aligned}$$

This proves the Theorem. □

Therefore, the new operator is auto-dual.

Theorem 3. *For arbitrary IFSs A, B, C the new operator represents all extended modal operators from Section 2.*

Proof. Following the definition of the new operator X, we can prove directly the following representations:

$$\square A = X_{E^*,O^*,R_1,E^*,E^*}(A),$$
$$\lozenge A = X_{E^*,E^*,E^*,E^*,R_2}(A),$$
$$D_B(A) = X_{E^*,B,E^*,E^*,E^*}(A),$$
$$F_B(A) = X_{E^*,B,E^*,E^*,E^*}(A),$$
$$G_{B,C}(A) = X_{B,U^*,R_3,C,R_4}(A),$$
$$H_{B,C}(A) = X_{B,C,R_3,E^*,E^*}(A),$$
$$H^*_{B,C}(A) = X_{B,C,R_4,E^*,B}(A),$$
$$J_{B,C}(A) = X_{E^*,B,E^*,C,O^*}(A),$$
$$J^*_{B,C}(A) = X_{E^*,B,C,C,O^*}(A),$$

where R_i, $i = 1, 2, 3, 4$ denotes an arbitrary IFS.

It is important to mention that in the case of the operator D_B, the set B is a fuzzy set, and in the cases of the operators $H_{B,C}$ and $H^*_{B,C}$, in the set C for each $x \in E : \mu_C(x) = 0$ and $\nu_C(x) \geq 0$. □

Having in mind the definition of operation @ over two IFSs A and Z, which has the form

$$A@Z = \left\{ \left\langle x, \frac{\mu_A(x) + \mu_Z(x)}{2}, \frac{\nu_A(x) + \nu_Z(x)}{2} \right\rangle \,\middle|\, x \in E \right\}$$

(see, e.g. [1, 2]), we can prove the following theorem.

Theorem 4. *For every seven IFSs A, B, C, D, F, G, Z over universe E, that satisfy the conditions (1) and (2):*

$$X_{B,C,D,F,G}(A@Z) = X_{B,C,D,F,G}(A)@X_{B,C,D,F,G}(Z).$$

Proof. Let the sets A, B, C, D, F, G, Z be given. Then

$$\begin{aligned}
&X_{B,C,D,F,G}(A@Z)\\
&= X_{B,C,D,F,G}\left(\left\{ \left\langle x, \frac{\mu_A(x) + \mu_Z(x)}{2}, \frac{\nu_A(x) + \nu_Z(x)}{2} \right\rangle \,\middle|\, x \in E \right\}\right)\\
&= \Bigg\{ \Bigg\langle x, \mu_B(x)\frac{\mu_A(x) + \mu_Z(x)}{2} + \mu_C(x)\Bigg(1 - \frac{\mu_A(x) + \mu_Z(x)}{2} - \mu_D(x)\frac{\nu_A(x) + \nu_Z(x)}{2}\Bigg),\\
&\qquad \nu_F(x)\frac{\nu_A(x) + \nu_Z(x)}{2} + \nu_C(x)\Bigg(1 - \nu_G(x)\frac{\mu_A(x) + \mu_Z(x)}{2} - \frac{\nu_A(x) + \nu_Z(x)}{2}\Bigg)\Bigg\rangle \,\Bigg|\, x \in E \Bigg\}\\
&= \Bigg\{ \Bigg\langle x, \frac{1}{2}\Bigg(\mu_B(x)\mu_A(x) + \mu_C(x)(1 - \mu_A(x) - \mu_D(x)\nu_A(x))\\
&\qquad\qquad + \mu_B(x)\mu_Z(x) + \mu_C(x)(1 - \mu_Z(x) - \mu_D(x)\nu_Z(x))\Bigg),\\
&\qquad \frac{1}{2}\Bigg(\nu_F(x)\nu_A(x) + \nu_C(x)(1 - \nu_G(x)\mu_A(x) - \nu_A(x))\\
&\qquad\qquad \nu_F(x)\nu_Z(x) + \nu_C(x)(1 - \nu_G(x)\mu_Z(x) - \nu_Z(x))\Bigg)\Bigg\rangle \,\Bigg|\, x \in E \Bigg\}\\
&= X_{B,C,D,F,G}(A)@X_{B,C,D,F,G}(Z),
\end{aligned}$$

that proves the theorem. □

4 Conclusion

We finish with the following **Open problems**:

1. What other interesting properties does the operator $X_{B,C,D,F,G}$ have?
2. Can the intuitionistic fuzzy modal operators from a second type be extended in a similar manner?
3. Can the operator $X_{B,C,D,F,G}$ be represented as a composition of some of the operators F, G, H, H^*, J, J^*?

Acknowledgements

The author is thankful for the support provided by the Bulgarian National Science Fund under Grant Ref. No. KP-06-N72/8.

References

[1] Atanassov, K. (1999). *Intuitionistic Fuzzy Sets: Theory and Applications*. Springer, Heidelberg.

[2] Atanassov, K. (2012). *On Intuitionistic Fuzzy Sets Theory*. Springer, Berlin.

[3] Atanassov, K. (2015). A property of the intuitionistic fuzzy modal logic operator $X_{a,b,c,d,e,f}$. *Notes on Intuitionistic Fuzzy Sets*, 21(1), 1–5.

[4] Atanassov, K. (2017). New intuitionistic fuzzy extended modal operators. *Notes on Intuitionistic Fuzzy Sets*, 23(4), 40–45.