Notes on Intuitionistic Fuzzy Sets
Print ISSN 1310–5132, Online ISSN 2367–8283
2025, Volume 31, Number 1, 1–8
DOI: 10.7546/nifs.2025.31.1.1-8

On commutativity of intuitionistic L-fuzzy groups

Poonam Kumar Sharma

P. G. Department of Mathematics, D.A.V. College
Jalandhar, Punjab, India
e-mail: pksharma@davjalandhar.com

Received: 28 September 2024 **Accepted:** 11 December 2024 **Online First:** 27 January 2025

Abstract: In this paper, we discuss the commutativity of an intuitionistic L-fuzzy subgroup of a group. Some necessary and sufficient conditions for an intuitionistic L-fuzzy subgroup to be commutative are derived. The relationship between commutativity and normality of intuitionistic L-fuzzy subgroups is briefly studied. It is also proved that any commutative intuitionistic L-fuzzy subgroup of a finite group admits a decomposition as a direct product of intuitionistic L-fuzzy subgroups of Sylow subgroups.
Keywords: Intuitionistic L-fuzzy subgroup (ILFSG); Level-cut subgroup; Normal intuitionistic L-fuzzy subgroup (NILFSG); Commutative intuitionistic L-fuzzy subgroup (CILFSG).
2020 Mathematics Subject Classification: 03F55, 08A72, 20XX .

1 Introduction

Applying the concept of intuitionistic fuzzy sets introduced by Atanassov [2, 3] to group theory Biswas [5] defined intuitionistic fuzzy subgroups of a given group and derived some of their properties. Ahn, Hur, and Jang [1] introduced the concept of level subgroups of an intuitionistic fuzzy subgroup and studied some of their properties. These concepts were further investigated by Sharma [12] and characterized some more properties of intuitionistic fuzzy subgroups by their (α, β)-cut sets. He further studied intuitionistic fuzzy abelian subgroups of a group in terms of (α, β)-cut subgroups in [13, 14]. Fathi and Salleh [6] investigate intuitionistic fuzzy subgroups based on the notion of intuitionistic fuzzy space. Rasuli [10] studied the concept of intuitionistic fuzzy subgroups with the help of norms. He also introduced and studied the notion of normalization, commutativity, and centralization in multi-fuzzy sets of a group G under t-norm in [11]. In all these studies, the closed unit interval $[0, 1]$ is taken as the membership lattice. However, the notion of intuitionistic fuzzy sets is generalized to an arbitrary lattice L in [4] by Atanassov and Stoeva. Palaniappan, Naganathan, and Arjunan [9] introduced and studied the notion of an intuitionistic L-fuzzy subgroup. Meena and Thomas [8] developed the notion of intuitionistic L-fuzzy subrings and ideals. Kanchan, Sharma, and Pathania studied the concept of intuitionistic L-fuzzy submodules in [7]. Sreedevi and Joseph [15] developed the concept of an intuitionistic L-fuzzy graph.

In this paper, we extend these concepts to the intuitionistic L–fuzzy case, where L is an arbitrary lattice, and derive some more properties. We also introduce commutativity for intuitionistic L-fuzzy subgroups and obtain some characterizations.

2 Intuitionistic L-fuzzy groups

In this section we list some basic concepts and well-known results of intuitionistic L-fuzzy sets. Throughout this paper $(L, \leq, \wedge, \vee)$ denotes a complete distributive lattice with maximal element 1 and minimal element 0, respectively, with an evaluative order reversing operation $N : L \to L$ such that $N(0) = 1$, $N(1) = 0$; If $\alpha \leq \beta$, then $N(\beta) \leq N(\alpha)$. Also, $N(N(\alpha)) = \alpha$. Moreover, $N(\vee_{i=1}^{n}\alpha_i) = \wedge_{i=1}^{n} N(\alpha_i)$ and $N(\wedge_{i=1}^{n}\alpha_i) = \vee_{i=1}^{n} N(\alpha_i)$.

Definition 2.1. ([4]) An intuitionistic L-fuzzy set A in X is defined as an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where $\mu_A : X \to L$ and $\nu_A : X \to L$ define the degree of membership and the degree of non membership for every $x \in X$ satisfying $\mu_A(x) \leq N(\nu_A(x))$.

