Notes on Intuitionistic Fuzzy Sets
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On intuitionistic fuzzy characteristic and Frattini subgroups

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Abstract: This paper presents intuitionistic fuzzy characteristic subgroups and discusses some of its properties. It is established that an intuitionistic fuzzy subgroup of an intuitionistic fuzzy group is characteristic provided its cuts are characteristic subgroups. In addition, it is proven that every intuitionistic fuzzy characteristic subgroup of an intuitionistic fuzzy group is an intuitionistic fuzzy normal subgroup. Furthermore, the notions of intuitionistic fuzzy maximal subgroups and intuitionistic fuzzy Frattini subgroups are established. It is shown that every intuitionistic fuzzy Frattini subgroup is an intuitionistic fuzzy characteristic subgroup as well as an intuitionistic fuzzy normal subgroup, respectively. Finally, some results on intuitionistic fuzzy Frattini subgroups are presented with regards to the level sets and cuts of an intuitionistic fuzzy groups.
Keywords: Intuitionistic fuzzy sets, Intuitionistic fuzzy groups, Intuitionistic fuzzy subgroups, Intuitionistic fuzzy normal subgroup, Intuitionistic fuzzy Frattini subgroups, Intuitionistic fuzzy characteristic subgroups.
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1 Introduction

The study of fuzzy set theory (FST) [35] has revolutionized decision problems in real-life and as well has open the study of fuzzy algebra. Rosenfeld [27] introduced fuzzy groups by direct application of fuzzy sets to the classical group theory and fuzzified many group theoretic concepts. The characterization of fuzzy subgroups was presented in [6], Frattini fuzzy subgroups were discussed in [17], solvable groups was established in fuzzy multigroup domain [16], and fuzzy characteristic subgroup was presented in [4, 22, 23].

Due to the inadequacy of FST in the sense that it admits only the membership grade, intuitionistic fuzzy sets (IFSs) was introduced [8, 9] by incorporating membership and non-membership grades with the chance for hesitation margin. Various properties of IFSs were presented [10, 15] and IFSs have been applied in real-life problems [1, 7, 14, 26]. Consequently, Biswas [12, 13] introduced intuitionistic fuzzy subgroups (IFSGs) based on IFSs, Ahn *et al.* [3] deliberated on sublattice of lattice of IFSGs of a group, some properties of IFSGs were discoursed in [2], Fathi and Salleh [18] discussed intuitionistic fuzzy groups (IFGs) based on intuitionistic fuzzy space. In addition, Yuan *et al.* [34] shared some light on the description of IFSGs, Bal *et al.* [11] presented a brief note of kernel subgroups on IFGs and derived some properties of IFGs, Sharma [31] discussed the direct product of IFSGs, and the homomorphism of IFGs was presented in [28].

In addition, the idea of (α, β)-cuts of IFGs was presented in [30] and the t-IFSGs was introduced in [32]. Certain essential theorems of t-intuitionistic fuzzy isomorphism of t-IFSGs and the fuzzification of the famous Lagrange's theorem were presented in [5, 24]. The idea of intuitionistic anti-fuzzy subgroups was discussed in [25] and Husban *et al.* [20] presented complex intuitionistic fuzzy group (CIFG) by allowing the membership and non-membership degrees to include complex numbers. Afterwards, Husban *et al.* [21] established the idea of normality in CIFGs with some properties.

Although several group theoretic notions have been discussed in IFGs, the concepts of characteristic subgroups and Frattini subgroups have not been studied under IFGs. In the light of this, the concepts of intuitionistic fuzzy characteristic subgroups (IFCSGs) and intuitionistic fuzzy Frattini subgroups (IFFSGs) are presented in this article and several of their properties are discussed. The rest of the work are thus outlined: Section 2 presents some preliminaries on fuzzy subgroups, IFGs, IFSGs, among others. Section 3 discusses the notion of IFCSGs and presents IFFSGs. Finally, Section 4 concludes the work with recommendations for future investigations.

2 Preliminaries

Throughout the article, the symbols S and G represent a non-empty set and a group, respectively.

Definition 2.1 ([35])**.** A fuzzy subset ϱ of S is of the form:

$$\varrho = \{\langle s, M_\varrho(s)\rangle \mid s \in S\},$$

where $M_\varrho\colon X \to [0, 1]$ is the membership grade of $s \in X$.

Definition 2.2 ([27]). A fuzzy subset ϱ of G is a fuzzy subgroup of G if

(i) $M_\varrho(gh) \geq \min\Big\{M_\varrho(g), M_\varrho(h)\Big\} \; \forall\, g, h \in G$,

(ii) $M_\varrho(g^{-1}) = M_\varrho(g) \; \forall\, g \in G$,

where "min" is a minimum operation. Furthermore, $M_\varrho(e) = M_\varrho(gg^{-1}) \geq \min\Big\{M_\varrho(g), M_\varrho(g)\Big\}$ $= M_\varrho(g) \; \forall\, g \in G$, where e is the unit element of G. Then, $M_\varrho(e)$ is the upper bound or tip of ϱ.

Definition 2.3 ([8, 9]). An IFS η of S is of the form:

$$\eta = \{\langle s, M_\eta(s), N_\eta(s)\rangle \mid s \in S\},$$

where $M_\eta \colon X \to [0,1]$ and $N_\eta \colon X \to [0,1]$ are the membership and non-membership grades of $s \in X$ and $0 \leq M_\eta(s) + N_\eta(s) \leq 1$.

Definition 2.4 ([10]). Let η and γ be IFSs of S. Then,

(i) $\eta = \gamma \Longleftrightarrow M_\eta(s) = M_\gamma(s)$ and $N_\eta(s) = N_\gamma(s) \; \forall s \in S$,

(ii) $\eta \subseteq \gamma \Longleftrightarrow M_\eta(s) \leq M_\gamma(s)$ and $N_\eta(s) \geq N_\gamma(s) \; \forall s \in S$,

(iii) $\eta \cap \gamma = \Big\{\langle s, \min\{M_\eta(s), M_\gamma(s)\}, \max\{N_\eta(s), N_\gamma(s)\}\rangle \mid s \in S\Big\}$,

(iv) $\eta \cup \gamma = \Big\{\langle s, \max\{M_\eta(s), M_\gamma(s)\}, \min\{N_\eta(s), N_\gamma(s)\}\rangle \mid s \in S\Big\}$.

Definition 2.5 ([12]). An IFS η in G is an IFG/IFSG of G if

(i) $M_\eta(gh) \geq \min\Big\{M_\eta(g), M_\eta(h)\Big\}$ and $N\eta(gh) \leq \max\Big\{N_\eta(g), N_\eta(h)\Big\} \; \forall\, g, h \in G$,

(ii) $M_\eta(g^{-1}) = M_\eta(g)$ and $N_\eta(g^{-1}) = N_\eta(g) \; \forall\, g \in G$,

where "min" and "max" are minimum and maximum operations, respectively. In addition,

$$M_\eta(e) = M_\eta(gg^{-1}) \geq \min\Big\{M_\eta(g), M_\eta(g)\Big\} = M_\eta(g),$$
$$N_\eta(e) = N_\eta(gg^{-1}) \leq \max\Big\{N_\eta(g), N_\eta(g)\Big\} = N_\eta(g)$$

$\forall\, g \in G$, where e is the unit element of G. Then, $M\eta(e)$ is the upper bound or tip of η and $N_{eta}(e)$ is the lower bound of η, respectively.

Definition 2.6 ([12]). If η and γ are IFGs of G, then η is an IFSG of γ if $\eta \subseteq \gamma$, and η is a proper IFSG of γ if $\eta \subset \gamma$, which implies η is strictly contained in γ.

Definition 2.7 ([30]). Let η be an IFG of G. Then, the level set of η is:

$$\eta_* = \{g \in G \mid M_\eta(s) \geq 0 \text{ and } N_\eta(s) \leq 0\},$$

and it is a subgroup of G.

Definition 2.8 ([30]). Suppose η is an IFG of G, then the strong $[\alpha, \beta]$-cuts of η is:

$$\eta_{[\alpha,\beta]} = \{g \in G \mid M_\eta(s) \geq \alpha \text{ and } N_\eta(s) \leq \beta\},$$

and it a subgroup of G, where $\alpha, \beta \in [0, 1]$.

Similarly, the weak (α, β)-cuts of η is:

$$\eta_{(\alpha,\beta)} = \{g \in G \mid M_\eta(s) > \alpha \text{ and } N_\eta(s) < \beta\},$$

and it a subgroup of G.

Definition 2.9 ([19]). An IFG η of G is commutative if $M_\eta(gh) = M_\eta(hg)$ and $N_\eta(gh) = N_\eta(hg)$ $\forall g, h \in G$. However, an IFG η is commutative if G is a commutative group.

Definition 2.10 ([19]). Let η be an IFSG of an IFG γ of G. Then, η is normal in γ, denoted as $\eta \triangleleft \gamma$ if $M_\eta(gh) = M_\eta(hg)$ and $N_\eta(gh) = N_\eta(hg) \iff M_\eta(h) = M_\eta(g^{-1}hg)$ and $N_\eta(h) = N_\eta(g^{-1}hg)$ $\forall\, g, h \in G$.

Definition 2.11 ([19]). Two IFGs η and γ of G are conjugate of each other if:

$$M_\eta(h) = M_\gamma(ghg^{-1}) = M_{\gamma^g}(h),\ N_\eta(h) = N_\gamma(ghg^{-1}) = N_{\gamma^g}(h),$$

and

$$M_\gamma(h) = M_\eta(ghg^{-1}) = M_{\eta^g}(h),\ N_\gamma(h) = N_\eta(ghg^{-1}) = N_{\eta^g}(h),$$

$\forall g, h \in G$.

Definition 2.12 ([29]). Suppose η is an IFSG of G. Then, an IFS $h\eta$ for $h \in G$ defined by $M_{(h\eta)}(g) = M_\eta(h^{-1}g)$ and $N_{(h\eta)}(g) = N_\eta(h^{-1}g)$ $\forall\, g \in G$ is called the left intuitionistic fuzzy coset of G. Similarly, an IFS ηh for $h \in G$ defined by $M_{(\eta h)}(g) = M_\eta(gh^{-1})$ and $N_{(\eta h)}(g) = N_\eta(gh^{-1})$ $\forall\, g \in G$ is called the right intuitionistic fuzzy coset of G.

Definition 2.13 ([33]). If η and γ are IFGs of G and $\eta \triangleleft \gamma$. Then, the collection of the left intuitionistic fuzzy cosets/right intuitionistic fuzzy cosets of η such that $g\eta \circ h\eta = gh\eta\, \forall g, h \in G$ is called an intuitionistic fuzzy quotient group (IFQG) of γ by η, denoted by γ/η.

Definition 2.14 ([28]). Assume G and H are groups and $\theta \to H$ is a homomorphism. Suppose η and γ are IFGs of G and H, respectively. Then, θ induces a homomorphism from η to γ that fulfills

(i)

$$M_\eta(\theta^{-1}(h_1h_2)) \geq \min\{M_\eta(\theta^{-1}(h_1)), M_\eta(\theta^{-1}(h_2))\}$$

and

$$N_\eta(\theta^{-1}(h_1h_2)) \leq \max\{N_\eta(\theta^{-1}(h_1)), N_\eta(\theta^{-1}(h_2))\}$$

$\forall h_1, h_2 \in H$,

(ii) $M_\gamma(\theta(g_1g_2)) \geq \min\{M_\gamma(\theta(g_1)), M_\gamma(\theta(g_2))\}$ and $N_\gamma(\theta(g_1g_2)) \leq \max\{N_\gamma(\theta(g_1)),$ $M_\gamma(\theta(g_2))\}$ $\forall g_1, g_2 \in G$, where the image of η under θ, denoted by $\theta(\eta)$, is an IFS in H defined by

$$M_{\theta(\eta)}(h) = \begin{cases} \bigvee_{g\in\theta^{-1}(h)} M_\eta(g), & \theta^{-1}(h) \neq \varnothing \\ 0, & \text{otherwise} \end{cases}$$

$$N_{\theta(\eta)}(h) = \begin{cases} \bigwedge_{g\in\theta^{-1}(h)} N_\eta(g), & \theta^{-1}(h) \neq \varnothing \\ 0, & \text{otherwise} \end{cases}$$

for each $h \in H$.

In addition, the inverse image of γ under θ, denoted by $\theta^{-1}(\gamma)$, is an IFS in G defined by $M_{\theta^{-1}(\gamma)}(g) = M_\gamma(\theta(g))$ and $N_{\theta^{-1}(\gamma)}(g) = N_\gamma(\theta(g))\ \forall g \in G$.

Hence, the homomorphism θ of η onto γ is an automorphism of η onto η if θ is both one-one and onto (i.e., bijective).

3 Main results

Under the main results, we present the concepts of IFCSGs and IFFSGs, respectively.