We write an intuitionistic L-fuzzy set by ILFS and the set of all ILFSs on X by $(L \times L)^X$.

Remark 2.2. (i) When $\mu_A(x) = N(\nu_A(x))$, for all $x \in X$, then A is called L-fuzzy set.

(ii) When $L = [0, 1]$, $N(\alpha) = 1 - \alpha, \forall \alpha \in L$. Then the ILFS is called an intuitionistic fuzzy set.

(iii) We use the notion $A = (\mu_A, \nu_A)$ to denote the ILFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$.

Proposition 2.3. ([4]) *If $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two ILFSs of X, then:*

(i) $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$;

(ii) $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$;

(iii) $A^c = (\mu_{A^c}, \nu_{A^c})$, where $\mu_{A^c}(x) = \nu_A(x)$ and $\nu_{A^c}(x) = \mu_A(x)$;

(iv) $A \cap B = (\mu_{A\cap B}, \nu_{A\cap B})$ where $\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x)$ and $\nu_{A\cap B}(x) = \nu_A(x) \vee \nu_B(x)$;

(v) $A \cup B = (\mu_{A\cup B}, \nu_{A\cup B})$ where $\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x)$ and $\nu_{A\cup B}(x) = \nu_A(x) \wedge \nu_B(x)$.

(vi) $A \times B = (\mu_{A\times B}, \nu_{A\times B})$ where $\mu_{A\times B}(x, y) = \mu_A(x) \wedge \mu_B(y)$ and $\nu_{A\times B}(x, y) = \nu_A(x) \vee \nu_B(y)$.

Definition 2.4. ([7]) Let $A \in (L \times L)^X$ and $\alpha, \beta \in L$ with $\alpha \leq N(\beta)$, then (α, β)-cut set of A is a crisp set denoted by $A_{(\alpha,\beta)}$ and is defined as $A_{(\alpha,\beta)} = \{x \in X : \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta\}$ and support of A in X is denoted by $\operatorname{supp}(A)$ and is defined as

$$\operatorname{supp}(A) = \{x \in X : \mu_A(x) > 0 \text{ and } \nu_A(x) < 1\}.$$

Definition 2.5. ([8]) Let Y be any subset of X. Then the intuitionistic L-fuzzy characteristic function $\chi_Y = (\mu_{\chi_Y}, \nu_{\chi_Y})$ on Y is defined as

$$\mu_{\chi_Y}(x) = \begin{cases} 1, & \text{if } x \in Y \\ 0, & \text{otherwise} \end{cases} ; \quad \nu_{\chi_Y}(x) = \begin{cases} 0, & \text{if } x \in Y \\ 1, & \text{otherwise.} \end{cases}$$

Remark 2.6. When $Y = \emptyset$ or $Y = X$, then χ_Y is denoted by $\widetilde{0}$ and $\widetilde{1}$, respectively.

Definition 2.7. ([8]) Let $A = (\mu_A, \nu_A)$ be an ILFS of X and $Y \subseteq X$. Then the restriction of A to the set Y is an ILFS $A|_Y = (\mu_{A|_Y}, \nu_{A|_Y})$ of Y and is defined as

$$\mu_{A|_Y}(y) = \begin{cases} \mu_A(y), & \text{if } y \in Y \\ 0, & \text{otherwise} \end{cases} ; \quad \nu_{A|_Y}(x) = \begin{cases} \nu_A(y), & \text{if } y \in Y \\ 1, & \text{otherwise.} \end{cases}$$

Definition 2.8. ([9]) An intuitionistic L-fuzzy set A of a multiplicative group G is said to be an intuitionistic L-fuzzy group of G (or an intuitionistic L-fuzzy subgroup of G) if for every $x, y \in G$, the following holds:

(i) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$

(ii)$\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$

(iii)$\mu_A(x^{-1}) \geq \mu_A(x)$

(iv) $\nu_A(x^{-1}) \leq \nu_A(x)$.

or equivalently, $\mu_A(xy^{-1}) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy^{-1}) \leq \nu_A(x) \vee \nu_A(y)$ holds.

Remark 2.9. If G is a group and L is a lattice, then $(G, L \times L)$ shall denote the collection of all intuitionistic L-fuzzy subgroups of G.