3.1 Intuitionistic fuzzy characteristic subgroups

In the classical setting, we say a subgroup H of a group G is characteristic if $H^\theta = H$ for every automorphism, θ of G, where $H^\theta = \theta(H)$. Now, we define the analogue of this concept as follows:

Definition 3.1. Let η be an IFSG of an IFG γ in G. Then, η is characteristic (f-invariant) in γ if $M_{\eta^\theta}(g) = M_\eta(g)$, $N_{\eta^\theta}(g) = N_\eta(g)\ \forall g \in G$ for every automorphism, θ of G. Hence, $\theta(\eta) \subseteq \eta$ for every $\theta \in Aut(G)$.

Definition 3.2. Let η be an IFG in G and θ a mapping from G into G. Define an IFS η^θ in G by $M_{\eta^\theta}(g) = M_\eta(g^\theta)$, $N_{\eta^\theta}(g) = N_\eta(g^\theta)$, where $g^\theta = \theta(g) = g\ \forall g \in G$.

Proposition 3.3. *Let η be an IFG in G and $g \in G$. Suppose θ is an automorphism of G for $\theta(g) = hgh^{-1}\ \forall g \in G$, then $\eta^h = \eta^\theta$.*

Proof. For η be an IFG in G and $g \in G$, and assume $\theta\colon G \to G$ is defined by $\theta(g) = hgh^{-1}$ $\forall g \in G$. Then, $M_{\eta^h}(g) = M_\eta(hgh^{-1}) = M_\eta(\theta(g)) = M_{\eta^\theta}(g)$, $N_{\eta^h}(g) = N_\eta(hgh^{-1}) = N_\eta(\theta(g)) = N_{\eta^\theta}(g)$ by Definition 2.11, and the result follows. □

Proposition 3.4. *Let η be an IFG in G and θ be an automorphism of G. Then, η^θ is an IFG in G.*

Proof. For $g, h \in G$, we have $M_{\eta^\theta}(gh) = M_\eta((gh)^\theta) = M_\eta(g^\theta h^\theta)$ and $N_{\eta^\theta}(gh) = N_\eta((gh)^\theta) = N_\eta(g^\theta h^\theta)$ because θ is a homomorphism. Since η is an IFG in G, then

$$M_\eta(g^\theta h^\theta) \geq \min\{M_\eta(g^\theta), M_\eta(h^\theta)\} = \min\{M_{\eta^\theta}(g), M_{\eta^\theta}(h)\}$$

and

$$N_\eta(g^\theta h^\theta) \leq \max\{M_\eta(g^\theta), M_\eta(h^\theta)\} = \max\{M_{\eta^\theta}(g), M_{\eta^\theta}(h)\}.$$

Thus, $M_{\eta^\theta}(gh) \geq \min\{M_{\eta^\theta}(g), M_{\eta^\theta}(h)$ and $N_{\eta^\theta}(gh) \leq \max\{M_{\eta^\theta}(g), M_{\eta^\theta}(h)$. In addition,

$$M_{\eta^\theta}(g^{-1}) = M_\eta((g^{-1})^\theta) = M_\eta((g^\theta)^{-1}) = M_\eta(g^\theta) = M_{\eta^\theta}(g)$$

and

$$N_{\eta^\theta}(g^{-1}) = N_\eta((g^{-1})^\theta) = N_\eta((g^\theta)^{-1}) = N_\eta(g^\theta) = N_{\eta^\theta}(g).$$

Hence, η^θ is an IFG in G. □

Theorem 3.5. *Suppose $\theta\colon G \to G$ is an automorphism and η is an IFG in G, then η^θ is an IFG in G if and only if η is an IFG in G.*

Proof. Suppose η is an IFG in G, then using the logic in the proof of Proposition 3.4, it is certain that η^θ is an IFG in G.

Conversely, if η^θ is an IFG in G, then

$$M_{\eta^\theta}(gh) \geq \min\{M_{\eta^\theta}(g), M_{\eta^\theta}(y)\} \text{ and } M_{\eta^\theta}(g^{-1}) = M_{\eta^\theta}(g)$$

and

$$N_{\eta^\theta}(gh) \leq \max\{N_{\eta^\theta}(g), N_{\eta^\theta}(y)\} \text{ and } N_{\eta^\theta}(g^{-1}) = N_{\eta^\theta}(g)$$

$\forall g, h \in G$. Thus

$$\begin{aligned} M_{\eta^\theta}(gh) = M_\eta((gh)^\theta) &= M_\eta(\theta(gh)) \\ &= M_\eta(gh), \end{aligned}$$

$$\Rightarrow M_\eta(gh) \geq \min\{M_\eta(g), M_\eta(h)\}$$

and

$$\begin{aligned} N_{\eta^\theta}(gh) = N_\eta((gh)^\theta) &= N_\eta(\theta(gh)) \\ &= N_\eta(gh), \end{aligned}$$

$$\Rightarrow N_\eta(gh) \leq \max\{N_\eta(g), N_\eta(h)\}\ \forall g, h \in G.$$

In addition,

$$\begin{aligned} M_{\eta^\theta}(g^{-1}) = M_\eta((g^{-1})^\theta) &= M_\eta((g^\theta)^{-1}) \\ &= M_\eta((\theta(g))^{-1}) = M_\eta(g^{-1}), \end{aligned}$$

$$\Rightarrow M_\eta(g^{-1}) = M_\eta(g).$$

and

$$\begin{aligned} N_{\eta^\theta}(g^{-1}) = N_\eta((g^{-1})^\theta) &= N_\eta((g^\theta)^{-1}) \\ &= N_\eta((\theta(g))^{-1}) = N_\eta(g^{-1}), \end{aligned}$$

$$\Rightarrow N_\eta(g^{-1}) = N_\eta(g)\ \forall g \in G.$$

Hence, η is an IFG in G. □

Theorem 3.6. *Every IFCSG of an IFG is an intuitionistic fuzzy normal subgroup (IFNSG).*

Proof. Let $g, h \in G$ and let η be an IFCSG of an IFG γ in G. To show that η is an IFNSG in γ, we need to verify that $M_\eta(gh) = M_\eta(hg)$ and $N_\eta(gh) = N_\eta(hg)$ $\forall g, h \in G$. Assume θ is the automorphism of G defined by $\theta(h) = g^{-1}h'g$ $\forall h' \in G$. Now, since η is an IFCSG of γ, $\eta^\theta = \eta$. Then,

$$\begin{aligned} M_\eta(gh) &= M_{\eta^\theta}((gh)) = M_\eta((gh)^\theta) \\ &= M_\eta(g^{-1}(gh)g) \\ &= M_\eta(hg) \end{aligned}$$

and

$$\begin{aligned} N_\eta(gh) &= N_{\eta^\theta}((gh)) = N_\eta((gh)^\theta) \\ &= N_\eta(g^{-1}(gh)g) \\ &= N_\eta(hg). \end{aligned}$$

Hence, η^θ is normal in γ. □

Remark 3.7. Let η, γ, and μ be IFGs in G in which $\mu \subseteq \gamma \subseteq \eta$.

(i) If μ is an IFCSG of γ and γ is an IFCSG of η, then μ is an IFCSG of η.

(ii) If μ is an IFCSG of γ and γ is an IFNSG of η, then μ is an IFNSG of η.

Proposition 3.8. *If γ is an IFG in G and η is an IFCSG of γ, then η_* is a characteristic subgroup of G. In addition, η_* is a characteristic subgroup of γ_*.*

Proof. It is certain that η_* is a subgroup of G (Definition 2.7). To show that η_* is characteristic in G, it is sufficient to verify that $\theta(\eta_*) \subseteq \eta_*$ $\forall \theta \in Aut(G)$. Let $\theta \in Aut(G)$, then $M_{\eta^\theta}(g) = M_\eta(g)$ and $N_{\eta^\theta}(g) = N_\eta(g)$, since η is an IFCSG of γ. Let $g \in \eta_*$, then $M_\eta(g) \geq 0$ and $N_\eta(g) \leq 0$, which implies $M_{\eta^\theta}(g) = M_\eta(\theta(g)) = M_\eta(g) \geq 0$ and $N_{\eta^\theta}(g) = N_\eta(\theta(g)) = N_\eta(g) \leq 0$, so $\theta(g) \in \eta_*$. Hence, $\theta(\eta_*) \subseteq \eta_*$, which concludes the proof.

In addition, because η is an IFCSG of γ and η_* is a characteristic subgroup of G, it means η_* is a characteristic subgroup of γ_*. □

Theorem 3.9. *Suppose G is finite and η is an IFCSG of an IFG γ in G. Then $\eta_{[\alpha,\beta]}$ and $\eta_{(\alpha,\beta)}$ are characteristic subgroups of G.*

Proof. It is certain that, $\eta_{[\alpha,\beta]}$ is a subgroup of G (Definition 2.8). Next, we show that $\eta_{[\alpha,\beta]}$ is characteristic in G by proving that $\theta(\eta_{[\alpha,\beta]}) \subseteq \eta_{[\alpha,\beta]}$ $\forall \theta \in Aut(G)$. For $\theta \in Aut(G)$, we have $M_{\eta^\theta}(g) = M_\eta(g)$ and $N_{\eta^\theta}(g) = N_\eta(g)$, because η is an IFCSG. Let $g \in \eta_{[\alpha,\beta]}$, then $M_\eta(g) \geq \alpha$ and $N_\eta(g) \leq \beta$, which mean $M_{\eta^\theta}(g) = M_\eta(\theta(g)) = M_\eta(g) \geq \alpha$ and $N_{\eta^\theta}(g) = N_\eta(\theta(g)) = N_\eta(g) \leq \beta$, so $\theta(g) \in A_{[\alpha,\beta]}$. Thus, $\theta(\eta_{[\alpha,\beta]}) \subseteq \eta_{[\alpha,\beta]}$. Hence, $\eta_{[\alpha,\beta]}$ is a characteristic subgroup of G. Similarly, $\eta_{(\alpha,\beta)}$ is a characteristic subgroup of G. □

Remark 3.10. Because η is an IFCSG of γ, and $\eta_{[\alpha,\beta]}$ and $\eta_{(\alpha,\beta)}$ are characteristic subgroups of G, it follows that $\eta_{[\alpha,\beta]}$ is a characteristic subgroup of $\gamma_{[\alpha,\beta]}$ and $\eta_{(\alpha,\beta)}$ is a characteristic subgroup of $\gamma_{(\alpha,\beta)}$.

Theorem 3.11. *Let η be an IFSG of an IFG γ in G, where G is finite. If $\eta_{[\alpha,\beta]}$ and $\eta_{(\alpha,\beta)}$ are characteristic subgroups of G, then η is an IFCSG of γ.*

Proof. Because G is finite, $|\eta| < \infty$. Let $Im(\eta) = \{(\alpha_0, \beta_0), \ldots, (\alpha_n, \beta_n)\}$ with $\alpha_0 > \cdots > \alpha_n$ and $\beta_0 < \cdots < \beta_n$. By hypothesis, $\eta_{[\alpha,\beta]} = \{g \in G | \, M_\eta(g) \geq \alpha_j, N_\eta(g) \leq \beta_j\}$ is a characteristic subgroup of G, $\forall j = 0, \ldots, n$. Let $\theta \in Aut(X)$. Because

$$M_{\eta^\theta}(g) = M_\eta(\theta(g)) = M_{\theta^{-1}(\eta)}(g) = M_\eta(g) \text{ and}$$
$$N_{\eta^\theta}(g) = N_\eta(\theta(g)) = N_{\theta^{-1}(\eta)}(g) = N_\eta(g),$$

then $Im(\eta^\theta) = Im(\eta)$.

Moreover, $\forall q = 0, \ldots, n$, we have $(\eta^\theta)_{[\alpha_q,\beta_q]} = \eta_{[\alpha_q,\beta_q]}$, since $g \in (\eta^\theta)_{[\alpha_q,\beta_q]} \Leftrightarrow M_{\eta^\theta}(g) \geq \alpha_q$, $N_{\eta^\theta}(g) \leq \beta_q \Leftrightarrow M_\eta(\theta(g)) \geq \alpha_q$, $N_\eta(\theta(g)) \leq \beta_q \Leftrightarrow \theta(g) \in \eta_{[\alpha_q,\beta_q]} \Leftrightarrow g \in \theta^{-1}(\eta_{[\alpha_q,\beta_q]}) \Leftrightarrow g \in \eta_{[\alpha_q,\beta_q]}$. Thus $\eta^\theta = \eta$ and hence, η is an IFCSG of γ. □

In what follows, we validate that Theorems 3.9 and 3.11 are still true by discarding the finite order of G.

Theorem 3.12. *Suppose η is an IFSG of an IFG γ in G. Then, the following statements are the same:*

(i) η is an IFCSG of γ.

(ii) $\eta_{[\alpha,\beta_q]}$ and $\eta_{(\alpha,\beta_q)}$ are IFCSGs in G.