Proposition 2.10. ([8]) *Let G be a group and $A \in (L \times L)^G$. Then $A \in (G, L \times L)$ if and only if $A_{(\alpha,\beta)}$ is a subgroup of G, for all $\alpha, \beta \in L$ such that $\alpha \leq N(\beta)$.*

Definition 2.11. ([10]) Let G be a group and $A \in (G, L \times L)$. Then A is called a normal intuitionistic L-fuzzy subgroup (NILFSG) of G if $A(xyx^{-1}) = A(y)$, for all $x, y \in G$ or equivalently, $A(xy) = A(yx)$, for all $x, y \in G$.

Definition 2.12. ([7]) The lattice L is said to be regular if $a \wedge b > 0$ for every $a > 0, b > 0$ and $a \vee b < 1$ for every $a < 1, b < 1$.

Every chain is a regular lattice. In particular, $[0, 1]$ is a regular lattice.

In [13] author has considered the following definition of an intuitionistic L-fuzzy abelian group of a group G, when $L = [0, 1]$ and $N : L \to L$ defined as $N(\alpha) = 1 - \alpha, \forall \alpha \in L$.

Definition 2.13. Let G be a group and $A \in (L \times L)^G$. Then A is called an intuitionistic L-fuzzy abelian group of G if $A_{(\alpha,\beta)}$ is an abelian subgroup of G, for all $\alpha, \beta \in L - \{0\}$ such that $\alpha \leq N(\beta)$.

3 Commutativity of intuitionistic L-fuzzy groups

Throughout this section, we assume that L is a regular lattice. If $A\ (\neq \tilde{0})$ is an ILFS of G, then the restriction of A to $\text{supp}(A)$ is denoted by $A|_{\text{supp}(A)}$ is an ILFS of $\text{supp}(A)$. We shall denote $A|_{\text{supp}(A)}$ by $\tilde{A}$.

Definition 3.1. An ILFSG A of a group G is said to be commutative if $\tilde{A_{xy}} = \tilde{A_{yx}}$ for all $x, y \in G$ with $x, y \in \text{supp}(A)$, where $\tilde{A_{xy}} = A|_{\{xy\}}$, where $\{xy\}$ is a singleton subset of G.

It may be noted that commutativity of A requires xy and yx to coincide whenever $x, y \in \text{supp}(A)$. Observe that this definition actually generalizes the notion of commutativity of ordinary subgroup. That is, for any non-empty subset H of G, χ_H is a commutative intuitionistic L-fuzzy subgroups (CILFSG) of G if and only if H is a commutative subgroup of G.

Example 3.2. Let $L = [0, 1]$, $N(\alpha) = 1 - \alpha, \forall \alpha \in L$, and $G = S_3 = \{i, (abc), (acb), (ab), (bc), (ca)\}$ be a symmetric group on three elements $\{a, b, c\}$. Define an ILFS A of G as follows:

$$\mu_A(x) = \begin{cases} 1, & \text{if } x = i \\ 0.5, & \text{if } x = (abc), (acb) \\ 0, & \text{if } x = (ab), (bc), (ca) \end{cases} \quad ; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x = i \\ 0.3, & \text{if } x = (abc), (acb) \\ 0.5, & \text{if } x = (ab), (bc), (ca). \end{cases}$$

Clearly, A is an ILFSG of G with $\text{supp}(A) = \{i, (abc), (acb)\}$.

Notice that for each $x, y \in \text{supp}(A)$, we have $\tilde{A_{xy}} = \tilde{A_{yx}}$. Hence A is a CILFSG of G.

Theorem 3.3. *Let $A (\neq \tilde{0})$ be an ILFSG of a group G. Then the following are equivalent:*

(a) A is a CILFSG of G;

(b) $\text{supp}(A)$ is a commutative subgroup of G;

(c) The (α, β)-cut $A_{(\alpha,\beta)}$ are commutative subgroups of G for every $\alpha, \beta \in L \setminus \{0\}$ such that $\alpha \leq N(\beta)$.

Proof.

(a) $\Rightarrow$ (b) Let A be a CILFSG of G, then $\text{supp}(A)$ is a subgroup of G. Let $x, y \in \text{supp}(A)$. Since A is commutative, therefore, $\tilde{A_{xy}} = \tilde{A_{yx}}$. Thus $xy = yx$.