Proof. We first proof (i) ⇒ (ii). Let $\alpha, \beta \in Im(\eta)$, $\theta \in Aut(G)$ and $g \in \eta_{[\alpha,\beta]}$. Assume η is an IFCSG of γ, then $M_\eta(\theta(g)) = M_\eta(g) \geq \alpha$ and $N_\eta(\theta(g)) = N_\eta(g) \leq \beta$. Thus, $\theta(g) \in \eta_{[\alpha,\beta]}$ and so, $\theta(\eta_{[\alpha,\beta]}) \subseteq \eta_{[\alpha,\beta]}$. But $\eta_{[\alpha,\beta]} \subseteq \theta(\eta_{[\alpha,\beta]})$ by symmetry. Let $g \in \eta_{[\alpha,\beta]}$ and let $h \in G$ wherein $\theta(h) = g$. Then, $M_\eta(h) = M_\eta(\theta(h)) = M_\eta(g) \geq \alpha$ and $N_\eta(h) = N_\eta(\theta(h)) = N_\eta(g) \leq \beta$ imply that $h \in \eta_{[\alpha,\beta]}$, therefore $g \in \theta(\eta_{[\alpha,\beta]})$. Hence, $\eta_{[\alpha,\beta]} \subseteq \theta(\eta_{[\alpha,\beta]})$. Thus, $\eta_{[\alpha,\beta_q]}$ is an IFCSG in G. Similarly, $\eta_{(\alpha,\beta_q)}$ is an IFCSG in G.

Next, we proof (ii) ⇒ (i). Let $g \in G$, $\theta \in Aut(G)$ and $M_\eta(g) = \alpha$, $N_\eta(g) = \beta$. Then, $g \in \eta_{[\alpha,\beta]}$ and $g \notin \eta_{[\hat{\alpha},\hat{\beta}]}$ for $\hat{\alpha}, \hat{\beta} > \alpha, \beta$. By hypothesis, $\theta(\eta_{[\alpha,\beta]}) = \eta_{[\alpha,\beta]}$, so $\theta(g) \in \eta_{[\alpha,\beta]}$ and hence $M_\eta(g) = M_\eta(\theta(g)) \geq \alpha$ and $N_\eta(g) = N_\eta(\theta(g)) \leq \beta$.

Let $\hat{\alpha} = M_\eta(\theta(g))$ and $\hat{\beta} = N_\eta(\theta(g))$. Assume $\hat{\alpha}, \hat{\beta} > \alpha, \beta$, then $\theta(g) \in \eta_{[\hat{\alpha},\hat{\beta}]} = \theta(\eta_{[\hat{\alpha},\hat{\beta}]})$. Because θ is injective, we have $g \in \eta_{[\hat{\alpha},\hat{\beta}]}$, which is a contradiction. Thus, $M_\eta(\theta(g)) = \alpha = M_\eta(g)$ and $N_\eta(\theta(g)) = \beta = N_\eta(g)$, implying that η is an IFCSG of γ. □

Theorem 3.13. *Let η be an IFG that is normal in an IFG γ in G and assume θ is an automorphism of G which leaves an invariant subgroup, η_*. Then θ induces an automorphism $\bar{\theta}$ of γ/η defined by $\bar{\theta}(g\eta) = \theta(g)\eta \; \forall g \in G$.*

Proof. To begin with, we verify that $\bar{\theta}$ is well defined. Suppose $g, h \in G$ such that $g\eta = h\eta$. Then, it suffices to show that $\theta(g)\eta = \theta(h)\eta$. Since $g\eta = h\eta$, we have $M_{g\eta}(g) = M_{h\eta}(g)$ and $M_{g\eta}(h) = M_{h\eta}(h) \Rightarrow M_\eta(e) = M_\eta(h^{-1}g)$ and $M_\eta(g^{-1}h) = M_\eta(e) \Rightarrow M_\eta(h^{-1}g) = M_\eta(g^{-1}h) = M_\eta(e) \Rightarrow h^{-1}g, g^{-1}h \in \eta_*$.

Similarly, $N_{g\eta}(g) = N_{h\eta}(g)$ and $N_{g\eta}(h) = N_{h\eta}(h) \Rightarrow N_\eta(e) = N_\eta(h^{-1}g)$ and $N_\eta(g^{-1}h) = N_\eta(e) \Rightarrow N_\eta(h^{-1}g) = N_\eta(g^{-1}h) = N_\eta(e) \Rightarrow h^{-1}g, g^{-1}h \in \eta_*$.

By hypothesis, $\theta(\eta_*) = \eta_*$, then $\theta(h^{-1}g), \theta(g^{-1}h) \in \eta_*$. Thus, $M_\eta(\theta(h^{-1}g)) = M_\eta(\theta(g^{-1}h))$ and $N_\eta(\theta(h^{-1}g)) = N_\eta(\theta(g^{-1}h))$. Now, let $\hat{g} \in G$, then

$$\begin{aligned} M_{\theta(g)\eta}(\hat{g}) &= M_\eta(\theta(g^{-1})\hat{g}) \\ &= M_\eta(\theta(g^{-1})\theta(h)\theta(h^{-1})\hat{g}) \\ &\geq \min\{M_\eta(\theta(g^{-1})\theta(h)), M_\eta(\theta(h^{-1})\hat{g})\} \\ &= \min\{M_\eta(\theta(g^{-1}h)), M_\eta(\theta(h^{-1})\hat{g})\} \\ &= M_{\theta(h)\eta}(\hat{g}) \end{aligned}$$

and

$$\begin{aligned} N_{\theta(g)\eta}(\hat{g}) &= N_\eta(\theta(g^{-1})\hat{g}) \\ &= N_\eta(\theta(g^{-1})\theta(h)\theta(h^{-1})\hat{g}) \\ &\leq \max\{N_\eta(\theta(g^{-1})\theta(h)), N_\eta(\theta(h^{-1})\hat{g})\} \\ &= \max\{N_\eta(\theta(g^{-1}h)), N_\eta(\theta(h^{-1})\hat{g})\} \\ &= N_{\theta(h)\eta}(\hat{g}). \end{aligned}$$

Thus, $M_{\theta(g)\eta}(\hat{g}) \geq M_{\theta(h)\eta}(\hat{g})$ and $N_{\theta(g)\eta}(\hat{g}) \leq N_{\theta(h)\eta}(\hat{g})$. Similarly, $M_{\theta(h)\eta}(\hat{g}) \geq M_{\theta(g)\eta}(\hat{g})$ and $N_{\theta(h)\eta}(\hat{g}) \leq N_{\theta(g)\eta}(\hat{g})$. Since $\hat{g} \in G$ is arbitrary, then $M_{\theta(g)\eta}(\hat{g}) = M_{\theta(h)\eta}(\hat{g})$ and $N_{\theta(g)\eta}(\hat{g}) = N_{\theta(h)\eta}(\hat{g})$, implying that $\theta(g)\eta = \theta(h)\eta$. Thus, $\bar{\theta}$ is well defined.