(b) $\Rightarrow$ (c) Assume that $\text{supp}(A)$ is a commutative subgroup of G. Let $\alpha, \beta \in L \setminus \{0\}$ such that $\alpha \leq N(\beta)$. Then $A_{(\alpha,\beta)}$ is a subgroup of G. Let $x, y \in A_{(\alpha,\beta)}$. Then $\mu_A(x) \geq \alpha > 0, \nu_A(x) \leq \beta < 1$ and $\mu_A(y) \geq \alpha > 0, \nu_A(y) \leq \beta < 1$ implies that $x, y \in \text{supp}(A)$ and hence $xy = yx$.

(c) $\Rightarrow$ (a) Assume that (c) hold. Let $x, y \in G$ such that $\mu_A(x) > 0, \nu_A(x) < 1$ and $\mu_A(y) > 0, \nu_A(y) < 1$. Let $\mu_A(x) = \alpha_1, \mu_A(y) = \alpha_2$ and $\nu_A(x) = \beta_1, \nu_A(y) = \beta_2$. Then $x \in A_{(\alpha_1,\beta_1)}$ and $y \in A_{(\alpha_2,\beta_2)}$. Put $\alpha^{'} = \alpha_1 \wedge \alpha_2 > 0$ and $\beta^{'} = \beta_1 \vee \beta_2 < 1$. Also $\alpha^{'} \leq \alpha_1, \alpha^{'} \leq \alpha_2$ and $\beta^{'} \geq \beta_1, \beta^{'} \geq \beta_2$ so that $A_{(\alpha',\beta')} \supseteq A_{(\alpha_1,\beta_1)}$ and $A_{(\alpha',\beta')} \supseteq A_{(\alpha_2,\beta_2)}$. Therefore $x, y \in A_{(\alpha',\beta')}$. But $A_{(\alpha',\beta')}$ is a commutative subgroup of G. Hence $xy = yx$ and therefore, $\tilde{A_{xy}} = \tilde{A_{yx}}$, i.e., A is CILFSG of G. □

Theorem 3.4. *If A is a CILFSG of G and $\mathrm{supp}(A)$ is a normal subgroup of G, then A is a NILFSG of G.*

Proof. Let $x, y \in G$. We have the following cases:

Case 1. When $x, y \in \mathrm{supp}(A)$. Then $\tilde{A_{xy}} = \tilde{A_{yx}}$. Hence $A(xy) = A(yx)$.

Case 2. When $x \in \mathrm{supp}(A)$ and $y \notin \mathrm{supp}(A)$. Then both xy and yx does not belongs to $\mathrm{supp}(A)$. Hence $A(xy) = A(yx) = (0, 1)$.

Case 3. When $x, y \notin \mathrm{supp}(A)$. Then xy and yx may or may not belongs to $\mathrm{supp}(A)$. Since $\mathrm{supp}(A)$ is a normal subgroup of G, either xy and yx both belongs to $\mathrm{supp}(A)$ or both does not belongs to $\mathrm{supp}(A)$.
(i) If $xy, yx \notin \mathrm{supp}(A)$, then $A(xy) = A(yx) = (0, 1)$.
(ii)If $xy, yx \in \mathrm{supp}(A)$, then by Theorem (3.3), $xy = yx$ and hence $A(xy) = A(yx)$. □

We observe from the following example that the converse of Theorem (3.4) is not true.

Example 3.5. Let $L = [0, 1]$, $N(\alpha) = 1-\alpha, \forall \alpha \in L$ and G be any non-commutative group. Since G is a normal subgroup of itself, χ_G is a NILFSG of G. But $\mathrm{supp}(\chi_G) = G$ is not commutative. Hence χ_G is not a CILFSG of G.

Theorem 3.6. *For any group G, the following are equivalent:*

(a) G is commutative,

(b) All ILFSGs of G are commutative,

(c) χ_G is a CILFSG of G.

Proof. (a) $\Rightarrow$ (b) Let G be a commutative group and A be any ILFSG of G. For any $x, y \in G$, $xy = yx$ and hence $\tilde{A_{xy}} = \tilde{A_{yx}}$. Hence any ILFSG of G is commutative.

(b) $\Rightarrow$ (c) Trivial, since χ_G itself is an ILFSG of G.

(c) $\Rightarrow$ (a) Let χ_G be a CILFSG of G. Then $\mathrm{supp}(A) = G$ is commutative. □

Definition 3.7. Let $G_i (i = 1, 2, 3, \ldots, n)$ be groups, $G = \prod_{i=1}^{n} G_i$ be their direct product and $\pi_i : G \to G_i$ be the projections defined by $\pi_i(x_1, x_2, \ldots, x_i, \ldots, x_n) = x_i$. The direct product of ILFSs A_i of $G_i (i = 1, 2, 3, \ldots, n)$ is defined as an ILFS $A = \prod_{i=1}^{n} A_i$ of G given by

$$\mu_A(x) = \bigwedge_{i=1}^{n} \mu_{A_i}(\pi_i(x)) \text{ and } \nu_A(x) = \bigvee_{i=1}^{n} \nu_{A_i}(\pi_i(x)), \forall x = (x_1, x_2, \ldots, x_i, \ldots, x_n) \in G.$$

Proposition 3.8. *If A_i is a (commutative) ILFSG of G_i for each $i = 1, 2, 3, \ldots, n$, then $\prod_{i=1}^{n} A_i$ is a (commutative) ILFSG of $\prod_{i=1}^{n} G_i$.*

Proof. Let $A = \prod_{i=1}^{n} A_i$ and $G = \prod_{i=1}^{n} G_i$. Then, for $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in G$, we have

$$\begin{aligned} \mu_A(xy^{-1}) &= \bigwedge_{i=1}^{n} \{\mu_{A_i}(x_i y_i^{-1})\} \\ &\geq \bigwedge_{i=1}^{n} \{\mu_{A_i}(x_i) \wedge \mu_{A_i}(y_i)\} \\ &= (\bigwedge_{i=1}^{n} \mu_{A_i}(x_i)) \wedge (\bigwedge_{i=1}^{n} \mu_{A_i}(y_i)) \\ &= \mu_A(x) \wedge \mu_A(y). \end{aligned}$$

i.e., $\mu_A(xy^{-1}) \geq \mu_A(x) \wedge \mu_A(y)$. Similarly, we can show that $\nu_A(xy^{-1}) \leq \nu_A(x) \vee \nu_A(y)$. Hence A is an ILFSG of G. □

Now, let A_i be CILFSG of G_i and $x, y \in \mathrm{supp}(A)$. Then
$\mu_A(x) = \bigwedge\{\mu_{A_i}(x_i) : i = 1, 2, 3, \ldots, n\} > 0$ and $\nu_A(x) = \bigvee\{\nu_{A_i}(x_i) : i = 1, 2, 3, \ldots, n\} < 1$.
Therefore, $\mu_{A_i}(x_i) > 0, \nu_{A_i}(x_i) < 1$, for each $i = 1, 2, 3, \ldots, n$.
Similarly, $\mu_{A_i}(y_i) > 0, \nu_{A_i}(y_i) < 1, \forall i = 1, 2, 3, \ldots, n$. Hence $x_i, y_i \in \mathrm{supp}(A_i), \forall i = 1, 2, 3, \ldots, n$. Since each A_i is CILFSG, by Theorem (3.3), $x_i y_i = y_i x_i, \forall i = 1, 2, 3, \ldots, n$. Hence $xy = yx$. Thus $\mathrm{supp}(A)$ is commutative; and hence A is a CILFSG of G.

The converse of the above proposition is not true.

Example 3.9. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_3$, where $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are additive groups of integers $\mathrm{mod}\ 2$ and $\mathrm{mod}\ 3$, respectively, and $L = [0, 1]$. Define ILFSs A_1 and A_2 of G as follows:

$$\mu_{A_1}(x) = \begin{cases} 0.75, & \text{if } x = 0 \\ 1, & \text{if } x = 1 \end{cases}; \quad \nu_{A_1}(x) = \begin{cases} 0.2, & \text{if } x = 0 \\ 0, & \text{if } x = 1. \end{cases}$$

and

$$\mu_{A_2}(x) = \begin{cases} 0.5, & \text{if } x = 0 \\ 0.25, & \text{if } x = 1, 2 \end{cases}; \quad \nu_{A_2}(x) = \begin{cases} 0.3, & \text{if } x = 0 \\ 0.7, & \text{if } x = 1, 2. \end{cases}$$