In addition, let $g, h \in G$. Because θ is homomorphic, $\theta(gh) = \theta(g)\theta(h)$ and so, $\theta(gh)\eta = \theta(g)\eta\theta(h)\eta$. Hence, $\bar{\theta}(gh\eta) = \theta(g)\eta\theta(h)\eta$ and $\bar{\theta}(g\eta h\eta) = \bar{\theta}(g\eta)\bar{\theta}(h\eta)$. Therefore, $\bar{\theta}$ is a homomorphism. □

Corollary 3.14. *Using the information of Theorem 3.13, $\bar{\theta}$ is an automorphism if G is finite and θ is an automorphism.*

Proof. Since G is finite, then θ has a finite order. Assume the order of θ is n, then $\theta^n = j$, where j is an identity map. From Theorem 3.13, $\bar{\theta}$ is a homomorphism, then we prove that θ is injective. Let $g, h \in G$ such that $\bar{\theta}(g\eta) = \bar{\theta}(h\eta)$. Then, $\theta(g)\eta = \theta(h)\eta$. Thus, $\bar{\theta}(\theta(g)\eta) = \bar{\theta}(\theta(h)\eta)$. In continuation with the definition of $\bar{\theta}$, we get $\theta^2(g)\eta = \theta^2(h)\eta$ and thus $j(g)\eta = \theta^n(g)\eta = \theta^n(h)\eta = j(h)\eta$, which implies $g\eta = h\eta$. Hence, $\bar{\theta}$ is injective, which finishes the proof. □

Corollary 3.15. *With the hypothesis in Theorem 3.13, if $\bar{\theta}$ is an automorphism and $\eta_* = \{e\}$, then θ is an automorphism of G.*

Proof. Let $\theta(g) = \theta(h)$ for $g, h \in G$. Then, $\theta(g)\eta = \theta(h)\eta$, implying $\bar{\theta}(g\eta) = \bar{\theta}(h\eta)$. Hence, $g\eta = h\eta$ since $\bar{\theta}$ is injective. Thus, $M_{g\eta}(h) = M_{h\eta}(h) \Rightarrow M_\eta(g^{-1}h) = M_\eta(e)$ and $N_{g\eta}(h) = N_{h\eta}(h) \Rightarrow N_\eta(g^{-1}h) = N_\eta(e)$. So $g^{-1}h \in \eta_*$ and $g^{-1}h = e$ since $\eta_* = \{e\}$. Thus, $M_\eta(g) = M_\eta(h), N_\eta(g) = N_\eta(h) \Rightarrow g = h$. Hence θ is injective, which finishes the proof. □

Corollary 3.16. *Let an IFG η be normal in an IFG γ in G. If θ is an automorphism of G and $\eta^\theta = \eta$, then θ induces an automorphism $\bar{\theta}$ of γ/η define by $\bar{\theta}(g\eta) = \theta(g)\eta\ \forall g \in G$.*

Proof. Let $g, h \in G$, then $g\eta = h\eta \Leftrightarrow g\eta^\theta = h\eta^\theta \Leftrightarrow M_{g\eta^\theta}(\hat{g}) = M_{yA^\theta}(\hat{g})$, $N_{g\eta^\theta}(\hat{g}) = N_{yA^\theta}(\hat{g})$ $\forall \hat{g} \in G \Leftrightarrow M_{\eta^\theta}(g^{-1}\hat{g}) = M_{\eta^\theta}(h^{-1}\hat{g})$, $N_{\eta^\theta}(g^{-1}\hat{g}) = N_{\eta^\theta}(h^{-1}\hat{g})$ $\forall \hat{g} \in G \Leftrightarrow M_\eta(\theta(g^{-1}\hat{g})) = M_\eta(\theta(h^{-1}\hat{g}))$, $N_\eta(\theta(g^{-1}\hat{g})) = N_\eta(\theta(h^{-1}\hat{g}))\forall \hat{g} \in G \Leftrightarrow M_\eta(\theta(g^{-1})\theta(\hat{g})) = M_\eta(\theta(h^{-1})\theta(\hat{g}))$, $N_\eta(\theta(g^{-1})\theta(\hat{g})) = N_\eta(\theta(h^{-1})\theta(\hat{g}))$ $\forall \hat{g} \in G \Leftrightarrow M_{\theta(g)\eta}(\theta(\hat{g})) = M_{\theta(h)\eta}(\theta(\hat{g}))$, $N_{\theta(g)\eta}(\theta(\hat{g})) = N_{\theta(h)\eta}(\theta(\hat{g}))$ $\forall \hat{g} \in G \Leftrightarrow \theta(g)\eta = \theta(h)\eta \Leftrightarrow \bar{\theta}(g\eta) = \bar{\theta}(h\eta)$. Thus, $\bar{\theta}$ is well defined and injective. Certainly, $\bar{\theta}$ maps γ/η onto itself. Because $\bar{\theta}$ is a homomorphism, then the proof is completed by the similar part of the proof of Theorem 3.13. □

Theorem 3.17. *Suppose $\theta : G \to H$ is a homomorphism, and η and γ are IFGs of G and H, respectively. Then $\theta(\eta) \subseteq \gamma \Leftrightarrow M_\gamma(\theta(g)) \geq M_\eta(g)$ and $N_\gamma(\theta(g)) \leq N_\eta(g)\ \forall\ g \in G$.*

Proof. Let $\theta(\eta) \subseteq \gamma$ and $g \in G$. Then

$$\begin{aligned} M_\gamma(\theta(g)) &\geq M_{\theta(\eta)}(\theta(g)) \\ &= \bigvee\{M_\eta(\hat{h}) \mid \theta(\hat{h}) = \theta(g)\} \\ &= M_\eta(g) \end{aligned}$$

and

$$\begin{aligned} N_\gamma(\theta(g)) &\leq N_{\theta(\eta)}(\theta(g)) \\ &= \bigwedge\{N_\eta(\hat{h}) \mid \theta(\hat{h}) = \theta(g)\} \\ &= N_\eta(g). \end{aligned}$$

Conversely, let $M_\gamma(\theta(g)) \geq M_\eta(g)$ and $N_\gamma(\theta(g)) \leq N_\eta(g)\ \forall\ g \in G$. Then

$$\begin{aligned} M_{\theta(\eta)}(h) &= \bigvee\{M_\eta(g) \mid \theta(g) = h\} \\ &\leq \bigvee\{M_\gamma(\theta(g)) \mid \theta(g) = h\} \\ &= M_\gamma(h) \end{aligned}$$

and

$$\begin{aligned} N_{\theta(\eta)}(h) &= \bigwedge\{N_\eta(g) \mid \theta(g) = h\} \\ &\leq \bigwedge\{N_\gamma(\theta(g)) \mid \theta(g) = h\} \\ &= N_\gamma(h) \end{aligned}$$

$\forall\ h \in H$. Hence, $\theta(\eta) \subseteq \gamma$. □

3.2 Intuitionistic fuzzy Frattini subgroups

Here, the concept of IFFSGs is presented and the idea of intuitionistic fuzzy maximal subgroups (IFMSGs) is defined.

Definition 3.18. Let γ be an IFG in G and η be a proper subgroup of γ. Then, η is an IFMSG if there exists a proper IFSG ζ_j, for $j = 1, \ldots, n$ of γ such that $M_{\zeta_i}(g) \leq M_\eta(g)$, $N_{\zeta_i}(g) \leq N_\eta(g)$ $\forall g \in G$. Hence, η contains all the non-trivial posible IFSGs of γ.

Definition 3.19. Let γ be an IFG in G, and assume $\eta_1, \eta_2, \ldots, \eta_n$ are IFMSGs of γ. Then, the IFFSG of γ denoted by $\Phi(\eta_j)$, $j = 1, \ldots, n$ is the intersection of η_j defined by:

$$\left.\begin{array}{rl} M_{\Phi(\eta_j)}(g) = & \min\{M_{\eta_1}(g), M_{\eta_2}(g), \ldots, M_{\eta_n}(g), \\ N_{\Phi(\eta_j)}(g) = & \max\{N_{\eta_1}(g), N_{\eta_2}(g), \ldots, N_{\eta_n}(g) \end{array}\right\},$$

$\forall g \in G$, or

$$\left.\begin{array}{r} M_{\Phi(\eta_j)}(x) = \bigwedge_{j=1}^{n} M_{\eta_j}(g), \\ N_{\Phi(\eta_j)}(x) = \bigvee_{j=1}^{n} N_{\eta_j}(g) \; \forall g \in G \end{array}\right\}.$$

Remark 3.20. Certainly, every IFFSG is an IFSG.

Proposition 3.21. *Suppose η and γ are IFGs in G, then the following are equivalent:*

(i) $\eta = \bigcap_{i=1}^{n} \eta_j$, *where η_j are the IFMSGs of γ.*

(ii) *η is an IFFSG of γ.*

Proof. Straightforward. □

Theorem 3.22. *If η is an IFG in a finite group G, then an IFFSG $\Phi(\eta_j)$ of η is an IFNSG.*

Proof. Suppose $\Phi(\eta_j)$ is an IFFSG of η, then $M_{\Phi(\eta_j)}(gh^{-1}) \geq \min\{M_{\Phi(\eta_j)}(g), M_{\Phi(\eta_j)}(h)\}$ and $N_{\Phi(\eta_j)}(gh^{-1}) \leq \max\{N_{\Phi(\eta_j)}(g), N_{\Phi(\eta_j)}(h)\}$ $\forall g, h \in G$ (Remark 3.20), which implies that $\Phi(\eta_j)$ is an IFSG of G.

To prove that $\Phi(\eta_j)$ is normal in η, let $g, h \in G$, and then we have

$$\begin{aligned} M_{\Phi(\eta_j)}(hgh^{-1}) &= M_{\Phi(\eta_j)}((hg)h^{-1}) \\ &= M_{\Phi(\eta_j)}(g(hh^{-1})) \\ &= M_{\Phi(\eta_j)}(g) \end{aligned}$$

and

$$\begin{aligned} N_{\Phi(\eta_j)}(hgh^{-1}) &= N_{\Phi(\eta_j)}((hg)h^{-1}) \\ &= N_{\Phi(\eta_j)}(g(hh^{-1})) \\ &= N_{\Phi(\eta_j)}(g). \end{aligned}$$

Hence, $\Phi(\eta_j)$ is normal in η by Definition 2.10.

□

Proposition 3.23. *Every IFFSG of an IFG is commuative.*

Proof. Let η be an IFG in G and $\Phi(\eta_j)$ be an IFFSG of η. Then, $\Phi(\eta_j)$ is an IFNSG of η by Theorem 3.22. Thus, we have $M_{\Phi(\eta_j)}(ghg^{-1}) = M_{\Phi(\eta_j)}(h)$, $N_{\Phi(\eta_j)}(ghg^{-1}) = N_{\Phi(\eta_j)}(h)$ $\forall g, h \in G \Leftrightarrow M_{\Phi(\eta_j)}(gh) = M_{\Phi(\eta_j)}(hg)$, $N_{\Phi(\eta_j)}(gh) = N_{\Phi(\eta_j)}(hg)$ $\forall g, h \in G$. Hence, the proof is completed by Definition 2.9. □

Theorem 3.24. *Let η_j be an IFMSG of an IFG η in G, then an IFFSG $\Phi(\eta_j)$ of η is an IFCSG.*

Proof. Because $\Phi(\eta_j)$ is an IFFSG of η, then

$$M_{\Phi(\eta_j)}(g) = \bigwedge_{j=1}^{n} M_{\eta_j}(g) \text{ and } N_{\Phi(\eta_j)}(g) = \bigvee_{j=1}^{n} N_{\eta_j}(g)$$

$\forall g \in G$. As $\Phi(\eta_j) = \bigcap_{i=1}^{n} \eta_j$, and η_j for $j = 1, \dots, n$ are IFMSGs of η, then $\Phi(\eta_j) \subseteq \eta_j$. Thus, $\Phi(\eta_j)$ is an IFCSG of η by Definition 3.1. □

Theorem 3.25. *Let η be an IFG in G and $\Phi(\eta_j)$ be an IFFSG of η. Then $[\Phi(\eta_j)]_*$ is a normal subgroup of G.*

Proof. First, we show that $[\Phi(\eta_j)]_*$ is a subgroup of G. Let $g, h \in [\Phi(\eta_j)]_*$, then $M_{\Phi(\eta_j)}(g) \geq 0$, $M_{\Phi(\eta_j)}(h) \geq 0$ and $N_{\Phi(\eta_j)}(g) \leq 0$, $M_{\Phi(\eta_j)}(h) \leq 0$. Using Definition 2.7, we have $M_{\Phi(\eta_j)}(gh^{-1}) \geq \min\{M_{\Phi(\eta_j)}(g), M_{\Phi(\eta_j)}(h) \geq 0$ and $N_{\Phi(\eta_j)}(gh^{-1}) \leq \max\{N_{\Phi(\eta_j)}(g), N_{\Phi(\eta_j)}(h) \leq 0$. Thus, $gh^{-1} \in [\Phi(\eta_j)]_* \Rightarrow [\Phi(\eta_j)]_*$ is a subgroup of G.