Then $A = A_1 \times A_2$ is defined as

$$\mu_{A_1 \times A_2}((x, y)) = \begin{cases} 0.75, & \text{if } (x, y) = (0, 0) \\ 1, & \text{if } (x, y) \neq (0, 0) \end{cases}; \quad \nu_{A_1 \times A_2}((x, y)) = \begin{cases} 0.2, & \text{if } (x, y) = (0, 0) \\ 0, & \text{if } (x, y) \neq (0, 0). \end{cases}$$

It is easy to verify that A is an ILFSG of G. But A_1 is not an IFLSG of $\mathbb{Z}_2$.

Theorem 3.10. *Let $G = \prod_{i=1}^{n} G_i$, where $|G_i|$ and $|G_j|$ are relatively prime for every $i \neq j$, and A is an ILFSG of G. Then there exists ILFSGs A_i of G_i ($i = 1, 2, \ldots, n$) such that $A = \prod_{i=1}^{n} A_i$.*

Proof. Define $A_i \in (L \times L)^{G_i} (i = 1, 2, \ldots, n)$ by $A_i(x_i) = A(1, 1, \ldots, x_i, 1, 1, \ldots, 1)$.
Let $x_i, x_i^{'} \in G_i$, we have

$$\begin{aligned}
\mu_{A_i}(x_i x_i^{'}) &= \mu_A((1, 1, \ldots, x_i x_i^{'}, 1, 1, \ldots, 1)) \\
&= \mu_A[(1, 1, \ldots, x_i, 1, 1, \ldots, 1)(1, 1, \ldots, x_i^{'}, 1, 1, \ldots, 1)] \\
&\geq \mu_A(1, 1, \ldots, x_i, 1, 1, \ldots, 1) \wedge \mu_A(1, 1, \ldots, x_i^{'}, 1, 1, \ldots, 1) \\
&= \mu_{A_i}(x_i) \wedge \mu_{A_i}(x_i^{'}).
\end{aligned}$$

Thus, $\mu_{A_i}(x_i x_i^{'}) \geq \mu_{A_i}(x_i) \wedge \mu_{A_i}(x_i^{'})$. Similarly, we can show that $\nu_{A_i}(x_i x_i^{'}) \leq \nu_{A_i}(x_i) \vee \nu_{A_i}(x_i^{'})$.

Further, $\mu_{A_i}(x_i^{-1}) = \mu_A((1, 1, \ldots, x_i^{-1}, 1, 1, \ldots, 1)) \geq \mu_A((1, 1, \ldots, x_i, 1, 1, \ldots, 1)) = \mu_{A_i}(x_i)$. Replacing x_i with x_i^{-1} we obtain $\mu_{A_i}(x_i) \geq \mu_{A_i}(x_i^{-1})$. Thus $\mu_{A_i}(x_i^{-1}) = \mu_{A_i}(x_i)$. Similarly, we can show that $\nu_{A_i}(x_i^{-1}) = \nu_{A_i}(x_i)$. Hence A_i is an ILFSG of G_i.

Moreover, we have $\gcd(|G_i|, |G_j|) = 1$ for $i \neq j$. So we can write

$$\begin{aligned}
&(x_1, x_2, \ldots, x_i, \ldots, x_n) \\
&= (x_1, 1, 1, \ldots, 1)(1, x_2, 1, \ldots, 1) \cdots (1, 1, \ldots, 1, x_i, 1, \ldots, 1) \cdots (1, 1, 1, \ldots, x_n).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mu_A(x_1, x_2, \ldots, x_i, \ldots, x_n) \\
= &\mu_A((x_1, 1, 1, \ldots, 1)(1, x_2, 1, \ldots, 1) \cdots (1, 1, \ldots, 1, x_i, 1, \ldots, 1) \cdots (1, 1, 1, .., x_n)) \\
= &\mu_A(x_1, 1.., 1) \wedge \mu_A(1, x_2, 1, \ldots, 1) \wedge \cdots \wedge \mu_A(1.., 1, x_i, 1, \ldots, 1) \wedge \cdots \wedge \mu_A(1, \ldots, x_n) \\
= &\mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2) \wedge \cdots \wedge \mu_{A_i}(x_i) \wedge \cdots \wedge \mu_{A_n}(x_n) \\
= &\mu_{A_1 \times A_2 \times \cdots \times A_i \times \cdots \times A_n}(x_1, x_2, \ldots, x_i, \ldots, x_n).
\end{aligned}$$

Similarly, we can show that

$$\mu_A(x_1, x_2, \ldots, x_i, \ldots, x_n) = \mu_{A_1 \times A_2 \times \cdots \times A_i \times \cdots \times A_n}(x_1, x_2, \ldots, x_i, \ldots, x_n).$$

This implies that $A = \prod_{i=1}^{n} A_i$. □

Theorem 3.11. *Let A be a CILFSG of a finite group G. Then there exists a decomposition of A into the direct product of ILFSGs of Sylow subgroups of* $\text{supp}(A)$.

Proof. By Theorem (3.3), $\text{supp}(A)$ is a commutative subgroup of G. Let $|\text{supp}(A)| = p_1^{t_1} p_2^{t_2} \cdots p_m^{t_m}$, where $p_i^{t_i}$ are distinct primes and t_i are positive integers, $\forall i = 1, 2, \ldots, m$. Then

$$\text{supp}(A) = \prod_{i=1}^{m} S(p_i),$$

where $S(p_i)$ are the p_i-Sylow subgroup of $\text{supp}(A)$ of order $p_i^{t_i}$. Since $|S(p_i)|$ and $|S(p_j)|$ $(i \neq j)$ are relatively prime, there exists ILFSGs A_i of $S(p_i)$ such that $A = \prod_{i=1}^{m} A_i$. □

References

[1] Ahn, T. C., Hur, K., & Kang, H. W. (2006). Intuitionistic fuzzy subgroups and level subgroups. *International Journal of Fuzzy Logic and Intelligent Systems*, 6(3), 240–246.

[2] Atanassov, K. T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1), 87–96.

[3] Atanassov, K. T. (1999). *Intuitionistic Fuzzy Sets Theory and Applications.* Studies on Fuzziness and Soft Computing, 35, Physica-Verlag, Heidelberg.

[4] Atanassov, K., & Stoeva, S. (1984). Intuitionistic L-fuzzy sets, *Cybernetics and Systems Research, Vol. 2*, R. Trappl (ed.). Elsevier Science Publishers B.V., North-Holland, pp. 539–540.

[5] Biswas, R. (1996). Intuitionistic fuzzy subgroups. *Math. Forum*, 10, 39–44.

[6] Fathi, M., & Salleh, A.R. (2009). Intuitionistic fuzzy subgroups, *Asian Journal of Algebra,* 2(1), 1–10.

[7] Kanchan, Sharma, P. K., & Pathania, D. S. (2020). Intuitionistic L-fuzzy submodules. *Advances in Fuzzy Sets and Systems*, 25(2), 123–142.

[8] Meena, K., & Thomas, K. V. (2011). Intuitionistic L-fuzzy subrings. *International Mathematical Forum*, 6(52), 2561-2572.

[9] Palaniappan, N., Naganathan, S., & Arjunan, K. (2009). A Study on Intuitionistic L-Fuzzy Subgroups. *Applied Mathematical Sciences*, 3(53), 2619–2624.

[10] Rasuli, R. (2020). Intuitionistic fuzzy subgroups with respect to norms (T, S). *Engineering and Applied Science Letters*, 3(2), 40–53.

[11] Rasuli, R. (2023). Normalization, commutativity and centralization of TFSM(G). *Journal of Discrete Mathematical Sciences & Cryptography*, 26(4), 1027–1050.

[12] Sharma, P. K. (2011). Intuitionistic fuzzy groups. *International Journal of Data Warehousing and Mining*, 1(1), 86–94.

[13] Sharma, P. K. (2012). On intuitionistic fuzzy Abelian subgroups. *Advances in Fuzzy Sets and Systems,* 12(1), 1–16.

[14] Sharma, P. K. (2016). On intuitionistic fuzzy Abelian subgroups-II. *Advances in Fuzzy Sets and Systems*, 21(1), 1–6.

[15] Sreedevi, V. S. & Joseph, B. (2020). Intuitionistic L-fuzzy graph. *Journal of Mathematical and Computational Science*, 10(5), 1748–1754.