Next, we show that $[\Phi(\eta_j)]_*$ is a normal subgroup of G. By Theorem 3.22, we get

$$M_{\Phi(\eta_j)}(ghg^{-1}) = M_{\Phi(\eta_j)}(h) \text{ and } N_{\Phi(\eta_j)}(ghg^{-1}) = N_{\Phi(\eta_j)}(h).$$

Then, $M_{\Phi(\eta_j)}(ghg^{-1}) = M_{\Phi(\eta_j)}(h) \geq 0$ and $N_{\Phi(\eta_j)}(ghg^{-1}) = N_{\Phi(\eta_j)}(h) \leq 0$ by Definition 2.7. Thus, $ghg^{-1} \in \Phi(\eta_j)_*$ and so $[\Phi(\eta_j)]_*$ is a normal subgroup of G. □

Proposition 3.26. *Let η be an IFG in G and $\Phi(\eta_j)$ be an IFFSG of η. Then, $[\Phi(\eta_j)]_*$ is a characteristic subgroup of G.*

Proof. By Theorem 3.24, $\Phi(\eta_j)$ is an IFCSG of η. Using Proposition 3.8, it follows that $[\Phi(\eta_j)]_*$ is characteristic in G. □

Proposition 3.27. *Suppose η is an IFG in G and $\Phi(\eta_j)$ is an IFFSG of η. Then, $[\Phi(\eta_j)]_{[\alpha,\beta]}$ and $[\Phi(\eta_j)]_{(\alpha,\beta)}$ are subgroups of G.*

Proof. Let $g, h \in [\Phi(\eta_j)]_{[\alpha,\beta]}$. By the Definition of $[\Phi(\eta_j)]_{[\alpha,\beta]}$, we have $M_{\Phi(\eta_j)}(g) \geq \alpha$, $M_{\Phi(\eta_j)}(h) \geq \alpha$ and $N_{\Phi(\eta_j)}(g) \leq \beta$, $N_{\Phi(\eta_j)}(h) \leq \beta$. Because $\Phi(\eta_j)$ is also an IFG, we have

$$M_{\Phi(\eta_j)}(gh^{-1}) \geq \min\{M_{\Phi(\eta_j)}(g), M_{\Phi(\eta_j)}(h)\} \geq \alpha, \text{ and}$$
$$N_{\Phi(\eta_j)}(gh^{-1}) \leq \max\{M_{\Phi(\eta_j)}(g), M_{\Phi(\eta_j)}(h)\} \leq \beta,$$

which means $M_{\Phi(\eta_j)}(gh^{-1}) \geq \alpha$ and $N_{\Phi(\eta_j)}(gh^{-1}) \leq \beta$. Thus, $gh^{-1} \in [\Phi(\eta_j)]_{[\alpha,\beta]}$. Hence, $[\Phi(\eta_j)]_{[\alpha,\beta]} \subseteq G$. In the same vein, $[\Phi(\eta_j)]_{(\alpha,\beta)} \subseteq G$. □

Proposition 3.28. *Let η be an IFG in G and $\Phi(\eta_j)$ be an IFFSG of η. Then $[\Phi(\eta_j)]_{[\alpha,\beta]} \lhd G$ and $[\Phi(\eta_j)]_{(\alpha,\beta)} \lhd G$.*

Proof. By Theorem 3.22, it follows that $\Phi(\eta_i) \lhd \eta$. By Definition 2.8, $[\Phi(\eta_j)]_{[\alpha,\beta]} \subseteq G$. Now, assume $g \in G$ and $h \in [\Phi(\eta_j)]_{[\alpha,\beta]}$, then $M_{\Phi(\eta_j)}(ghg^{-1}) = M_{\Phi(\eta_j)}(h)$, $N_{\Phi(\eta_j)}(ghg^{-1}) = N_{\Phi(\eta_j)}(h)$ (Theorem 3.22). Then $M_{\Phi(\eta_j)}(ghg^{-1}) = M_{\Phi(\eta_j)}(h) \geq \alpha$ and $N_{\Phi(\eta_j)}(ghg^{-1}) = N_{\Phi(\eta_j)}(h) \leq \beta$. Thus, $ghg^{-1} \in [\Phi(\eta_i)]_{[\alpha,\beta]}$ and so, $[\Phi(\eta_j)]_{[\alpha,\beta]} \lhd G$. Similarly, $[\Phi(\eta_j)]_{(\alpha,\beta)} \lhd G$. □

Proposition 3.29. *Suppose η is an IFG in G and $\Phi(\eta_j)$ is an IFFSG of η, then $[\Phi(\eta_j)]_{[\alpha,\beta]}$ and $[\Phi(\eta_j)]_{(\alpha,\beta)}$ are characteristic subgroups of G.*

Proof. The proof is straightforward by synthesizing Theorem 3.24 and Proposition 3.27. □

Proposition 3.30. *Let η, γ, and ζ be IFGs in G. If η is an IFFSG of γ and γ is an IFFSG of ζ, then η is an IFFSG of ζ.*

Proof. The proof follows from the principle of transitivity. □

Corollary 3.31. *Let η, γ, and ζ be IFGs in G. If η is an IFFSG of γ and γ is an IFFSG of ζ, then η is an IFCSG of ζ.*

Proof. By Proposition 3.30, η is an IFFSG of ζ. Using Theorem 3.24, it follows that η is an IFCSG of ζ. □

Corollary 3.32. *Let η, γ, and ζ be IFGs in G. If η is an IFFSG of γ and γ is an IFFSG of ζ, then η is an IFNSG of ζ.*

Proof. By Proposition 3.30, η is an IFFSG of ζ. Using Theorem 3.22, it follows that η is an IFNSG of ζ. □

4 Conclusion

In this paper, the concepts of IFCSGs and IFFSGs were explored. Several results concerning IFCSGs were obtained and proved. In addition, the definition of IFFSG was presented and some results of IFFSGs in conjunction to IFCSGs, IFNSGs, cuts of IFGs, and level sets were discussed. In future, more results on IFCSGs and IFFSGs could be discussed.

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